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Periodic Integral Transforms and Associated Noncommutative Orbifold Projections

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Dedicated to George Elliott on his seventieth birthday

Presented by Pierre Miman, FRSC

Abstract. We report on recent results on the existence of Cubic and Hexic integral transforms on self-dual locally compact groups (orders 3 and 6 analogues of the classical Fourier transform) and their application in constructing a canonical continuous section of smooth projections $\mathcal{E}(t)$ of the continuous field of rotation C*-algebras $\{A_t\}_{0<t<1}$ that is invariant under the noncommutative Hexic transform automorphism. This leads to invariant matrix (point) projections of the irrational noncommutative tori A_θ. We also present a quick method for computing the (quantized) topological invariants of such projections using techniques from classical Theta function theory.

Résumé. On décrit des résultats récents sur l'existence d'une transformation intégrale d'ordre trois (ou d'ordre six) sur un groupe localement compact abélien self-dual. On étudie l'application possible à la construction d'un champs continu de projecteurs invariants sous l'automorphisme associé du champs de C*-algèbres de rotation. On calcule certains invariants topologiques de ces projecteurs.

1. Introduction

In this announcement paper we discuss recent results on the existence of Cubic and Hexic integral transforms on self-dual locally compact Abelian groups [14] (these are, respectively, orders 3 and 6 analogues of the classical Fourier transform) and their application [15] in constructing a canonical continuous section $\mathcal{E}(t)$ of smooth projections (in fact, analytic projections in Sakai's sense [10]) of the continuous field of rotation C*-algebras $\{A_t\}_{0<t<1}$, as studied by Elliott [5], that is invariant under the noncommutative Hexic transform automorphism, i.e. under the canonical order 6 automorphism $\rho = \rho_t$ of A_t defined by

$$\rho(U_t) = V_t, \qquad \rho(V_t) = e^{-\pi i t} U_t^{-1} V_t. \tag{1.1}$$

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(Which corresponds to the order six $\mathrm{SL}(2,\mathbb{Z})$ matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]$ on the K_1-group.) Here, U_t, V_t are the canonical unitary generators of the rotation C*-algebra A_t (noncommutative 2-torus) satisfying the commutation relation $V_tU_t = e^{2\pi it}U_tV_t$. The projection section $\mathcal{E}(t)$, with trace t, is shown to be the support projection of the following concrete section of positive C^∞-elements that could be interpreted as a noncommutative (2-dimensional) Theta function:

$$\mathbb{X}(t) = t \sum_{m,n=-\infty}^{\infty} e^{-\frac{\pi t}{\sqrt{3}}(m^2+n^2)} e^{-\pi t(\frac{1}{\sqrt{3}}-i)mn} U_t^n V_t^m \tag{1.2}$$

for $0 < t < 1$. It is ρ invariant and has the useful feature that

$$\lim_{t\to 0^+} \|\mathbb{X}(t) - \mathcal{E}(t)\| = 0.$$

In addition, in [15] we devise a new and quicker computational technique (compared with our older and longer methods in [2], [11], [12], [13]) for obtaining the topological invariants ("quantum numbers") of the projection $\mathcal{E}(t)$ according to the limit rule

$$\psi^t(\mathcal{E}(t)) = \lim_{s\to 0^+} \psi^s(\mathbb{X}(s))$$

which turns out to exist for each of the noncanonical traces ψ^t associated to the automorphism ρ as well as to those associated to the Cubic transform $\kappa = \rho^2$ given by

$$\kappa(U_t) = e^{-\pi it}U_t^{-1}V_t, \qquad \kappa(V_t) = U_t^{-1}. \tag{1.3}$$

(The noncanonical traces are given in Section 3.)[1]

Using the section $\mathcal{E}(t)$ we show that for a certain concrete class of irrationals $\theta \in \mathbb{G}$ and associated rational approximations p/q, the projections arising from $\mathcal{E}$ according to

$$e = \zeta_{q,\theta}(\mathcal{E}(q^2\theta - pq)) \tag{1.4}$$

(where the *-homomorphism $\zeta_{q,\theta}$ is given by equations (3.8)) are matrix (point) projections of trace $q^2\theta - pq$, in the sense that they are approximately central in A_θ and the cut downs eUe, eVe are close (for sufficiently large q) to order q unitary matrices in a ρ-invariant $q \times q$ matrix algebra $\mathfrak{M}$ contained in $eA_\theta e$ whose identity is e (see Theorem 3.2 below).

