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# Constructive Geometrization of Thurston Maps

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Abstract. We prove that every Thurston map can be constructively geometrized in a canonical fashion. According to Thurston's theorem, a map with hyperbolic orbifold has a canonical geometrization – a combinatorially equivalent postcritically finite rational map of the Riemann sphere – if and only if there is no Thurston obstruction. We follow Pilgrim's idea of a canonical decomposition of a Thurston map to handle the obstructed case. A key ingredient of our proof is a geometrization result for marked Thurston maps with parabolic orbifolds – an analogue of Thurston's theorem for the exceptional case not covered by it.

Résumé. On montre que toute application de Thurston peut être géométrisée de façon constructive et canonique. Selon le théoreme de Thurston, une telle application ayant un orbifold hyperbolique possède une géométrisation canonique, c'est-à-dire une fonction rationnelle combinatoriellement équivalente dont les orbites critiques sont finies, si et seulement s'il n'existe pas d'obstruction de Thurston. On traite le cas où il existe une obstruction en utilisant l'idée de Pilgrim d'une décomposition canonique d'une application de Thurston. L'ingrédient principal de la preuve est un résultat de géométrisation pour les applications de Thurston marquées ayant un orbifold parabolique – un analogue du théorème de Thurston pour le cas exceptionnel.

## 1. Introduction

*A brief summary of the results* The celebrated theorem of Thurston [3] on topological characterization of postcritically finite rational functions is a *geometrization* result: it gives a criterion for a postcritically finite branched cover $f : S^2 \to S^2$ (a *Thurston map*) to be realizable as a unique rational mapping $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This criterion – the absence of *Thurston obstructions*, described below – applies under an important technical assumption that the orbifold of $f$ be of hyperbolic type. Conveniently, "most" Thurston maps satisfy this assumption.

As was shown in [1] by the second author and others, given a combinatorial description of $f$, the criterion can be checked constructively, and in the case

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when $f$ is not obstructed, the geometrization $R$ can be found algorithmically. In the general case of an obstructed Thurston map, a major step towards finding a canonical geometrization was made by Pilgrim [6, 7], who introduced the concept of a *canonical Thurston obstruction*, and the corresponding *decomposition* of a Thurston map into a finite collection of branched covers with some marked pre-periodic orbits. Furthermore, the first author gave a topological characterization of canonical obstructions [9], and demonstrated that the elements of the decomposition which have hyperbolic orbifolds are not obstructed [8] (and hence can be constructively geometrized).

The main remaining obstacle to constructive geometrization of a Thurston map should now be apparent: it is to produce a geometrization result for a Thurston map with some marked pre-periodic orbits which has a parabolic orbifold. It is clear what a candidate geometrization should be: a finite quotient of a real affine map of $\mathbb{C}$ (as described in [3]) with some marked pre-periodic orbits. In this paper we produce a topological characterization of such maps, which is analogous to Thurston's characterization of rational maps. The reader familiar with Thurston's theorem will see that while the formulation of our result is similar, the proof is very different, and does not use the Teichmüller Theory toolbox. After characterizing marked maps with parabolic orbifolds in Section 3, we proceed to prove our main statement (whose precise formulation is given in Section 2), and give an algorithm for a constructive and canonical geometrization of an arbitrary Thurston map in Section 5.

In a forthcoming paper [10], we develop the theory further, and use canonical geometrization to give a partial resolution of the general problem of algorithmic decidability of Thurston equivalence of two branched covers. In accordance with a format of a brief communication, we have kept our introductory details and outlines of proofs short, while [10] will contain the expanded versions.

*Branched covering maps* We now proceed to recall the basic setting of Thurston's characterization of rational functions, and introduce the basic notations. Let $f : S^2 \to S^2$ be an orientation-preserving branched covering self-map of the two-sphere. We define the *postcritical set* $P_f$ by

$$P_f := \bigcup_{n>0} f^{\circ n}(\Omega_f),$$

where $\Omega_f$ is the set of critical points of $f$. When the postcritical set $P_f$ is finite, we say that $f$ is *postcritically finite.*

A *(marked) Thurston map* is a pair $(f, Q_f)$ where $f : S^2 \to S^2$ is a postcritically finite ramified covering of degree at least 2 and $Q_f$ is a finite collection of marked points $Q_f \subset S^2$ which contains $P_f$ and is $f$-invariant: $f(Q_f) \subset Q_f$. Thus, all elements of $Q_f$ are pre-periodic for $f$.

**Thurston equivalence.** Two marked Thurston maps $(f, Q_f)$ and $(g, Q_g)$ are *Thurston (or combinatorially) equivalent* if there are homeomorphisms $\phi_0, \phi_1$ :

$S^2 \to S^2$ such that the maps $\phi_0, \phi_1$ send $Q_f$ to $Q_g$ and are isotopic rel $Q_f$ and the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\phi_1} & S^2 \\ \big\downarrow f & & \big\downarrow g \\ S^2 & \xrightarrow{\phi_0} & S^2 \end{array}$$

commutes. We will use this notion of equivalence for Thurston maps throughout the article.

