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# Cubic and Hexic Integral Transforms for Locally Compact Abelian Groups

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Abstract. We prove that for locally compact, compactly generated self-dual Abelian groups $G$, there are canonical unitary integral operators on $L^2(G)$ analogous to the Fourier transform but which have orders 3 and 6. To do this, we establish the existence of a certain projective character on $G$ whose phase multiplication with the FT gives rise to the Cubic transform (of order 3). (Thus, although the Fourier transform has order 4, one can "make it" have order 3 (or 6) by means of a phase factor!)

Résumé. Soit $G$ un groupe localement compact, engendré par un sousensemble compact, et isomorphe à son groupe dual. On construit des operateurs intégrals unitaires canoniques qui sont analogues à la transformée de Fourier, mais qui sont d'ordres trois et six.

**1. Introduction** It is a well-known phenomenon that the Fourier transform of a 'spread out' function is a localized 'delta' type function and vice versa. For example, on the cyclic group $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$ of order $q$, the Fourier transform of the constant function $f(n) = 1$ is the delta function $\widehat{f}(n) = \sqrt{q}\delta_q^n$ (localized at 0), where $\delta_q^n$ denotes the divisor delta function defined by $\delta_q^n = 1$ if $q$ divides $n$, and 0 otherwise.

In this paper we show that there are canonical order 3 and 6 integral transforms quite similar to the Fourier transform on many locally compact Abelian self-dual groups. We refer to the order 3 (respectively, order 6) transform as the Cubic transform (respectively, Hexic transform), where the former is the square of the latter.

An interesting new feature with the Cubic (and Hexic) transform $f \to f^c$ (defined below) is illustrated by the following triangular diagram, which shows

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two iterations of the Cubic transform of the constant 1 function on $\mathbb{Z}_q$:

$$1 \longrightarrow K\sqrt{q}\delta_q^n \longrightarrow K^2\alpha(n) \longrightarrow 1$$

In the first line we see that the Cubic transform of 1 basically acts like the Fourier transform (up to a phase constant $K$). But on a second application of the Cubic transform $1^{cc}$, which is the same as the Hexic transform of 1, we see that it gives rise to a projective character which is neither localized nor 'flat', but is a 'helical' function twirling around the cyclic group $\mathbb{Z}_q$ (as shown in Figure 1). The elements of $\mathbb{Z}_q$ are represented by the complex numbers $e^{2\pi in/q}$ $(n = 0, 1, \ldots, q-1)$ around the unit circle.

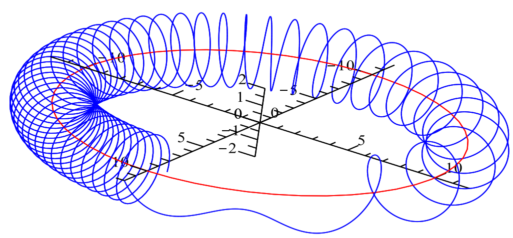

*Figure 1*: Plot of the second Cubic Transform $1^{cc} = K^2\alpha$ of the identity function 1 on the cyclic group $\mathbb{Z}_q$ (viewed as subset of the unit circle), as it winds around $\mathbb{Z}_q$.

We now proceed to set some background needed as well as define these integral transforms. Throughout the paper we shall use the notation

$$e(x) := e^{2\pi ix} \ \in \mathbb{T}$$

for real $x$, where $\mathbb{T}$ is the unit circle, and we use the standard definition

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e(-tx)dx$$

for the classical Fourier transform of a Schwartz (or $L^1$) function $f$ on the reals $\mathbb{R}$.

