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Commutativity Criteria in Banach Algebras

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Abstract. We consider complex Banach algebras satisfying the condition $(xy)^k = x^k y^k$ for all x, y in the algebra where k is an integer ($k \geq 2$).

We show that for $k = 2$ and $k = 3$, this condition yields commutativity in unital Banach algebras. For higher values of k, commutativity is obtained for semi-simple algebras and the conclusions are quite similar to the ones in [3].

The extension of the results to wider classes of algebras is also considered.

Résumé. Nous considérons des algèbres de Banach complexes vérifiant la condition $(xy)^k = x^k y^k$ pour tout x, y dans l'algèbre, k étant un entier ($k \geq 2$).

Nous montrons que pour $k = 2$ et $k = 3$, cette condition entraine la commutativité dans les algèbres de Banach unitaires. Pour les valeurs plus elevées de k, la commutativité est établie dans les algèbres semi-simples avec des résultats similaires à ceux obtenus dans [3].

L'extension des résultats à d'autres classes d'algèbres topologiques est également considérée.

1. Introduction The study of metric and algebraic conditions implying commutativity in Banach algebras has been inaugurated by C. Le Page in [5]. Since then, many characterizations for different levels of commutativity in Banach (and topological) algebras were obtained by several authors, including [6], [2], [3] and others.

In addition to their possible direct applications, these characterizations uncover interesting properties of commutative Banach algebras.

In this line of ideas, this note prolongs the study of commutativity in Banach algebras undertaken in [3] and continued in [4]. Using simple algebraic methods and also classical methods of complex analysis [2], we obtain characterizations for various levels of commutativity in a complex Banach algebra.

The main characterizations of commutativity in Banach algebras in this paper are stated in Theorem 2.1 and Theorem 2.3 below. Their proof is based on the following commutativity criterion.

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LEMMA 1.1. *The unital Banach algebra A is commutative if and only if $e^x e^y = e^y e^x$ for all $x, y \in A$.*

PROOF. It is enough to show that the condition is sufficient. This will be done in two steps.

- Let $x, y \in A$ and μ a complex number. For each bounded linear functional φ on A, define the complex valued function f by:

$$f(\lambda) = \varphi(e^{\lambda x} e^{\mu y} e^{-\lambda x}); \qquad (\lambda \text{ a complex number}).$$

The function f is an entire function that is bounded since, by the assumption,

$$|f(\lambda)| \leq \|\varphi\| \left\| e^{\lambda x} e^{\mu y} e^{-\lambda x} \right\| = \|e^{\mu y}\|; \qquad (\lambda \text{ a complex number}).$$

Thus, by Liouville theorem, f is constant. Hence

$$0 = f'(\lambda) = \varphi(x e^{\lambda x} e^{\mu y} e^{-\lambda x} - e^{\lambda x} e^{\mu y} x e^{-\lambda x}).$$

Taking $\lambda = 0$ yields $\varphi(x e^{\mu y} - e^{\mu y} x) = 0$ for each bounded linear functional φ on A. By the Hahn-Banach theorem, we get:

$$x e^{\mu y} = e^{\mu y} x, \quad \mu \text{ a complex number.} \tag{1}$$

- For $x, y \in A$ and for each bounded linear functional φ on A, define the complex valued function g by:

$$g(\lambda) = \varphi(e^{\lambda x} y e^{-\lambda x}); \qquad (\lambda \text{ a complex number}).$$

The function g is entire and bounded by (1). Using the same arguments as before, we conclude that:

$$xy = yx \quad \textit{for all } x, y \in A \tag{2}$$

and the algebra is commutative.

□

It is well known that in a Banach algebra the set of invertible elements is open and therefore the requirement $xy = yx$ for all invertible elements x, y is equivalent to commutativity. Lemma 1.1 shows that this requirement can be restricted to invertible elements of the form e^x, $x \in A$.

2. Main Results

Our first characterization of commutativity in complex Banach algebras is:

THEOREM 2.1. *The unital Banach algebra A is commutative if and only if:*

$$(xy)^2 = x^2y^2 \text{ for all } x, y \in A.$$

PROOF. If x and y are invertible elements of A, we multiply $(xy)^2 = x^2y^2$ by x^{-1} on the left and y^{-1} on the right to obtain $yx = xy$. Hence $e^x e^y = e^y e^x$, $for\ all\ x,\, y \in A$, and the conclusion follows from Lemma 1.1. □

Note that the assumption "unital" in Theorem 2.1 cannot be dispensed with as shown by the following example.

EXAMPLE 2.2. Let e_1 and e_2 be two symbols. Set ${e_1}^2 = {e_2}^2 = 0$ and $e_ie_je_i = 0$ for $i, j = 1, 2$. Under these conditions, the complex algebra A generated by e_1 and e_2 coincides with the 4-dimensional vector space whose canonical basis is $\{e_1, e_2, e_1e_2, e_2e_1\}$.
For $x \in A$, $x = \lambda_1e_1 + \lambda_2e_2 + \lambda_3e_1e_2 + \lambda_4e_2e_1$ define $\|x\| = \sum_1^4 |\lambda_i|$ to obtain a Banach algebra norm on A. The Banach algebra A satisfies $(xy)^2 = x^2y^2$ for all $x, y \in A$, but is not commutative.