*Thurston linear transformation.* Let $Q$ be a finite collection of points in $S^2$. We recall that a simple closed curve $\gamma \subset S^2 - Q$ is *essential* if it does not bound a disk, is *non-peripheral* if it does not bound a punctured disk.

DEFINITION 1.1. *A multicurve $\Gamma$ on $(S^2, Q)$ is a set of disjoint, nonhomotopic, essential, nonperipheral simple closed curves on $S^2 \backslash Q$. Let $(f, Q_f)$ be a Thurston map, and set $Q = Q_f$. A multicurve $\Gamma$ on $S \setminus Q$ is f-stable if for every curve $\gamma \in \Gamma$, each component $\alpha$ of $f^{-1}(\gamma)$ is either trivial (meaning inessential or peripheral) or homotopic rel $Q$ to an element of $\Gamma$.*

To any multicurve is associated its *Thurston linear transformation* $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$, best described by the following transition matrix

$$M_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f : \alpha \to \delta)}$$

where the sum is taken over all the components $\alpha$ of $f^{-1}(\delta)$ which are isotopic rel $Q$ to $\gamma$. Since this matrix has nonnegative entries, it has a leading eigenvalue $\lambda(\Gamma)$ that is real and nonnegative (by the Perron-Frobenius theorem). When a multicurve $\Gamma$ has a leading eigenvalue $\lambda(\Gamma) \geq 1$, we call it a *Thurston obstruction* for $f$.

We can now state Thurston's theorem:

**Thurston's Theorem.** *Let $f : S^2 \to S^2$ be a marked Thurston map with a hyperbolic orbifold. Then $f$ is Thurston equivalent to rational function $g$ with a finite set of marked pre-periodic orbits if and only if $\lambda(\Gamma) < 1$ for every $f$-stable multicurve $\Gamma$. The rational function $g$ is unique up to conjugation by an automorphism of $\mathbb{P}^1$.*

The proof of Thurston's Theorem for Thurston maps without additional marked points is given in [3], for the proof for marked maps see e.g. [2].

DEFINITION 1.2. *A Levy cycle is a multicurve*

$$\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$$

*such that each $\gamma_i$ has a nontrivial preimage $\gamma_i'$, where the topological degree of $f$ restricted to $\gamma_i'$ is 1 and $\gamma_i'$ is homotopic to $\gamma_{(i-1) \bmod n}$ rel $Q$. A Levy cycle is degenerate if each $\gamma_i'$ bounds a disk $D_i$ such that the restriction of $f$ to $D_i$ is a homeomorphism and $f(D_i)$ is homotopic to $D_{(i+1) \bmod n}$ rel $Q$.*

*Thurston maps with parabolic orbifolds* Maps with parabolic orbifolds are the exceptional case of the previous theorem. A complete classification of postcritically finite branched covers with parabolic orbifolds has been given in [3]. Postcritically finite rational functions with parabolic orbifolds have been extensively described in [5].

We are mostly interested in the case of Thurston maps that have orbifold with signature $(2,2,2,2)$ which will be called $(2,2,2,2)$-*maps.* An orbifold with signature $(2,2,2,2)$ is a quotient of a torus $T$ by an involution $i$; the four fixed points of the involution $i$ correspond to the points with ramification weight 2 on the orbifold. The corresponding branched cover $T \to \mathbb{S}^2$ has exactly 4 simple critical points which are the fixed points of $i$. It follows that a $(2,2,2,2)$-map $f$ can be lifted to a covering self-map $\hat{f}$ of $T$. In particular, we can define the associated matrix $\hat{f}_*$ to be the action of $\hat{f}$ on $H_1(T)$.

An orbifold with signature $(2,2,2,2)$ has a unique affine structure of the quotient $\mathbb{R}^2/G$ where the orbifold group $G$ is given by:

$$G = < z \mapsto z+1, z \mapsto z+i, z \mapsto -z > . \tag{1.1}$$

We will denote this quotient by the symbol ⌑, which graphically represents a "pillowcase" – a sphere with four corner points. Up to equivalence, we may thus assume that a $(2,2,2,2)$-map is a self-map of the $\mathcal{O}_f =$ ⌑.

The proof of the following statement is essentially contained in [3].

THEOREM 1.3. *Let $f$ be a Thurston map with postcritical set $P = P_f$ and no extra marked points ($Q_f = P_f$) with a parabolic orbifold. Then $f$ is equivalent to a quotient of a real affine map by the action of the orbifold group. This affine map is unique up to affine conjugation.*

PROOF. Since $\mathcal{O}_f$ is parabolic there are three cases: $\#P$ is either 2, 3 or 4. In the first two cases, the orbifold has a unique complex structure and $f$ is equivalent to a quotient of a complex affine map (see [3]). In the third case, the orbifold $\mathcal{O}_f =$ ⌑, so it is the quotient of $\mathbb{R}^2$ by the action of $G$ (1.1). Note that the elements of $G$ are either translations by an integer vector or symmetries around a preimage of a marked point. We will denote

$$S_w \cdot z = 2w - z$$

the symmetry around a point $w \in \mathbb{R}^2$. Consider a lift $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ of $f$ and denote

$$\tilde{P} = \{1/2(\mathbb{Z} + i\mathbb{Z})\}$$

the full preimage of $P$ by the projection map. To proceed, we need to state two easy lemmas (see Lemmas 3.6 and 3.7 in [10]).