In [3] we proved that for the additive group of the reals $\mathbb{R}$ there are canonical integral transforms quite similar to the Fourier transforms but which have orders 6 and 3. There, we obtained the *Hexic* transform

$$(Hf)(t) = i^{1/6} \int_{\mathbb{R}} f(x)e(tx - \tfrac{1}{2}x^2)dx \tag{1.1}$$

and its square $H^2$, the *Cubic* transform

$$f^c(t) := (H^2 f)(t) = i^{-1/6}\, e(\tfrac{1}{2}t^2) \int_{\mathbb{R}} f(x)e(tx)dx = i^{-1/6}\, e(\tfrac{1}{2}t^2)\widehat{f}(-t). \tag{1.2}$$

for $f \in \mathcal{S}(\mathbb{R})$ (the Schwartz space on the reals).[1] Just as in the case for the Fourier transform, these transforms extend to unitary operators on $L^2(\mathbb{R})$. Further, just as $\widehat{\widehat{f}}(t) = f(-t)$, we showed that

$$(H^3 f)(t) = f(-t) \tag{1.3}$$

for all $f$, so that $H$ commutes with the flip unitary $f \to \widetilde{f}$ where

$$\widetilde{f}(x) = f(-x)$$

on $L^2$. From this it followed that $H$ has order 6 and that $H^2 : f \to f^c$ has order 3.

Remark 1.1. In [3] we used these transforms to construct finitely generated projective modules over orbifolds associated to fixed point C*-subalgebras related to order 3 and 6 automorphisms of the noncommutative tori, or irrational rotation C*-algebras. But that is not the subject of this paper.

For our purposes it will be convenient for us to work with the Cubic transform since the Hexic transform can be written in terms of it according to

$$Hf = \widetilde{f}^{cc} = H^4(\widetilde{f}). \tag{1.4}$$

One reason we refer to these transforms (including the Fourier) as being 'canonical' is that they transform translation and phase multiplication operators into very simple combinations of themselves. Indeed, letting $(T_x f)(t) = f(t-x)$ denote a translation operator, and $(E_x f)(t) = e(-xt)f(t)$ denote phase multiplication operator, one checks the following relations:

$$T_x H = HE_x, \qquad E_x H T_x = e(-\tfrac{1}{2}x^2)T_x H.$$

[1] In [3] we have written down a one-parameter family of these transforms parametrized by a parameter $\mu > 0$ which here we have taken to be $\mu = 1/2$ as the ideal choice.

In this paper we prove that these results in the case of $\mathbb{R}$ extend to locally compact, compactly generated, self-dual Abelian groups $G$. (All Abelian groups here, except for the unit circle, will be written additively with identity 0.) For such self-dual groups $G$, the canonical pairing $\langle\ ,\ \rangle : G \times G \to \mathbb{T}$ will turn out to be symmetric: $\langle x, y\rangle = \langle y, x\rangle$.

A projective character on $G$ is described by the following theorem which is needed for Theorem 1.3 and which we also prove in this paper.

THEOREM 1.2. Let $G$ be a locally compact, compactly generated self-dual Abelian group with symmetric pairing $\langle\ ,\ \rangle$. There exists a continuous (in fact, $C^\infty$ if $G$ is a Lie group) map $\alpha : G \to \mathbb{T}$ such that

$$\alpha(x+y) = \alpha(x)\alpha(y)\langle x, y\rangle, \qquad \alpha(-x) = \alpha(x), \qquad \alpha(0) = 1$$

for all $x, y \in G$.

The condition $\alpha(-x) = \alpha(x)$ ensures that (1.3) holds, so that the Cubic transform for $G$ commutes with the flip unitary for the more general groups we're considering.

Our main result is the following.

THEOREM 1.3. Let $G$ be a locally compact, compactly generated self-dual Abelian group. Then there exists a projective character $\alpha : G \to \mathbb{T}$ and a constant $K, |K| = 1$, such that the transform

$$f^c(t) := K\,\alpha(t)\widehat{f}(-t) = K\,\alpha(t)\int_G f(x)\langle t, x\rangle\,dx$$

for $f \in \mathcal{S}(G)$ (the Schwartz space of $G$), defines a unitary operator of order 3 on $L^2(G)$ which commutes with the flip $f \to \widetilde{f}$, and so defines an order 6 unitary operator $H$ by $Hf = \widetilde{f}^{cc}$. Further, $H^3 f = \widetilde{f}, H^2 f = f^c$.

The Fourier transform for $G$ is, of course,

$$\widehat{f}(t) = \int_G f(x)\overline{\langle t, x\rangle}\,dx$$

for $f \in \mathcal{S}(G)$, where $dx$ is Haar measure on $G$.