The noncanonical traces, being discontinuous linear functionals defined on the infinitely differentiable elements A_t^∞ of the rotation C*-algebra field $\{A_t\}$, would constitute the differential topology of C*-algebras and their orbifolds –

[1] The noncanonical twisted traces are defined in a natural way on the continuous subfield of smooth noncommutative tori $\{A_t^\infty\}_{0<t<1}$, so that, for instance, if $p(t)$ is a polynomial section of this field and ψ^t is such a twisted trace, then $\psi^t(p(t))$ is a continuous complex-valued function of t.

much as topological invariants related to curvature, Chern characters, and Gauss-Bonnet Theorem in differential topology and geometry are derived from canonical analytic structures like covariant differentiation.

The toroidal orbifolds associated to the symmetry groups $\mathbb{Z}_3$ and $\mathbb{Z}_6$ are the fixed point C*-subalgebras $A_\theta^\kappa := \{x \in A_\theta \,|\, \kappa(x) = x\}$ and A_θ^ρ of A_θ. When θ is rational these orbifold algebras take the concrete form of a 2-sphere with 3 or 4 singularities (see [1] for the Flip case, and [7] [6] [8] for the orders 3, 4, 6 cases) each of which takes the form of multiple non-Hausdorff points. When θ is irrational, it was proved in [4] that these fixed point algebras and their respective (strongly Morita equivalent) C*-crossed products $A_\theta \rtimes_\kappa \mathbb{Z}_3$ and $A_\theta \rtimes_\rho \mathbb{Z}_6$ are approximately finite-dimensional. Such orbifolds associated to the Flip and Fourier transform have been used in the work of Konechny and Schwarz [9] in studies of compactification of M(atrix) theory – e.g., the topological invariants being related to number of D-branes associated to noncommutative orbifold singularities (see, for example, Section 9.3 of [9]). It would be interesting to see if their results extend to the symmetries discussed here. It would also be interesting to see what these results look like for higher dimensional noncommutative tori.

It will be convenient to use the notation $e(t) := e^{2\pi i t}$.

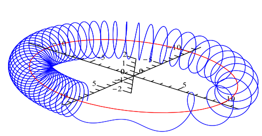
2. The Integral Transforms In this section we summarize results from [14], the main result of which is Theorem 2.2. This theorem was used in obtaining the results stated in Section 3.

THEOREM 2.1. Let G be a locally compact, compactly generated self-dual Abelian group with symmetric pairing $\langle\ ,\ \rangle$. There exists a continuous (in fact, C^∞ if G is a Lie group) map $\alpha : G \to \mathbb{T}$ such that

$$\alpha(x+y) = \alpha(x)\alpha(y)\langle x,\, y\rangle, \qquad \alpha(-x) = \alpha(x), \qquad \alpha(0) = 1$$

for all $x, y \in G$.

The condition $\alpha(-x) = \alpha(x)$ ensures that the Cubic integral transform $f \to f^c$ (given by the following theorem) commutes with the flip unitary $f \to \widetilde{f}$, where $\widetilde{f}(x) = f(-x)$, for the groups concerned. The image shown below on the left is a rough graph of α in the cyclic group case $\mathbb{Z}_q$



$$\begin{tikzcd}
1 \arrow[rr] & & K\sqrt{q}\,\delta_q^n \arrow[dl] \\
& K^2\alpha(n) \arrow[ul] &
\end{tikzcd}$$

whose elements are represented by the q^{th} roots of unity in the unit circle - it cycles around the circle in a quadratic manner since it has the form $\alpha(m) =$

$(-1)^m e(\frac{m^2}{2q})$ (see [14], Section 2). And the triangular diagram shows what happens to the constant function 1 on the group $\mathbb{Z}_q$ under two iterations of the Cubic transform (reflecting an interesting contrast with how the Fourier transform treats it, which is essentially just the first row). (The divisor δ-function δ_q^n is defined in Section 3.)