LEMMA 1.4. *A lift $F$ of a continuous map $f \colon$ ⌑ $\to$ ⌑ is affine on $\tilde{P}$.*

LEMMA 1.5. *Let $l(z)$ be a quotient of an affine map $L(z) = Az + b$ where $A$ is an integer matrix and $b \in 1/2(\mathbb{Z} + i\mathbb{Z})$ by the action of $G$, and $\phi$ be an element of PMCG(⌑). If $l(z) \circ \phi$ has a lift $L'$ to $\mathbb{R}^2$ such that $L'(z) = Az + b$ for all points in $\tilde{P}$, then $\phi$ is trivial.*

By Lemma 1.4, $F(z)$ agrees with an affine map $L(z) = Az + b$ on $\tilde{P}$, where $A$ is an integer matrix and $b \in 1/2(\mathbb{Z} + i\mathbb{Z})$ and $F_* g = L_* g$ for all $g \in G$. Therefore the map $\tilde{\phi} = L^{-1} \circ F$ is $G$-equivariant and projects to a self-homeomorphism $\phi$ of $\mathcal{O}_f$ which fixes $P$.

By Lemma 1.5, the homeomorphism $\phi$ represents the trivial element of the pure mapping class group PMCG(⌑) and, hence, is homotopic to the identity relative to $P$. Define $l$ to be the quotient of $L$ by the action of $G$. Then the commutative diagram

$$\begin{array}{ccc} R^2 & \xrightarrow{\tilde{\phi}} & R^2 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle L} \\ R^2 & \xrightarrow{\text{id}} & R^2 \end{array}$$

projects to the commutative diagram

$$\begin{array}{ccc} \text{⌑} & \xrightarrow{\phi} & \text{⌑} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle l} \\ \text{⌑} & \xrightarrow{\text{id}} & \text{⌑} \end{array}$$

which realizes Thurston equivalence between $f$ and $l$.

On the other hand, suppose that $l_1$ and $l_2$ are quotients of two affine maps, which are Thurston equivalent. Then $l_1$ and $l_2$ are conjugate on $P$, hence lifts thereof are conjugate on $\tilde{P}$ by an affine map (in the case when $\mathcal{O}_f =$ ⌑ this follows from Lemma 1.4; the other cases are similar) and the uniqueness part of the statement follows. □

*Topological characterization of canonical obstructions* For the definitions of the decomposition of a Thurston map along an obstruction see [6, 7]. Every obstructed Thurston map has a unique *canonical Thurston obstruction* [6, 7]. Instead of Pilgrim's original definition of canonical obstructions we will use the following chracterization proved by the first author [9]:

THEOREM 1.6 (Characterization of Canonical Obstructions). *The canonical obstruction $\Gamma$ is a unique minimal obstruction with the following properties.*

- *If the first-return map $F$ of a cycle of components in $\mathcal{S}_\Gamma$ is a $(2,2,2,2)$-map, then every curve of every simple Thurston obstruction for $F$ has two postcritical points of $f$ in each complementary component and the two eigenvalues of $\hat{F}_*$ are equal or non-integer.*
- *If the first-return map $F$ of a cycle of components in $\mathcal{S}_\Gamma$ is not a $(2,2,2,2)$-map or a homeomorphism, then there exists no Thurston obstruction of $F$.*

**2. Statement of the main result** We are now ready to formulate our constructive geometrization result.

THEOREM 2.1. *There exists an algorithm which for any Thurston map $f$ finds its canonical obstruction $\Gamma_f$.*

*Furthermore, consider the collection of the first return maps of the canonical decomposition of $f$ along $\Gamma_f$. Then the algorithm outputs the following information:*

- *for every first return map with a hyperbolic orbifold, the unique (up to Möbius conjugacy) marked rational map equivalent to it;*
- *for every first return map with a parabolic orbifold, the unique (up to affine conjugacy) affine map of the form $z \mapsto Az + b$ where $A \in SL(2, \mathbb{Z})$ and $b \in \frac{1}{2}\mathbb{Z}^2$ with marked points which is equivalent to $f$ after quotienting by the orbifold group.*

**3. Classification of marked $(2, 2, 2, 2)$-maps** Throughout this section we assume that $(f, Q)$ is a Thurston $(2, 2, 2, 2)$-map with the postcritical set $P$ and a forward-invariant marked set $Q \supset P$, such that $(f, P)$ is equivalent to the quotient $l$ of a real affine map $L(z) = Az + b$ by the orbifold group (1.1) where both eigenvalues of $A$ are not equal to $\pm 1$. The goal of this section is to sketch a proof of the following theorem, see [10] for the omitted proofs of the statements below.