## 2. Proof of Theorem 1.2

Here we discuss the proof of Theorem 1.2.

Let $G$ be a locally compact, compactly generated Abelian group. It is known that $G$ is (topologically) isomorphic to a direct sum $\mathbb{R}^n \oplus \mathbb{Z}^m \oplus K$ for some nonnegative integers $n, m$ and some compact Abelian group $K$ – see Hewitt and Ross [2], Theorem 9.8. The integers $m, n$ are unique (*ibid.*, Corollary 9.13). Further, it is known that each compact Abelian group is (topologically) isomorphic to $\mathbb{T}^k \oplus F$ for some nonnegative integer $k$ and some finite Abelian group $F$ (*ibid.*, Theorem 9.5 or 9.6). This means that $G \cong \mathbb{R}^n \oplus \mathbb{Z}^m \oplus \mathbb{T}^k \oplus F$. But since $G$ is assumed to be self-dual in this paper, $m = k$, and we have the topological isomorphism $G \cong \mathbb{R}^n \oplus (\mathbb{Z} \oplus \mathbb{T})^m \oplus F$ (as $\mathbb{R}$ and finite Abelian groups are already self-dual). By the fundamental theorem for finite Abelian groups, $F$ is a direct sum of a finite number of finite cyclic groups. Therefore, to show the existence of the map $\alpha$ asserted by Theorem 1.2 it is enough to show that such a map exists for each of the groups $\mathbb{R}, \mathbb{Z} \oplus \mathbb{T}$, and $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

In the case of group $\mathbb{R}$, such map $\alpha$ already exists by what we recalled above regarding the Hexic/Cubic transforms, namely $\alpha(x) = e(\frac{1}{2}x^2)$. It is easy to verify the properties in Theorem 1.2 for the case $G = \mathbb{R}$, as the (symmetric) pairing in this case is $\langle x, y \rangle = e(xy)$. (The normalizing constant in this case is $K = i^{-1/6}$.)

In the case of self-dual group $\mathbb{Z} \oplus \mathbb{T}$, whose (symmetric) pairing is

$$\langle (n, z), (m, w) \rangle = z^m w^n$$

for $m, n \in \mathbb{Z}$ and $z, w \in \mathbb{T}$, the map $\alpha$ can simply be taken to be $\alpha(n, z) = z^n$. Again, Theorem 1.2 is easy to verify in this case as well. The constant $K$ in Theorem 1.3 is $K = 1$.

For the cyclic group case $\mathbb{Z}_q$, whose (symmetric) pairing is $\langle m, n \rangle = e(\frac{mn}{q})$, we define $\alpha : \mathbb{Z}_q \to \mathbb{T}$ by

$$\alpha(n) = (-1)^n e(\frac{n^2}{2q}).$$

It is straightforward to check that this map is well defined on $\mathbb{Z}_q$ (the replacement $n \to n + q$ leaves the expression unchanged), and that the equalities in Theorem 1.2 are satisfied.

It is now plain that if $\alpha, \alpha'$ are projective characters on groups $G, G'$ (respectively) as in Theorem 1.2, then their direct product $\alpha\alpha'(x, y) = \alpha(x)\alpha'(y)$ defines a projective character on the direct sum $G \oplus G'$ as well. This proves Theorem 1.2.

## 3. Proof of Theorem 1.3

We now proceed with the proof of the main theorem.

Since the Haar measure on a finite direct sum of locally compact Abelian groups is just the product of the Haar measures of each summand, it is enough to prove Theorem 1.1 for each of the groups $\mathbb{R}, \mathbb{Z} \oplus \mathbb{T}, \mathbb{Z}_q$ separately (in view of what we stated earlier).

For $\mathbb{R}$, this was already dealt with in [3]. The case of the group $\mathbb{Z} \oplus \mathbb{T}$ is easier and so is left to the end. Now we proceed to deal with the case of the cyclic group since it involves some computations with Gaussian sums.

