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The Pólya-Schur Problem on the Unit Circle

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Abstract. The Pólya-Schur problem for a region Z in the complex plane is to characterize the semigroup of linear operators $A : \mathbb{C}[z] \to \mathbb{C}[z]$ that map polynomials whose zeros are confined to Z to polynomials of the same type, or to 0. We give a constructive solution to the Pólya-Schur problem in the case where Z is the unit circle. This shows that the associated semigroup is qualitatively simpler than in the classical case where Z is the real line, whereas recent results have not clearly distinguished the two cases.

Résumé. Le problème Pólya-Schur pour une région Z dans le plan complexe est de charactériser le semigroupe des opérateurs linéaires A : $\mathbb{C}[z] \to \mathbb{C}[z]$ envoyant chaque polynôme dont les racines appartiennent à Z vers un polynôme du même type, ou vers 0. Nous présentons une solution constructive au problème Pólya-Schur dans le cas où Z est le cercle unité. Cela démontre que le semigroupe associé est qualitativement plus simple que dans le cas classique de la ligne réelle, tandis que les résultats récents n'ont pas distingué les deux cas.

1. Introduction Let $\Omega \subset \mathbb{C}$ denote a set in the complex plane, and let $Z = \mathbb{C} \setminus \Omega$ denote its complement. A univariate polynomial $p \in \mathbb{C}[z]$ is said to be Ω-stable if it has no zeros in Ω, i.e., if its zeros are contained in Z. Let $\mathcal{S}(\Omega)$ denote the set of all Ω-stable polynomials, together with 0. A linear mapping $A : \mathbb{C}[z] \to \mathbb{C}[z]$ is said to be stability preserving with respect to Ω if $A(\mathcal{S}(\Omega)) \subset \mathcal{S}(\Omega)$; we denote the semigroup of all such mappings by $\mathscr{S}(\Omega)$.

The Pólya-Schur problem for a given region Ω is to determine the semigroup $\mathscr{S}(\Omega)$ of stability-preserving linear mappings. The name refers to early work [6] on multiplier sequences, corresponding to the classical case $Z = \mathbb{R}$, i.e., $\Omega = \mathbb{C} \setminus \mathbb{R}$. Pólya-Schur problems have recently come into prominence in relation to statistical mechanics, [2], [7]. An essential result in this connection concerns circular regions Ω, whose boundary in $\mathbb{C}$ is a circle or line, [3]. In the latter work, determining $\mathscr{S}(\Omega)$ is interpreted as characterizing it indirectly, such as by giving an infinite set X of test functions—i.e., A belongs to $\mathscr{S}(\Omega)$ if and only if $Ap \in \mathcal{S}(\Omega)$ for every $p \in X \subset \mathcal{S}(\Omega)$—or by describing certain characteristic functions of operators in $\mathscr{S}(\Omega)$. In particular, letting $\mathbb{T}$ denote the unit circle, the cases $Z = \mathbb{T}$ and $Z = \mathbb{R}$ are handled together in a single statement, [3, Cor. 4],

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suggesting that the semigroups $\mathscr{S}(\mathbb{C} \setminus \mathbb{T})$ and $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$ should have similar structure.

In the present paper we derive an explicit, constructive characterization of $\mathscr{S}(\mathbb{C} \setminus \mathbb{T})$ which, in conjunction with existing literature, reveals that $\mathscr{S}(\mathbb{C} \setminus \mathbb{T})$ and $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$ are qualitatively different. Indeed the structure of the semigroup $\mathscr{S}(\Omega)$ serves as a natural way to classify regions in the plane with respect to the Pólya-Schur problem, and from this point of view, circular regions are not a natural class.