THEOREM 3.1. *Let $(f, Q)$ be a Thurston $(2, 2, 2, 2)$-map with postcritical set $P$ and marked set $Q \supset P$, such that $(f, P)$ is equivalent to a quotient $l$ of a real affine map $L(z) = Az + b$ by the orbifold group where both eigenvalues of $A$ are not equal to $\pm 1$. Then $(f, Q)$ is equivalent to a quotient of a real affine map by the action of the orbifold group if and only if $f$ admits no degenerate Levy cycle.*

*Furthermore, in the former case the affine map is defined uniquely up to affine conjugacy.*

*Remark* 3.2. We note that in the case when the associated matrix has eigenvalue $\pm 1$, the two options are not mutually exclusive.

DEFINITION 3.3. *Let $f$ be a $(2, 2, 2, 2)$-map and let $z$ be an $f$-periodic point with period $n$. Fix a lift $F$ of $f$ to the universal cover and take a point $\tilde{z}$ in the fiber of $z$. If $z \notin P$, we define the* Nielsen index $ind_{F,n}(\tilde{z})$ *to be the unique element $g$ of the orbifold group $G$ such that $F^n(\tilde{z}) = g \cdot \tilde{z}$. If $z \in P$ then the Nielsen index of $z$ is defined up to pre-composition with the symmetry around $z$.*

Below, when we say that a point $z$ has a period $n$, we do not imply that $n$ is the minimal period of $z$.

DEFINITION 3.4. *Let $f$ be a $(2, 2, 2, 2)$-map and let $z_1, z_2$ be $f$-periodic points with period $n$. We say that $z_1$ and $z_2$ are in the same* Nielsen class of period $n$ *if there exists a lift $F_n$ of $f^n$ to the universal cover and points $\tilde{z}_1, \tilde{z}_2$ in the fibers* *of $z_1, z_2$ respectively, such that both $\tilde{z}_1$ and $\tilde{z}_2$ are fixed by $F_n$. We say that $z_1$ and $z_2$ are in the same* Nielsen class *if there exists an integer $n$ such that they are in the same class of period $n$.*

Note that if two points are in the same Nielsen class of period $n$, then they are in the same Nielsen class of period $mn$ for any $m \geq 1$. Clearly, being in the same Nielsen class (without specifying a period) is an equivalence relation, which is preserved under Thurston equivalence for points in $Q$.

LEMMA 3.5. *Periodic points $z_1$ and $z_2$ of period $n$ are in the same Nielsen class if and only if, for any lift $F$ of $f$ to the universal cover, there exist points $\tilde{z}_1, \tilde{z}_2$ in the fibers of $z_1, z_2$ respectively such that $ind_{F,n}(\tilde{z}_1) = ind_{F,n}(\tilde{z}_2)$.*

The following statement is straightfoward.

LEMMA 3.6. *A (2,2,2,2) map $f$ admits a degenerate Levy cycle if and only if so does its iterate $f^n$. Two points $z_1, z_2$ are in the same Nielsen class for $f$ with period $m$ if and only if they are in the same Nielsen class for $f^n$ with period $m/\gcd(m, n)$.*

We will also need the following lemma.

LEMMA 3.7. *Let $A$ be a $2 \times 2$ integer matrix with determinant greater than 1 and both eigenvalues not equal to $\pm 1$. If $v$ is a non-zero integer vector, then $A^{-n} \cdot v$ is non-integer for some $n > 0$.*

DEFINITION 3.8. *Suppose that one of the complementary components to a simple closed curve $\gamma$ in $(⌑, Q)$ contains at most one point of $P$ (so that $\gamma$ is trivial in $(⌑, P)$). We call that component $int(\gamma)$ the* interior *of $\gamma$.*

PROPOSITION 3.9. *Let $\{\gamma_n\}$ be a sequence of simple closed curves in $(⌑, Q)$ that are inessential in $(⌑, P)$ such that a $(2, 2, 2, 2)$-map $f$ sends $\gamma_{n+1}$ to $\gamma_n$ and $Q' = int(\gamma_n) \cap Q$ is the same for all $n$. Then there exits $m$ such that all points in $Q'$ are periodic with period $m$ and lie in the same Nielsen class.*

PROPOSITION 3.10. *A map $f$ admits a degenerate Levy cycle if and only if there exist two distinct periodic points in $Q$ in the same Nielsen class.*

LEMMA 3.11. *Let $\{\beta_n\}$ be a sequence of simple closed curves in $(⌑, Q)$ such that $f$ sends $\beta_{n+1}$ to $\beta_n$ with degree 1. Then all $\beta_n$ are inessential in $(⌑, P)$.*