*The Cyclic Group* $\mathbb{Z}_q$. We shall make use of the simple equation

$$\sum_{n=0}^{q-1} e(\frac{nm}{q}) = q\delta_q^m$$

for all integers $m$.

We now check that $f^{ccc} = f$ for all complex functions $f$ on $\mathbb{Z}_q$. But since the group is finite, it is enough to check this for delta functions $f(n) = \delta_q^{n-a}$ localized at any fixed element $a \in \mathbb{Z}_q$. The normalizing constant $K$ in the case of the cyclic group (determined in Section 4, see Equation (4.7) below) is not too trivial to find and is determined in due process. We have

$$f^c(m) = K\alpha(m)\widehat{f}(-m) = K\alpha(m)\frac{1}{\sqrt{q}}\sum_{n=0}^{q-1}\delta_q^{n-a}e(\frac{mn}{q}) = \frac{K}{\sqrt{q}}\alpha(m)e(\frac{ma}{q})$$

and

$$\begin{aligned} f^{cc}(k) &= K\alpha(k)\widehat{f^c}(-k) = K\alpha(k)\frac{1}{\sqrt{q}}\sum_{n=0}^{q-1} f^c(n)e(\frac{kn}{q}) \\ &= K\alpha(k)\frac{1}{\sqrt{q}}\sum_{n=0}^{q-1}\frac{K}{\sqrt{q}}\alpha(n)e(\frac{na}{q})e(\frac{kn}{q}) \\ &= \frac{K^2}{q}\alpha(k)\sum_{n=0}^{q-1}\alpha(n)e(\frac{na}{q})e(\frac{kn}{q}) \\ &= \frac{K^2}{q}\sum_{n=0}^{q-1}\alpha(k)\alpha(n)\langle k,\, n\rangle\, e(\frac{na}{q}) \\ &= \frac{K^2}{q}\sum_{n=0}^{q-1}\alpha(n+k)e(\frac{na}{q}) \end{aligned}$$

using translation invariance

$$\begin{aligned} &= \frac{K^2}{q}e(-\frac{ka}{q})\sum_{n=0}^{q-1}\alpha(n)e(\frac{na}{q}) \\ &= \frac{K^2}{\sqrt{q}}\widehat{\alpha}(-a)e(-\frac{ka}{q}). \end{aligned}$$

A third iteration gives

$$\begin{aligned} f^{ccc}(m) &= K\alpha(m)\widehat{f^{cc}}(-m) = \frac{K}{\sqrt{q}}\alpha(m)\sum_{n=0}^{q-1} f^{cc}(n)e(\frac{nm}{q}) \\ &= \frac{K^3}{q}\widehat{\alpha}(-a)\alpha(m)\sum_{n=0}^{q-1} e(\frac{n(m-a)}{q}) \\ &= K^3\widehat{\alpha}(-a)\alpha(m)\delta_q^{m-a} \\ &= K^3\widehat{\alpha}(-a)\alpha(a)f(m). \end{aligned}$$

We now note that $\alpha(a)\widehat{\alpha}(-a) = \widehat{\alpha}(0)$ for each $a$. This is easily seen by integrating both sides of the equation $\alpha(a+x) = \alpha(a)\alpha(x)\langle a,\, x\rangle$ with respect to $x$, giving $\widehat{\alpha}(0) = \alpha(a)\widehat{\alpha}(-a)$. Therefore, we have obtained $f^{ccc} = K^3\widehat{\alpha}(0)f$ for each $f$. Using the computation of Section 4, which shows that $\widehat{\alpha}(0)$ is a complex number of modulus 1, one chooses $K = \widehat{\alpha}(0)^{-1/3}$ as the normalizing constant giving $f^{ccc} = f$ so that the Cubic transform has order 3. In Section 4, $K(q) = K$ depends on the mod 4 class of $q$.