Multiplication operators and composition operators provide two readily constructed classes of stability preserving operators with respect to an arbitrary region Ω, as follows. Given any polynomial $\theta \in \mathcal{S}(\Omega)$, the associated multiplication operator

$$M_\theta : \mathbb{C}[z] \to \mathbb{C}[z], \qquad M_\theta p = \theta p \quad \forall p \in \mathbb{C}[z],$$

belongs to $\mathscr{S}(\Omega)$. Given a polynomial φ, let $C_\varphi : \mathbb{C}[z] \to \mathbb{C}[z]$ denote the composition operator defined by the formula

$$C_\varphi p = p \circ \varphi \quad \forall p \in \mathbb{C}[z].$$

It is straightforward to check that $C_\varphi \in \mathscr{S}(\Omega)$ if $\varphi(\Omega) \subset \Omega$; in the case where φ is non-constant, $C_\varphi \in \mathscr{S}(\Omega)$ if and only if $\varphi(\Omega) \subset \Omega$. We refer to any operator of the form $M_\theta C_\varphi$, for any θ, φ, as a product-composition operator. Our main result asserts that in the case $Z = \mathbb{T}$, the Pólya-Schur problem is solved essentially by product-composition operators.

THEOREM 1. *Let $\Omega = \mathbb{C} \setminus \mathbb{T}$. Then a linear mapping $A : \mathbb{C}[z] \to \mathbb{C}[z]$, not identically zero, belongs to $\mathscr{S}(\Omega)$ if and only if either A has rank 1, with range spanned by some $\theta \in \mathcal{S}(\Omega)$, or A is a product-composition operator of the form $A = M_\theta C_\varphi$, where $\theta \in \mathcal{S}(\Omega) \setminus \{0\}$ and $\varphi(z) = \gamma z^k$ for some $\gamma \in \mathbb{T}$ and some integer $k \geq 1$.*

Evidently $\theta \in \mathcal{S}(\Omega) \setminus \{0\}$ if and only if the zeros of θ belong to $\mathbb{T}$, i.e., if and only if

$$\theta(z) = \alpha \prod_{j=1}^{n} (z - r_j) \quad \text{for some } \alpha \in \mathbb{C}, n \geq 0 \text{ and } r_1, \ldots, r_n \in \mathbb{T}.$$

Thus the above result is completely explicit. Theorem 1 shows that $\mathscr{S}(\mathbb{C} \setminus \mathbb{T})$ is minimal in the sense that, apart from rank one operators, it includes only product-composition operators. The same result has been shown in [4] to hold for $\mathscr{S}(\Omega)$ for any bounded set Ω that has non-empty interior. By contrast, $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$ includes a family of operators such as differentiation which are not of rank one and which cannot be expressed as product-composition operators, viz. [1], [5, Ch. 8]; hence the classical case $Z = \mathbb{R}$ engenders a qualitatively more complicated structure than the circle.

2. Proof of the Main Result We refer to the moments of a given linear operator A using the following notation. For each integer $n \geq 0$, let μ_n denote the nth moment function, $\mu_n(z) = z^n$, and let $\psi_n = A\mu_n$ denote its image by A; we call ψ_n the nth moment of A. In order to prove Theorem 1, we consider an arbitrary $A \in \mathscr{S}(\Omega)$, and then study the possibilities for its moments ψ_n. Henceforth we assume that $\Omega = \mathbb{C} \setminus \mathbb{T}$ and that $A \in \mathscr{S}(\Omega)$ is fixed, with its moments denoted ψ_n. To begin, we consider the case where $\psi_0 = 0$.

2.1. The case $\psi_0 = 0$

Lemma 2.1. *Suppose that $\psi_0 = 0$. Then $\psi_n \in \mathcal{S}(\Omega)$ for each integer $n \geq 1$.*

Proof. For any $\lambda \in \mathbb{T}$ and $n \geq 1$, $p(z) = z^n - \lambda \in \mathcal{S}(\Omega)$. Therefore $Ap(z) = \psi_n \in \mathcal{S}(\Omega)$. □

Lemma 2.2. *Suppose that $\psi_0 = 0$, and let $n \geq 2$ be an integer. If*

$$dim\, span\, \{\psi_1, \dots, \psi_{n-1}\} \leq 1$$

then $dim\, span\, \{\psi_1, \dots, \psi_n\} \leq 1$.