In fact, the property stated in the theorem above for $k = 2$ is still true for $k = 3$, and we have:

THEOREM 2.3. *The unital Banach algebra A is commutative if and only if:*

$$(xy)^3 = x^3y^3 \text{ for all } x, y \in A.$$

PROOF. • First, note that if x and y are invertible elements of A, then multiplication of the relation $(xy)^3 = x^3y^3$ by x^{-1} on the left and y^{-1} on the right gives $(yx)^2 = x^2y^2$. Therefore $(yx)^4 = ((xy)^2)^2 = (x^2y^2)^2 = y^4x^4$. Multiplying anew by y^{-1} on the left and x^{-1} on the right, we obtain:

$$(xy)^3 = (yx)^3 \quad for\ all\ invertible\ x, y \in A. \tag{3}$$

• Let $x, y \in A$ and $z = e^x$. For each bounded linear functional φ on A, define the complex valued function h by:

$$h(\lambda) = \phi((e^{\lambda z}e^y e^{-\lambda z})^3); \qquad (\lambda \text{ a complex number}).$$

The entire function h is bounded since by (3) we have:

$$\|h(\lambda)\| \leq \|\varphi\| \, \|(e^y)\|^3 .$$

Using Liouville theorem, Hahn-Banach theorem and taking the derivative at $\lambda = 0$ we get $ze^y = e^y z$, hence:

$$e^x e^y = e^y e^x \quad for\ all\ x, y \in A. \tag{4}$$

Therefore, the algebra is commutative by Lemma 1.1.

□

The Banach algebra $\mathbb{C}^3$ with product defined by

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (0, 0, x_1 y_2 + 2x_2 y_1)$$

shows that the condition "unital" cannot be relaxed in Theorem 2.3.

EXAMPLE 2.4. Consider a unital Banach algebra A such that $A^3 \subseteq Z(A)$ where:

$$Z(A) = \{x \in A : xy = yx \text{ for all } y \in A\} \text{ is the center of the algebra.}$$

Then the algebra A is commutative since by associativity we have:

$$(xy)^3 = x(yxy)xy = x^2(y^2x)y = x^2y^3x = x^3y^3.$$

THEOREM 2.5. *Let $k \geq 2$ be a positive integer. If the Banach algebra A satisfies $(xy)^k = x^k y^k$ for all $x, y \in A$, then the algebra A is almost commutative [3], that is, $A/\mathcal{R}\mathrm{adA}$ is commutative .*

PROOF. In fact, the assumption guarantees that the spectral radius is submultiplicative [3]:

$$\begin{aligned} \rho(xy) &= \lim_{n \longrightarrow \infty} \left\| (xy)^{k^n} \right\|^{\frac{1}{k^n}} = \lim_{n \longrightarrow \infty} \left\| x^{k^n} y^{k^n} \right\|^{\frac{1}{k^n}} \\ &\leq \lim_{n \longrightarrow \infty} \left\| x^{k^n} \right\|^{\frac{1}{k^n}} \left\| y^{k^n} \right\|^{\frac{1}{k^n}} = \rho(x)\rho(y). \end{aligned}$$

Therefore, the algebra $A/\mathcal{R}\mathrm{adA}$ is commutative [3]. □

Remark 2.6. Theorem 2.4 shows that a semi-simple Banach algebra satisfying the condition $(xy)^k = x^k y^k$ for all $x, y \in A$, $(k \geq 2)$, is commutative.

The following corollary clarifies the relationship between the results obtained in the present note and those given in [3] and [4].

COROLLARY 2.7. *Suppose that $(xy)^k = x^k y^k$ for all x, y in the Banach algebra A, where k is a positive integer ($k \geq 2$). Then the algebra A satisfies the following equivalent properties:*

(1) $\rho(xy) \leq \rho(x) \|y\| ; \forall x, y \in E$.
(2) *If x is quasi-invertible with quasi-inverse x^0, the set $\left\{x + y : \|y\| < \frac{1}{1+\rho(x^0)}\right\}$ is contained in the set of quasi-invertible elements of E.*
(3) $\rho(x + y) \leq \rho(x) + \|y\| ; \forall x, y \in E$.
(4) *$E/\mathcal{R}\mathrm{adA}$ is commutative.*
(5) $|\rho(x) - \rho(y)| \leq \|x - y\| ; \forall x, y \in E$.
(6) $\rho(xy) \leq \rho(x)\rho(y); \forall x, y \in E$.
(7) $\rho(x + y) \leq \rho(x) + \rho(y); \forall x, y \in E$.
(8) *If x is quasi-invertible with quasi-inverse x^0, the set $\left\{x + y : \rho(y) < \frac{1}{1+\rho(x^0)}\right\}$ is contained in the set of quasi-invertible elements of E.*

3. Extensions The results obtained in the previous section for complex Banach algebras extend, without noteworthy modifications, to complex complete locally multiplicatively convex algebras (l.m.c.a.) [1].

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