PROOF. Since the degree of $f$ restricted to any $\beta_n$ is 1, the following holds:

$$A^n (p_n, q_n)^T = \pm (p_0, q_0)^T,$$

where $\beta_n$ corresponds to $\pm(p_n, q_n)^T$ in the first homology group of $(⌑, P)$. By Lemma 3.7, we see that $p_0 = q_0 = 0$. □

PROPOSITION 3.12. *Let $\{\gamma_n\}$ be a sequence of essential simple closed curves in $(⌑, Q)$ such that $f$ sends $\gamma_{n+1}$ to $\gamma_n$ with degree 1. Then $f$ admits a degenerate Levy cycle.*

Proof. By Lemma 3.11 all $\gamma_n$ are inessential in $(⌑, P)$. Replacing $\{\gamma_n\}$ by a subsequence $\{\gamma_{nk+l}\}$, for some integers $k, l$, we can always assume that $Q' = \operatorname{int}(\gamma_n) \cap Q$ is the same for all $n$. Since $\gamma_n$ are essential in $(⌑, Q)$, the set $Q'$ contains at least two points. By Proposition 3.9, these two points are in the same Nielsen class and Proposition 3.10 implies existence of a Levy cycle. □

Corollary 3.13. *If $f$ admits no Levy cycle, then for every simple closed curve $\gamma$ in $(⌑, Q)$, which is inessential in $(⌑, P)$, there exists an integer $d$ such that all connected components of $f^{-d}(\gamma)$ are inessential in $(⌑, Q)$.*

Proof. Define the *depth* of $\gamma$ to be the largest integer $d(\gamma)$ such that $f^{-d(\gamma)}(\gamma)$ has an essential component. The goal is to prove that $d_\gamma$ is finite for all inessential in $(⌑, P)$ curves. Clearly,

$$d(\alpha) = 1 + \max d(\alpha_i)$$

where $\alpha_i$ are the connected components of the preimage of a simple closed curve $\alpha$. Therefore, if $\gamma$ has infinite depth, so does at least one of its preimages $\gamma_1$. We construct thus an infinite sequence of essential in $(⌑, Q)$ curves $\gamma_n$ such that $f$ maps $\gamma_{n+1}$ to $\gamma_n$. Since a preimage of a trivial in $(⌑, P)$ curve is also trivial in $(⌑, P)$, truncating the sequence if necessary, we may assume that all $\gamma_n$ are either all trivial or all non-trivial in $(⌑, P)$. In both cases, the degree of $f$ restricted to $\gamma_n$ is 1 for all $n$ and the previous proposition yields the existence of a Levy cycle. □

We call $\Gamma$ a *simple* obstruction if no permutation of the curves in $\Gamma$ puts $M_\Gamma$ in the block form

$$M_\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix},$$

where the leading eigenvalue of $M_{11}$ is less than 1. The above result immediately implies:

Corollary 3.14. *If $f$ admits no Levy cycle, then every curve of every simple Thurston obstruction for $f$ is essential in $(⌑, P)$.*

Denote by $\tilde{Q}$ the full preimage of $Q$ by the covering map.

Definition 3.15. *Denote by $RMCG(⌑, Q)$ the* relative mapping class group *of $(⌑, Q)$, which is the group of all mapping classes $\phi$ for which there exists a lift $\tilde{\phi}$ to the universal cover that is identical on $\tilde{Q}$. We now need the following generalization of Lemma* 1.5*.*

Theorem 3.16. *The group $RMCG(⌑, Q)$ is generated by Dehn twists around trivial curves in $(⌑, P)$ and by second powers of Dehn twists around non-trivial inessential curves in $(⌑, P)$.*

The proof of the previous theorem is similar to the proof of classical results on generators of MCG, which makes use of the Birman exact sequence (cf. [4]).

DEFINITION 3.17. *Denote by* $\mathrm{Lift}(\phi)$ *the virtual endomorphism of* $PMCG(⌑, Q)$ *that acts by lifting by* $f$*, i.e. we write* $\mathrm{Lift}(\phi) = \psi$ *whenever there exists* $\psi \in PMCG(⌑, Q)$ *such that* $\phi \circ f = f \circ \psi$ *as mapping classes.*

PROPOSITION 3.18. $\mathrm{Lift}(\phi)\colon RMCG(⌑, Q) \to RMCG(⌑, Q)$ *is a well-defined endomorphism. If* $f$ *admits no Levy cycles, then for every* $\phi \in RMCG(⌑, Q)$*, there exist an* $n$ *such that* $\mathrm{Lift}^n(\phi) = \mathrm{id}$.

PROOF. It is enough to prove the statement for a generating set of $\mathrm{RMCG}(⌑, Q)$. By Theorem 3.16 we only need to consider two cases.