Proof. Fix $n \geq 2$ and suppose that $\dim \operatorname{span} \{\psi_1, \dots, \psi_{n-1}\} \leq 1$. The conclusion of the lemma follows immediately either if

$$\dim \operatorname{span} \{\psi_1, \dots, \psi_{n-1}\} = 0$$

or if $\psi_n = 0$. Therefore assume that $\psi_n \neq 0$ and that

$$\dim \operatorname{span} \{\psi_1, \dots, \psi_{n-1}\} = 1$$

which implies that there is some index j, with $1 \leq j \leq n-1$, such that $\psi_j \neq 0$ and

$$\operatorname{span}\{\psi_1, \dots, \psi_{n-1}\} = \operatorname{span}\{\psi_j\}.$$

If ψ_n and ψ_j are both constant, then the conclusion of the lemma follows; so assume further that ψ_n and ψ_j are not both constant.

Set $k = n - j$. By assumption, $\psi_k \in \operatorname{span}\{\psi_j\}$ so that $\psi_k = \alpha\psi_j$ for some $\alpha \in \mathbb{C}$. Consider polynomials of the form

$$p(z) = (z^j - \lambda_1)(z^k - \lambda_2) = z^n - \lambda_2 z^j - \lambda_1 z^k + \lambda_1\lambda_2$$

where $\lambda_1, \lambda_2 \in \mathbb{T}$. Since every such $p \in \mathcal{S}(\Omega)$ it follows that

$$\begin{aligned} Ap &= \psi_n - \lambda_2\psi_j - \lambda_1\psi_k + \lambda_1\lambda_2\psi_0 \\ &= \psi_n - \lambda_2\psi_j - \lambda_1\alpha\psi_j \\ &= \psi_n - (\lambda_2 + \alpha\lambda_1)\psi_j \in \mathcal{S}(\Omega). \end{aligned}$$

Since ψ_n and ψ_j are not both constant, the function Ap is constant for at most one value of $\mu = \lambda_2 + \alpha\lambda_1$. If there is such a value denote it μ_0; otherwise set $\mu_0 = \infty$. Thus for $\mu \neq \mu_0$, $\psi_n - \mu\psi_j \in \mathcal{S}(\Omega)$ is non-constant and therefore has its zeros on $\mathbb{T}$. Now, given an arbitrary $\lambda_2 \in \mathbb{T}$, set

$$\lambda_1 = \begin{cases} \frac{\overline{\alpha}}{|\alpha|}\lambda_2 & \text{if } \alpha \neq 0 \\ \lambda_2 & \text{if } \alpha = 0 \end{cases}.$$

With this choice of λ_1 we have that $\mu = (1+|\alpha|)\lambda_2 \neq 0$, and there is at most one value of λ_2 for which $\mu = \mu_0$. If there is a value of λ_2 for which $\mu = \mu_0$, denote this value λ_0, and otherwise set $\lambda_0 = \infty$. Thus with this choice of λ_1, and for $\lambda_2 \neq \lambda_0$, the function $\psi_n - \mu\psi_j \in \mathcal{S}(\Omega)$ is zero if and only if

$$R(z) = \frac{\psi_n(z)}{(1+|\alpha|)\psi_j(z)} = \lambda_2$$

in which case necessarily $z \in \mathbb{T}$.

But if the rational function $R(z)$ is not constant, it has at least one zero, say ζ, and one pole, say π, both belonging to the set $\mathbb{T} \cup \{\infty\}$. (Recall that both ψ_n and ψ_j belong to $\mathcal{S}(\Omega)$, by Lemma 2.1.) Observe that there exists a simple path $\gamma : [0,1] \mapsto \mathbb{C}$ going from $\gamma(0) = \zeta$ to $\gamma(1) = \pi$, such that

$$\gamma\big((0,1)\big) \subset \mathbb{C} \setminus \Big(\overline{\mathbb{D}} \cup R^{-1}(\lambda_0)\Big).$$