Theorem 2.2. ([14]) Let G be a locally compact, compactly generated self-dual Abelian group. Then the linear transform

$$f^c(t) := K\,\alpha(t)\widehat{f}(-t) = K\,\alpha(t)\int_G f(x)\langle t,\, x\rangle\, dx$$

for $f \in \mathcal{S}(G)$ (the Schwartz space of G) and for some constant $K, |K| = 1$, defines a unitary operator of order 3 on $L^2(G)$ which commutes with the flip $f \to \widetilde{f}$, and therefore gives an order 6 unitary operator H by $Hf = \widetilde{f}^{cc}$. Further, $H^3 f = \widetilde{f}$ and $H^2 f = f^c$.

The transforms f^c and Hf are directly related to the Cubic and Hexic automorphisms κ, ρ by means of C*-module actions and C*-inner products (see [15]).

3. Continuous Field of Projections and Matix Projections in $\mathbb{Z}_3, \mathbb{Z}_6$ Orbifolds In order to discuss our next results from [15], we need to state the noncanonical traces that give the topological invariants mentioned in the next theorem.

Given a (finite order) automorphism β of an algebra A, a (twisted) β-trace is a complex linear map $\psi : A \to \mathbb{C}$ such that

$$\psi(xy) = \psi(\beta(y)x)$$

for $x, y \in A$. The restriction of ψ to the β-orbifold $A^\beta = \{x \in A \,|\, \beta(x) = x\}$ (fixed point subalgebra) defines a trace, and therefore induces a morphism on K-theory $\psi_* : K_0(A^\beta) \to \mathbb{C}$ that gives a topological invariant for projections and modules ("bundles") over the orbifold A^β. (Invariably, A will be a dense *-subalgebra of a C*-algebra that is closed under the holomorphic functional calculus.)

In the case of noncommutative tori A_θ, such maps are defined on the canonical dense *-subalgebra A_θ^∞ of differentiable elements – namely, Schwartz series $\sum a_{mn} U^m V^n$ where $\{a_{mn}\}$ is rapidly decreasing.

In joint work with Julian Buck [2], we computed such twisted traces for the Cubic transform κ and showed ([2], Theorem 3.3) that they form a 3-dimensional complex vector space with basis given by the following basic κ-traces

$$\psi_j^\theta(U^m V^n) = e(\tfrac{\theta}{6}(m-n)^2)\, \delta_3^{m-n-j} \tag{3.1}$$

where $j = 0, 1, 2$ and δ_d^m is the *divisor delta function* given by $\delta_d^m = 1$ if d divides m, and $\delta_d^m = 0$ otherwise. (Here, $VU = e^{2\pi i\theta} UV$.) These three noncanonical traces, together with the usual canonical trace state τ, give rise to its Connes-Chern character invariant for the Cubic orbifold:

$$T_3 : K_0(A_\theta^\kappa) \to \mathbb{C}^4, \qquad T_3(x) = (\tau(x); \psi_0(x), \psi_1(x), \psi_2(x)).$$

For the identity element, for example, we have $T_3(1) = (1; 1, 0, 0)$. Recall that the canonical trace is defined by $\tau(\sum a_{mn} U^m V^n) = a_{00}$.

For the Hexic transform (Theorem 3.1 in [2]) there is a unique ρ-trace φ_1^θ (up to scalar multiples) defined on A_θ^∞, a pair of ρ-invariant ρ^2-traces φ_{2j}^θ, and a pair of ρ-invariant ρ^3-traces φ_{3j}^θ given by

$$\varphi_1^\theta(U^m V^n) = e(\tfrac{\theta}{2}(m^2 + n^2)) \tag{3.2}$$

$$\varphi_{20}^\theta(U^m V^n) = e(\tfrac{\theta}{6}(m-n)^2)\delta_3^{m-n}, \quad \varphi_{21}^\theta(U^m V^n) = e(\tfrac{\theta}{6}(m-n)^2), \tag{3.3}$$

$$\varphi_{30}^\theta(U^m V^n) = e(-\tfrac{\theta}{2}mn)\delta_2^m \delta_2^n, \qquad \varphi_{31}^\theta(U^m V^n) = e(-\tfrac{\theta}{2}mn). \tag{3.4}$$