For simplicity, suppose $\phi = T_\alpha$ where $\alpha$ is a simple closed curve in $(⌑, Q)$, which is trivial in $(⌑, P)$ (the other case can be considered using similar arguments). All connected components $\alpha_i$ of $f^{-1}(\alpha)$ are pairwise disjoint simple closed curves that are trivial in $(⌑, P)$ and are mapped by $f$ to $\alpha$ with degree 1. It is straightforward to see that

$$T_\alpha \circ f = f \circ \prod T_{\alpha_i}.$$

Thus

$$\mathrm{Lift}(T_\alpha) = \prod T_{\alpha_i} \in \mathrm{RMCG}(⌑, Q)$$

is well-defined. Similarly, denote by $\alpha_i^n$ all connected components of $f^{-n}(\alpha)$; then

$$T_\alpha \circ f^n = f^n \circ \prod T_{\alpha_i^n} \text{ and } \mathrm{Lift}^n(T_\alpha) = \prod T_{\alpha_i^n}.$$

By Corollary 3.13, there exists an integer $n$ such that all $\alpha_i^n$ are inessential in $(⌑, Q)$, implying

$$\mathrm{Lift}^n(T_\alpha) = \prod T_{\alpha_i^n} = \mathrm{id}.$$

□

LEMMA 3.19. *If* $\psi = \mathrm{Lift}(\phi)$ *for some* $\phi \in PMCG(⌑, Q)$*, then* $f \circ \phi$ *is Thurston equivalent to* $f \circ \psi$*.*

PROOF. $f \circ \psi = \phi \circ f = \phi \circ (f \circ \phi) \circ \phi^{-1}$. □

We arrive at the following statement.

THEOREM 3.20. *If* $f$ *admits no Levy cycle and* $\phi \in RMCG(⌑, Q)$ *then* $f \circ \phi$ *is Thurston equivalent to* $f$*.*

PROOF. By proposition 3.18 and the previous lemma, there exists $n$ such that $f \circ \phi$ is equivalent to $f \circ \mathrm{Lift}^n(\phi) = f \circ \mathrm{id} = f$. □

We can now prove the first part of the statement of Theorem 3.1.

PROOF. **Necessity.** Suppose a quotient $(l, Q)$ of a real affine map $L(z) = Az + b$ by the orbifold group $G$ admits a degenerate Levy cycle. By Proposition 3.10, there exist two distinct points $q_1, q_2 \in Q$ in the same Nielsen class and Lemma 3.5 implies that there exist points $\tilde{q}_1, \tilde{q}_2$ in the fibers of $q_1, q_2$ respectively such that

$$\mathrm{ind}_{L,n}(\tilde{q}_1) = \mathrm{ind}_{L,n}(\tilde{q}_2) = g \in G, \text{ i.e. } L^n(\tilde{q}_i) = g(\tilde{q}_i) \text{ for } i = 1, 2.$$

Since

$$L^n(z) = A^n z + b' \text{ and } g(z) = c \pm z$$

for some integer vectors $b'$ and $c$, the equation

$$L^n(\tilde{q}_i) = g(\tilde{q}_i)$$

is equivalent to

$$(A^n \pm I)z = c - b'$$

where $I$ denotes the identity matrix. By assumption, the eigenvalues of $A$ are not equal to $\pm 1$, hence the matrix $(A^n \pm I)$ is non-degenerate. This yields $\tilde{q}_1 = \tilde{q}_2$, which is a contradiction.

**Sufficiency.** Suppose $f$ admits no Levy cycles and, hence, no two distinct points of $Q$ are in the same Nielsen class by Proposition 3.10. Consider a lift $F$ of $f$ to the universal cover; by Lemma 1.4

$$F(z) = L(z) = A'z + b' \text{ for all } z \in \tilde{P}.$$

Conjugating the original $f$ by a quotient of an affine transformation, if necessary, we may assume that $A' = A$ and $b' = b$. Pick a point $\tilde{q}$ in the fiber of a periodic point $q \in Q$ of period $n$. Let $s$ be a unique solution of the equation

$$L^n(z) = \mathrm{ind}_{F,n}(\tilde{q}) \cdot z.$$

We can push the point $\tilde{q}$ by a $G$-equivariant isotopy $\Phi_t(z) \colon \mathbb{R}^2 \to \mathbb{R}^2$ along some path $\alpha$ in $\mathbb{R}^2 \setminus \tilde{Q}$ (except for the starting point $\tilde{q}$ and, possibly, the end point) that ends at $s$. Since $\Phi$ is $G$-equivariant, it pushes the point

$$F^n(\tilde{q}) = \mathrm{ind}_{F,n}(\tilde{q}) \cdot \tilde{q}$$

along the path $\mathrm{ind}_{F,n}(\tilde{q}) \cdot \alpha$ to the point

$$\mathrm{ind}_{F,n}(\tilde{q}) \cdot s = L^n(s).$$

Therefore, for $F_1 = \Phi_1 \circ F \circ \Phi_1^{-1}$, we have $F_1^n(s) = L^n(s)$. Let $s' = g \cdot s$, where $g \in G$ be any other point in the same fiber as $s$. Then $G$-equivariance of $\Phi$ implies

$$\begin{aligned} F_1^n(s') &= \Phi_1 \circ F^n \circ \Phi_1^{-1}(g \cdot s) = \Phi_1 \circ F^n(g \cdot \Phi_1^{-1}(s)) \\ &= \Phi_1(F_*^n g \cdot F^n \circ \Phi_1^{-1}(s)) = F_*^n g \cdot \Phi_1 \circ F^n \circ \Phi_1^{-1}(s) \\ &= F_*^n g \cdot F_1^n(s) = F_*^n g \cdot L^n(s). \end{aligned}$$