By the intermediate value theorem, there is a point $t \in (0,1)$ at which $|R(\gamma(t))| = 1$ (with $R(\gamma(t)) \neq \lambda_0$), since γ goes from a zero to a pole of R. And by construction $|\gamma(t)| \neq 1$, contradicting the fact that for any $\lambda_2 \in \mathbb{T}$ distinct from λ_0, $R(z) = \lambda_2$ only if $z \in \mathbb{T}$. This contradiction implies that the rational function $R(z)$ must in fact be constant, from which it follows that ψ_n and ψ_j are proportional. In other words,

$$\text{span}\{\psi_1, \ldots, \psi_n\} = \text{span}\{\psi_1, \ldots, \psi_{n-1}\},$$

proving the lemma. □

PROPOSITION 2.1. *Suppose that $\psi_0 = 0$. Then A has rank at most 1.*

PROOF. Starting from the fact that $\dim \text{span}\{\psi_1\} \leq 1$, induction using Lemma 2.2 yields that $\text{rank} A = \dim \text{span}\{\psi_1, \psi_2, \ldots\} \leq 1$. □

In the next section we examine the consequences of $\psi_0 \neq 0$.

2.2. The case $\psi_0 \neq 0$ We establish some additional notation associated with the case $\psi_0 \neq 0$. Note first that since $1 \in \mathcal{S}(\Omega)$ and $\psi_0 = A1$, it follows that $\psi_0 \in \mathcal{S}(\Omega)$, so the zeros of ψ_0 lie on $\mathbb{T}$. For each integer $n \geq 0$, set $\varphi_n = \psi_n/\psi_0$, so that in particular $\varphi_0 = 1$. Define a new linear operator

$$B = M_{\frac{1}{\psi_0}} A.$$

We shall see that each φ_n is in fact a monomial, and hence that

$$B : \mathbb{C}[z] \to \mathbb{C}[z]$$

maps polynomials to polynomials, whereby $B \in \mathscr{S}(\Omega)$ just as $A \in \mathscr{S}(\Omega)$.

Lemma 2.3. *If $p \in \mathcal{S}(\Omega)$ and $Bp \neq 0$, then the zeros of Bp necessarily belong to $\mathbb{T}$.*

Proof. Since $\psi_0 \neq 0$, $Bp \neq 0$ implies that $Ap = \psi_0 Bp \neq 0$. Every zero of Bp is a zero of Ap. Since $Ap \in \mathcal{S}(\Omega)$ and distinct from 0, it follows that every zero of Ap, and in particular every zero of Bp, belongs to $\mathbb{T}$. □

Lemma 2.4. *If $\psi_0 \neq 0$ then for every $n \geq 1$ either φ_n is constant or there exist $\gamma_n \in \mathbb{T}$ and and a positive integer $k_n \geq 1$ such that $\varphi_n(z) = \gamma_n z^{k_n}$.*

Proof. Fix $n \geq 1$ and suppose that φ_n is not constant. Note that, since $\psi_0 \in \mathcal{S}(\Omega)$, any poles that $\varphi_n = \psi_n/\psi_0$ has are confined to $\mathbb{T}$. For every $\lambda \in \mathbb{T}$,

$$p(z) = z^n + \lambda \in \mathcal{S}(\Omega)$$

and so

$$Ap = \psi_n + \lambda\psi_0 \in \mathcal{S}(\Omega) \quad \text{while} \quad Bp = \varphi_n + \lambda.$$

Now, $Ap = 0$ only if φ_n is constant; and if $Ap \neq 0$, then the zeros of $Ap(z)$ include those of $Bp(z)$. Since φ_n is assumed not to be constant, it has at least one zero and one pole, and the zeros of Bp lie on $\mathbb{T}$ (for every $\lambda \in \mathbb{T}$).