*The Group* $\mathbb{Z}\oplus\mathbb{T}$. Here, we write functions on the group as sequences $f(n, z) = f_n(z)$ of functions on the unit circle, so the Fourier transform becomes

$$\widehat{f}(m, w) = \int_{\mathbb{Z}\oplus\mathbb{T}} f(n, z)\overline{\langle (n, z),\, (m, w)\rangle}\, d(n, z) = \sum_n w^{-n} \int_{\mathbb{T}} f(n, z) z^{-m} dz.$$

The generic functions $f_{a,b}(n, z) = z^a\delta_{n,b}$ defined for any pair of integers $a, b$ form a total set of functions in $L^2(\mathbb{Z}\oplus\mathbb{T})$, and it is easy to check that

$$\widehat{f}_{a,b} = f_{-b,a}.$$

Therefore,

$$f^c_{a,b}(m, w) = w^m f_{-b,a}(-m, w^{-1}) = w^{m+b}\delta_{m,-a}.$$

We have

$$\begin{aligned} f^{cc}_{a,b}(m, w) &= w^m \widehat{f^c_{a,b}}(-m, w^{-1}) \\ &= w^m \sum_n w^n \int_{\mathbb{T}} f^c_{a,b}(n, z) z^m dz \\ &= w^m \sum_n w^n \int_{\mathbb{T}} z^{n+b}\delta_{n,-a} z^m dz \\ &= w^m w^{-a} \int_{\mathbb{T}} z^{b-a+m} dz = w^{m-a}\delta_{m,a-b} \end{aligned}$$

and therefore

$$\begin{aligned} f^{ccc}_{a,b}(k, v) &= v^k \widehat{f^{cc}_{a,b}}(-k, v^{-1}) \\ &= v^k \sum_n v^n \int_{\mathbb{T}} f^{cc}_{a,b}(n, z) z^k dz \\ &= v^k \sum_n v^n \int_{\mathbb{T}} z^{n-a}\delta_{n,a-b} z^k dz \\ &= v^k v^{a-b} \int_{\mathbb{T}} z^{k-b} dz = v^{k+a-b}\delta_{k,b} \\ &= v^a\delta_{k,b} = f_{a,b}(k, v) \end{aligned}$$

so that $f^{ccc} = f$ for all $f$.

**4. The Normalization Constant** $K(q)$ In this section we compute the Fourier transform of the character $\alpha_q(n) = (-1)^n e(n^2/2q)$ at 0, i.e. $\widehat{\alpha}(0)$, so as to obtain the constant $K(q) = \widehat{\alpha}(0)^{-1/3}$.

First, recall the (quadratic) Gaussian sum defined for a pair of relatively prime integers $p, q$, with $q \geq 1$, by

$$G(p,q) = \sum_{n=0}^{q-1} e(\frac{pn^2}{q}) = \sum_{n=0}^{q-1} e^{2\pi i p n^2/q}.$$

It is known that for $p = 1$, one has Gauss' (remarkable!) formula

$$G(1,q) = \sqrt{q}\left(\frac{1+i}{\sqrt{2}}\right)\frac{1+i^{-q}}{\sqrt{2}} = \sqrt{q}e(\tfrac{1}{8})\frac{1+i^{-q}}{\sqrt{2}} \tag{4.1}$$

for all $q \geq 1$. (See, for example, Section 9.10 of Apostol [1].)

When $q$ is odd one has $G(p,q) = (p|q)G(1,q)$ where $(p|q)$ is the Jacobi symbol of $p, q$ (*ibid.*, Section 9.10). Thus, by (4.1)

$$G(p,q) = \sqrt{q}(p|q)\begin{cases} 1 & \text{if } q \equiv 1 \mod 4 \\ i & \text{if } q \equiv 3 \mod 4 \end{cases} \tag{4.2}$$

for all odd $q$. We will also need the following formula for the Jacobi symbol when $p = 2$:

$$(2|q) = (-1)^{(q^2-1)/8} \tag{4.3}$$

which holds for *odd* $q$ (see [1], Theorem 9.10). So for odd $q$, we have

$$G(2,q) = (2|q)G(1,q) = \sqrt{q}(-1)^{(q^2-1)/8}\begin{cases} 1 & \text{if } q \equiv 1 \mod 4, \\ i & \text{if } q \equiv 3 \mod 4. \end{cases} \tag{4.4}$$