Since $F = L$ on $\tilde{P}$, their actions on the orbifold group are the same: $F_* = L_*$. Thus,

$$F_1^n(s') = F_*^n g \cdot L^n(s) = L_*^n g \cdot L^n(s) = L^n(g \cdot s) = L^n(s').$$

We repeat this procedure for each of the periodic points in $Q$ to obtain a $G$-equivariant isotopy $\Psi_t(z)\colon \mathbb{R}^2 \to \mathbb{R}^2$ and set $F_2 = \Psi_1 \circ F \circ \Psi_1^{-1}$, such that for any point $s = \Psi_1(\tilde{q})$, where $\tilde{q}$ is in the fiber of a periodic point of any period $n$ from $Q$, we have $F_2^n(s) = L^n(s)$. The only possible obstacle can occur when we need to push some point $\tilde{q}$ from the fiber of $q$ into the fiber of some other point $q'$, which has already been adjusted. This would immediately imply that $q$ and $q'$ are in the same Nielsen class, which contradicts our assumptions.

Note that our construction automatically implies $F_2(s) = L(s)$ for all $s = \Psi_1(\tilde{q})$, where $\tilde{q}$ is in the fiber of a periodic point $q$ of any period $n$. Indeed, if $F^n(z) = g \cdot z$, then

$$F^n(F(z)) = F(F^n(z)) = F(g \cdot z) = F_* g \cdot F(z),$$

hence

$$\mathrm{ind}_{F,n}(F(z)) = F_* \mathrm{ind}_{F,n}(z) = L_* \mathrm{ind}_{F,n}(z).$$

Therefore, if $s = \Psi_1(\tilde{q})$ is a unique solution of the equation

$$L^n(z) = \mathrm{ind}_{F,n}(\tilde{q}) \cdot z,$$

then $L(s)$ is a unique solution of the equation

$$L^n(z) = \mathrm{ind}_{F,n}(F(\tilde{q})) \cdot z = L_* \mathrm{ind}_{F,n}(\tilde{q}) \cdot z,$$

because

$$L^n(L(z)) = L(L^n(z)) = L(\mathrm{ind}_{F,n}(\tilde{q}) \cdot z) = L_* \mathrm{ind}_{F,n}(\tilde{q}) \cdot L(z).$$

This yields $\Psi_1(F(\tilde{q})) = L(s)$ and

$$F_2(s) = F_2 \circ \Psi_1(\tilde{q}) = \Psi_1 \circ F(\tilde{q}) = L(s).$$

Now we perform an analogous procedure on all strictly pre-periodic points. Let $q \in Q$ be a strictly pre-periodic point and $\tilde{q}$ be some point in its fiber. Denote by $n$ the pre-period of $q$, i.e. the smallest integer such that $f^n(q)$ is periodic. We find a $G$-equivariant isotopy that pushes $\tilde{q}$ to $L^{-n}(F_2^n(\tilde{q}))$ and leaves all fibers of other points of $Q$ in place. After repeating this process for all pre-periodic points of $f$, we construct a $G$-equivariant isotopy $\Xi_t(z)\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_3 = \Xi_1 \circ F \circ \Xi_1^{-1}$ agrees with $L(z)$ on $\Xi_1(\tilde{Q})$, in particular $F_{3*} = L_*$.

Denote by $f_3$ and $\xi$ the quotients of $F_3$ and $\Xi_1$ respectively by the action of $G$. Then $f_3 = \xi_1 \circ f \circ \xi_1^{-1}$, and $(f, Q)$ is conjugate (and, hence, Thurston equivalent) to $(f_3, \xi(Q))$. Set $\Theta(z) = L^{-1} \circ F_3(z)$; we see that

$$\Theta(g \cdot z) = L^{-1} \circ F_3(g \cdot z) = L^{-1}(F_{3*} g \cdot F_3(z)) = L^{-1}(L_* g \cdot F_3(z)) = g \cdot L^{-1} \circ F_3(z),$$

i.e. $\Theta$ is $G$-equivariant. Therefore $f_3 = l \circ \theta$ where $\theta$ is the quotient of $\Theta$ by the action of $G$. Since $F_3 = L$ on $\tilde{Q}$, the lift $\Theta$ of $\theta$ to the universal cover is identical on $\tilde{Q}$ so $\theta \in \mathrm{RMCG}(⌑, Q)$. By Theorem 3.20 $(f_3, \xi(Q))$ and $(l, \xi(Q))$ are Thurston equivalent, which concludes our proof. □