We claim that $\varphi_n(\mathbb{T}) \subset \mathbb{T}$. Suppose to the contrary that $\varphi_n(\xi) \notin \mathbb{T}$ for some $\xi \in \mathbb{T}$. If $|\varphi_n(\xi)| > 1$, then choose a zero ζ of φ_n and path $\gamma : [0,1] \to \mathbb{C}$ going from $\gamma(0) = \xi$ to $\gamma(1) = \zeta$ such that $\gamma(t) \notin \mathbb{T}$ for every $t \in (0,1)$. Then for some $t_0 \in (0,1)$, $|\varphi(\gamma(t_0))| = 1$. But, for $\lambda = -\varphi_n(\gamma(t_0))$, this means that $\gamma(t_0)$ is a zero of the corresponding Bp, contradicting the fact that such zeros lie on $\mathbb{T}$. Similarly, if $|\varphi(\xi)| < 1$ one may use a path from ξ to a pole π of φ_n (possibly $\pi = \infty$) to find a point off $\mathbb{T}$ which is a zero of Bp for an appropriate choice of λ, again contradicting that such zeros lie on $\mathbb{T}$. Therefore the only possibility is that $\varphi_n(\xi) \in \mathbb{T}$, proving the claim.

We have that φ_n is a rational function whose poles are confined to $\mathbb{T}$ and which maps $\mathbb{T}$ into $\mathbb{T}$. This means that φ_n is an inner function. The structure theorem for inner functions implies that φ_n is a product of a finite Blaschke product β with a monomial of the form $\gamma_n z^{k_n}$, where $\gamma_n \in \mathbb{T}$ and $k_n \geq 0$ is an integer. But a nontrivial Blaschke factor has poles outside $\overline{\mathbb{D}}$, which φ_n does not. So $\beta = 1$ is trivial. Furthermore, since we have assumed that φ_n is non-constant it follows that the integer $k_n \geq 1$, completing the proof. □

Lemma 2.5. *If φ_1 is constant then ϕ_n is constant for every $n \geq 1$.*

PROOF. Suppose that φ_1 is constant. If not every φ_n is constant, choose the least $n \geq 2$ such that φ_n is not constant. Then φ_n has the form $\varphi_n(z) = \gamma_n z^{k_n}$ for some $\gamma_n \in \mathbb{T}$ and some integer $k_n \geq 1$ by Lemma 2.4. Write γ_j for the constant value of φ_j for $1 \leq j \leq n-1$.

We claim that each $\gamma_j = 0$. To see this, fix j in the given range and set $k = n - j$. Consider polynomials of the form

$$p(z) = (z^j - \lambda)(z^k - \mu) = z^n - \mu z^j - \lambda z^k + \lambda\mu \quad \text{where} \quad \lambda, \mu \in \mathbb{T}.$$

Each such $p \in \mathcal{S}(\Omega)$ and therefore its image by B,

$$Bp(z) = \varphi_n - \mu\gamma_j - \lambda\gamma_k + \lambda\mu,$$

being non-constant, has all its zeros on $\mathbb{T}$ by Lemma 2.3. If γ_j and γ_k are not both zero, then it is possible to choose $\lambda, \mu \in \mathbb{T}$ such that

$$\eta = \mu\gamma_j + \lambda\gamma_k - \lambda\mu \notin \mathbb{T}.$$

(For instance, if $\gamma_j\gamma_k \neq 0$, set

$$\lambda = \pm\frac{\overline{\gamma_k}}{|\gamma_k|} \quad \text{and} \quad \mu = \pm\frac{\overline{\gamma_j}}{|\gamma_j|}.$$

Then at most one of the two resulting values of η can lie on $\mathbb{T}$.) But if $\eta \notin \mathbb{T}$, then, since $\varphi_n = \gamma_n z^{k_n}$, Bp has a zero not on $\mathbb{T}$, a contradiction. Therefore $\gamma_j = 0$ as claimed.

Given that $\gamma_j = 0$ for $1 \leq j \leq n-1$, consider polynomials of the form

$$q(z) = (z + \lambda)^{n+1} \quad \text{where} \quad \lambda \in \mathbb{T}.$$

Then $Bq = \varphi_{n+1} + (n+1)\lambda\varphi_n + \lambda^{n+1}$. Since Bq is non-constant irrespective of φ_{n+1} (consistent with Lemma 2.4), each zero of Bq lies on $\mathbb{T}$. Now,

$$\begin{aligned} Bq = 0 \Leftrightarrow \varphi_n &= \frac{-\lambda^{n+1} - \varphi_{n+1}}{\lambda(n+1)} \\ \Rightarrow |\varphi_n| &= \frac{|\varphi_{n+1} + \lambda^{n+1}|}{n+1}. \end{aligned}$$

Whatever φ_{n+1}, there is a choice of $\lambda \in \mathbb{T}$ for which the right-hand side of the latter equation is different from 1. Since $\varphi_n(z) = \gamma_n z^{k_n}$, this implies that Bq has a zero not in $\mathbb{T}$, a contradiction. The latter contradiction implies that φ_n is constant for every $n \geq 1$. $\square$

In light of the original definition of $\varphi_n = \psi_n/\psi_0$, the following result is an immediately corollary of Lemma 2.5.