When no confusion arises we simply write $\varphi_{jk}^\theta = \varphi_{jk}$. The Connes-Chern character invariant for the Hexic orbifold A_θ^ρ consists of these together with the canonical trace:

$$T_6 : K_0(A_\theta^\rho) \to \mathbb{C}^6, \quad T_6(x) = (\tau(x); \varphi_1(x), \varphi_{20}(x), \varphi_{21}(x), \varphi_{30}(x), \varphi_{31}(x))$$

When θ is irrational the Connes-Chern characters T_3, T_6 are one-to-one, therefore giving complete invariants for projections and modules over the orbifolds[2]. (See [3].)

We have the following results.

THEOREM 3.1. ([15]) There is a continuous section $\mathcal{E} : (0,1) \to \{A_t\}$ of C^∞-projections of the continuous field $\{A_t\}$ of rotation C*-algebras such that

(1) $\rho(\mathcal{E}(t)) = \mathcal{E}(t)$, $\kappa(\mathcal{E}(t)) = \mathcal{E}(t)$;
(2) $\mathcal{E}(t)$ has κ-topological numbers

$$\psi_0 = \psi_1 = \psi_2 = \omega := \frac{1}{2}(1 + \frac{i}{\sqrt{3}}); \tag{3.5}$$

(3) $\mathcal{E}(t)$ has ρ-topological numbers

$$\varphi_1 = 3\omega - 1, \quad \varphi_{20} = \omega, \quad \varphi_{21} = 3\omega, \quad \varphi_{30} = \frac{1}{2}, \quad \varphi_{31} = 2 \tag{3.6}$$

(4) $\mathcal{E}(t)$ is the support projection of the noncommutative 2D "Theta function" C^∞ positive element

$$\mathbb{X}(t) = t \sum_{m,n} e^{-\frac{\pi t}{\sqrt{3}}(m^2+n^2)} e^{-\pi t(\frac{1}{\sqrt{3}} - i)mn} U_t^n V_t^m \tag{3.7}$$

for $0 < t < 1$; further, one has $\lim_{t \to 0^+} \|\mathcal{E}(t) - \mathbb{X}(t)\| = 0$.

[2]When θ is rational one would have to include Connes' cyclic 2-cocycle which "picks out the label of the trace."

In the next theorem we use the homomorphism $\zeta_{q,\theta} : A_{\theta_q} \to A_\theta$, where $\theta_q := q^2\theta - pq$, defined by

$$\zeta_{q,\theta}(U_{\theta_q}) = U_\theta^q, \quad \zeta_{q,\theta}(V_{\theta_q}) = V_\theta^q. \tag{3.8}$$

Theorem 3.2. ([15]) Let θ be any irrational number such that there are infinitely many rational approximations p/q (in reduced form, with $p \geq 0$, $q \geq 1$) such that

$$0 < \theta - \frac{p}{q} < \frac{0.995}{q^2}$$

where p an even perfect square. Then the projection

$$e = \zeta_{q,\theta}(\mathcal{E}(q^2\theta - pq)) \tag{3.9}$$

in A_θ (with trace $q^2\theta - pq$) is ρ invariant, is approximately central, and there exists a ρ-invariant $q \times q$ matrix algebra $\mathfrak{M} \subset eA_\theta e$ with unit e such that: for any finite subset $F \subset A_\theta$ and each $\epsilon > 0$, there exists large enough q such that exe has distance less than ϵ from $\mathfrak{M}$ for each $x \in F$. (The same conclusions hold for the Cubic transform κ.)

We remark that the class of irrationals in this theorem contains dense G_δ sets, and that in the matrix approximation of this theorem, the cut downs of the canonical unitaries eUe, eVe are close to order q unitary matrices of $\mathfrak{M}$. (This is borne out in the proof of this theorem in [15] if not explicitly stated in the theorem.) Recall that $e = e_q$ is approximately central in A_θ if for any finite subset $F \subset A_\theta$ and $\epsilon > 0$ there exists large enough q such that $\|xe - ex\| < \epsilon$, $\forall x \in F$.

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