Theorem 3.1 and Corollary 3.14 imply the following:

Corollary 3.21. • *Let $f$ be marked $(2,2,2,2)$-map such that the corresponding matrix does not have eigenvalues $\pm 1$. Then $f$ is equivalent to a quotient of an affine map with marked pre-periodic orbits if and only if every curve of every simple Thurston obstruction for $f$ has two postcritical points of $f$ in each complimentary component.*
• *A marked Thurston map $f$ with a parabolic orbifold that is not $(2,2,2,2)$ is equivalent to a quotient of an affine map if and only it admits no Thurston obstruction.*

## 4. Constructive geometrization of Thurston maps with parabolic orbifolds

From this point on we will assume all objects to be piecewise linear (see [1]). Recall the main result of [1].

Theorem 4.1. *There exists an algorithm $\mathcal{A}_1$ which, given a finite description of a marked Thurston map $f$ with hyperbolic orbifold, outputs 1 if there exists a Thurston obstruction for $f$ and 0 otherwise. In the latter case, $\mathcal{A}_1$ also outputs a finite description which uniquely identifies the rational mapping $R$ which is Thurston equivalent to $f$, and the pre-periodic orbits of $R$ that correspond to points in $Q_f$.*

We start by proving the same result for Thurston maps with parabolic orbifolds.

Theorem 4.2. *There exists an algorithm $\mathcal{A}_2$ which for any marked Thurston map $f$ with a parabolic orbifold whose matrix does not have eigenvalues $\pm 1$ finds either a degenerate Levy cycle or an equivalence to a quotient of an affine map with marked pre-periodic orbits.*

Proof. The proof is completely analogous to the argument given in [1], where the detailed algorithms for procedures used below are provided. We begin by identifying the orbifold group $G$ and finding an affine map $L(x) = Ax + b$ such that $f$ without marked points is equivalent to the quotient $l$ of $L$ by $G$ (Theorem 1.3).

We now execute two sub-programs in parallel:

**(I)** We enumerate all $f$-stable multicurves $\Gamma_n$. We check whether $\Gamma_n$ is a degenerate Levy cycle. If yes, we output **degenerate Levy cycle found** and halt.

**(II)** We identify all forward invariant sets $S_k$ of pre-periodic orbits of $l$ of the same cardinality as the set of marked points of $f$. We enumerate the sequence $\psi_n$ of all elements of $\mathrm{PMCG}(S^2, Q)$. For every $\psi_n$ and each of the finitely many sets $S_k$ we check whether $h_k \circ \psi_n$ realizes Thurston equivalence between $f$ and $l$ with marked points $S_k$, where $h_k\colon (S^2, Q) \to (S^2, S_k)$ is an arbitrary chosen homeomorphism. If yes, we output **Thurston equivalence found**, list the maps $l$, $h_k \circ \psi_n$ and the set $S_k$ and halt.

By Theorem 3.1 either the first or the second sub-program, but not both, will halt and deliver the desired result. □

5. **Constructive canonical geometrization of a Thurston map** We are now ready to present the proof of Theorem 2.1. The result of [1] together with Theorem 4.2 implies the existence of the subprogram $\mathcal{P}$ which given a marked Thurston map $f$ does the following:

1. if $f$ has a hyperbolic orbifold and is obstructed, it outputs a Thurston obstruction for $f$;
2. if $f$ has a parabolic orbifold not of type $(2,2,2,2)$ and a degenerate Levy cycle it outputs such a Levy cycle;
3. if $f$ is a $(2,2,2,2)$ map such that the corresponding matrix has two distinct integer eigenvalues outputs a Thurston obstruction for $f$;
4. if $f$ is a $(2,2,2,2)$ map with a degenerate Levy cycle outputs such a Levy cycle;
5. in the remaining cases output a geometrization of $f$ as described in the statement of the theorem.

We apply the subprogram $\mathcal{P}$ recursively to decompositions of $f$ along the found obstructions until no new obstructions are generated (this will eventually occur by Theorem 1.6 and Corollary 3.21).

Denote by $\Gamma$ the union of all obstructions thus generated. Use algorithm $\mathcal{A}_2$ and sub-program $\mathcal{P}$ to find the set $S$ of all subsets $\Gamma' \subset \Gamma$ such that:

- $\Gamma'$ is a Thurston obstruction for $f$;
- denote by $\mathcal{F}'$ the set of first return maps obtained by decomposing along $\Gamma'$. Then no $h \in \mathcal{F}'$ is a $(2,2,2,2)$ map whose matrix has distinct integer eigenvalues, and every $h \in \mathcal{F}'$ which is not a homeomorphism is geometrizable.

Set

$$\Gamma_f = \cap_{\Gamma' \in S} \Gamma'.$$

By Theorem 1.6 and Corollary 3.21, $\Gamma_f$ is the canonical obstruction of $f$. The conclusion of Theorem 2.1 follows. □

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