Proposition 2.2. *If φ_1 is constant then the range of A is span$\{\psi_0\}$.*

We say that a function $q(z,\lambda)$ is constant with respect to z for a particular λ if $q(z,\lambda) = q(0,\lambda)$ for every $z \in \mathbb{C}$, i.e., if $q(\cdot,\lambda)$ is a constant function.

Lemma 2.6. *Suppose that φ_1 is non-constant and fix $n \geq 1$. Set $p(z,\lambda) = (z+\lambda)^n$, and define $q(z,\lambda)$ by $q = Bp$. Then q is constant with respect to z for at most n distinct values of λ.*

Proof. By definition,

$$q(z,\lambda) = \sum_{j=0}^{n} \binom{n}{j} \varphi_j(z) \lambda^{n-j}.$$

Write $u(\lambda) = (\lambda^n, \lambda^{n-1}, \ldots, 1) \in \mathbb{C}^n$, and

$$v(z) = \left(1, n\varphi_1(z), \binom{n}{2}\varphi_2(z), \ldots, \binom{n}{n-1}\varphi_{n-1}(z), \varphi_n(z)\right) \in \mathbb{C}^n,$$

so that $q(z,\lambda) = \langle u(\lambda), \overline{v(z)}\rangle$, expressed in terms of the standard scalar product on $\mathbb{C}^n$. Thus $q(z,\lambda)$ is constant with respect to z if and only if

$$\forall z \in \mathbb{C}, \qquad \frac{\partial}{\partial z} q(z,\lambda) = \langle u(\lambda), \overline{v'(z)}\rangle = 0.$$

Note that $v'(z) = \left(0, nk_1\gamma_1 z^{k_1-1}, \ldots\right) \neq 0$ if $z \neq 0$, since by Lemma 2.4 and the assumption that φ_1 is non-constant, $\varphi_1 = \gamma_1 z^{k_1}$ for some $\gamma_1 \in \mathbb{T}$ and $k_1 \geq 1$. Suppose that $q(z,\mu_j)$ is constant with respect to z for each of n distinct numbers $\mu_1, \ldots, \mu_n \in \mathbb{C}$. Construct the $(n+1)\times(n+1)$ Vandermonde matrix $V(\lambda)$ whose jth row is $u(\mu_j)$, for $1 \leq j \leq n$, and whose $(n+1)$st row is $u(\lambda)$. Since $V(\lambda)$ is non-singular for

$$\lambda \notin \{\mu_1, \ldots, \mu_n\},$$

it follows that $V(\lambda)v'(z)^T \neq 0$ if $z \neq 0$. By assumption, $\langle u(\mu_j), \overline{v'(z)}\rangle = 0$ for $1 \leq j \leq n$. Therefore $\langle u(\lambda), \overline{v'(z)}\rangle \neq 0$, proving that $q(z,\lambda)$ is non-constant for λ distinct from each μ_j. □

Lemma 2.7. *Let $n \geq 2$ and suppose that φ_j is non-constant for $1 \leq j \leq n$. Then $\varphi_j = (\varphi_1)^j$ for $2 \leq j \leq n$.*

Proof. Let $n \geq 2$ and suppose that φ_j is non-constant for $1 \leq j \leq n$. Define $q(z,\lambda)$ to be the image by B of the polynomial $p(z,\lambda) = (z+\lambda)^n$, as in Lemma 2.6. Then by Lemma 2.6 $q(z,\lambda)$ is non-constant with respect to z for almost every $\lambda \in \mathbb{T}$. Let $\lambda_0 \in \mathbb{T}$ be such that $q(z,\lambda_0)$ is non-constant. Then the equation $q(z,\lambda_0) = 0$ has at least one root, $z = z_0$, and this root belongs to $\mathbb{T}$ by Lemma 2.3.