With this in mind, we proceed to compute $\widehat{\alpha}(0)$ by computing the following sum

$$S(q) := \sqrt{q}\widehat{\alpha}(0) = \sum_{j=0}^{q-1} \alpha(j) = \sum_{j=0}^{q-1} (-1)^j e(\frac{j^2}{2q}).$$

Consider the sum

$$L(q) = \sum_{j=0}^{q-1} e(\frac{2j^2}{q}) = \begin{cases} 2G(1, \frac{q}{2}) & \text{if } q \text{ is even,} \\ G(2,q) & \text{if } q \text{ is odd.} \end{cases}$$

where both Gaussians $G(1, \frac{q}{2}), G(2,q)$ have been computed above.

LEMMA 4.1. One has $S(q) = L(q) - \frac{1}{2}G(1, 2q)$ for all $q$.

PROOF. Consider the sums

$$A=\sum_{j=0}^{2q-1}(-1)^j e(\frac{j^2}{2q}),\qquad G(1,2q)=\sum_{j=0}^{2q-1}e(\frac{j^2}{2q}).$$

Then by appropriate change of variables one has

$$A+G(1,2q)=\sum_{j=0}^{2q-1}(1+(-1)^j)e(\frac{j^2}{2q})=2\sum_{j=0}^{q-1}e(\frac{2j^2}{q})=2L(q).$$

Thus, $A=2L(q)-G(1,2q)$. Now let us break $A$ into two sums as follows

$$\begin{aligned}A&=\sum_{j=0}^{q-1}(-1)^j e(\frac{j^2}{2q})+\sum_{j=q}^{2q-1}(-1)^j e(\frac{j^2}{2q})\\&=S(q)+\sum_{k=0}^{q-1}(-1)^{k+q}e(\frac{(k+q)^2}{2q})=2S(q).\end{aligned}$$

Therefore, $S(q)=\frac{1}{2}A=L(q)-\frac{1}{2}G(1,2q)$ as desired. $\square$

Lemma 4.1 then gives

$$\begin{aligned}S(q)&=\begin{cases}2G(1,\frac{q}{2})-\frac{1}{2}G(1,2q) & \text{if } q \text{ is even,}\\ G(2,q)-\frac{1}{2}G(1,2q) & \text{if } q \text{ is odd,}\end{cases}\\ &=\begin{cases}\sqrt{q}e(\frac{1-q}{8}) & \text{if } q \text{ is even,}\\ G(2,q) & \text{if } q \text{ is odd.}\end{cases}\end{aligned}\tag{4.5}$$

where have noted that in the $q$ odd case $G(1,2q)=0$, and $G(2,q)$ is given by (4.4).

From (4.5) one immediately obtains the normalizing constant $K(q)$ to satisfy

$$K(q)^{-3}=\frac{1}{\sqrt{q}}S(q)=\begin{cases}e(\frac{1-q}{8}) & \text{if } q \text{ is even,}\\ (-1)^{(q^2-1)/8} & \text{if } q\equiv 1 \mod 4,\\ i(-1)^{(q^2-1)/8} & \text{if } q\equiv 3 \mod 4.\end{cases}\tag{4.6}$$

Or, more specifically,

$$K(q)=\begin{cases}e(\frac{q-1}{24}) & \text{if } q \text{ is even,}\\ e(\frac{1-q^2}{48}) & \text{if } q\equiv 1 \mod 4,\\ i^{-1/3}e(\frac{1-q^2}{48}) & \text{if } q\equiv 3 \mod 4.\end{cases}\tag{4.7}$$

If you have read the paper this far, you have done me a great honor. Thank you.

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## References

1. Tom Apostol, *Introduction to Analytic Number theory*, Springer-Verlag, Undergraduate Texts in Mathematics, 1976.
2. E. Hewitt and K. Ross, *Abstract Harmonic Analysis* (Vol. I), Springer Verlag, Second Edition, 1979.
3. S. Walters, *Periodic integral transforms and C*-algebras*, C. R. Math. Rep. Acad. Sci. Canada **26** (2004), no. 2, 55-61. arXiv: math.OA/0401340.

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