Consider the segment $\gamma(t) = tz_0$ for $t \in [0,1]$. Note that $q(tz_0, \lambda)$ is a polynomial in λ of degree n whose coefficients depend continuously on the real parameter t. Let $r_1(t), \dots, r_n(t)$ denote the n values of λ (including multiplicity) that satisfy the equation $q(tz_0, \lambda) = 0$, ordered such that each $r_j : [0,1] \to \mathbb{C}$ is continuous.

Note that $q(0, \lambda) = \lambda^n$ by Lemma 2.4, since each φ_n is non-constant. Therefore $r_j(0) = 0$ for each j. On the other hand,

$$\lambda_0 \in \{r_1(1), \dots, r_n(1)\}$$

by construction. We claim that $|r_j(t)| < 1$ for each $1 \le j \le n$ and each $t \in (0,1)$. Otherwise some such $|r_j(t)| = 1$, which implies that $q(tz_0, r_j(t)) = 0$. Lemma 2.3 then implies that $|tz_0| = 1$, contradicting the facts that $z_0 \in \mathbb{T}$ and $t \in (0,1)$. This proves the claim. It then follows by continuity that each $|r_j(1)| \le 1$. Thus $r_1(1), \dots, r_n(1)$ are n points lying in the closed unit disk.

We have that

$$q(z_0, \lambda) = \prod_{j=1}^{n} (\lambda - r_j(1)) = \lambda^n - (r_1(1) + \cdots + r_n(1))\lambda^{n-1} + \cdots$$

and

$$q(z_0, \lambda) = \sum_{j=0}^{n} \binom{n}{j} \varphi_j(z_0) \lambda^{n-j} = \lambda^n + n\varphi_1(z_0)\lambda^{n-1} + \cdots . \tag{2.1}$$

Comparing coefficients of λ^{n-1} shows that

$$\frac{|r_1(1) + \cdots + r_n(1)|}{n} = |\varphi_1(z_0)| = 1,$$

since $\varphi_1(z) = \gamma_1 z^{k_1}$ for some $\gamma_1 \in \mathbb{T}$ and some $k_1 \ge 1$. In other words, the centroid of the $r_j(1)$ lies on the unit circle, with each $r_j(1)$ belonging to the closed unit disk. The only way for the centroid not to belong to the open unit disk is if

$$r_1(1) = r_2(1) = \cdots = r_n(1) = \lambda_0.$$

Thus $q(z_0, \lambda) = (\lambda - \lambda_0)^n$. Comparing this to (2.1) shows that

$$\varphi_j(z_0) = (-\lambda_0)^j,$$

for each $1 \le j \le n$. In particular,

$$\varphi_1(z_0) = \gamma_1 z_0^{k_1} = -\lambda_0$$

so that $\varphi_j(z_0) = (\varphi_1(z_0))^j$ for each $1 \le j \le n$. All but finitely many points in $\mathbb{T}$ could have been chosen as λ_0, it follows that the latter equation holds for infinitely many $z_0 \in \mathbb{T}$, proving that $\varphi_j = (\varphi_1)^j$. □

LEMMA 2.8. *If φ_1 is non-constant, then φ_n is non-constant for every $n \geq 1$.*

PROOF. Suppose that φ_1 is non-constant, so that $\varphi_1(z) = \gamma_1 z^{k_1}$ for some $\gamma_1 \in \mathbb{T}$ and $k_1 \geq 1$ as per Lemma 2.4. Suppose further that there is a least index $n \geq 2$ such that $\varphi_n = \gamma_n \in \mathbb{C}$ is constant. We shall obtain a contradiction by showing in turn that each of $|\gamma_n| = 1$, $|\gamma_n| < 1$ and $|\gamma_n| > 1$ is impossible.

Note that φ_j has the form $\varphi_j(z) = \gamma_1^j z^{jk_1}$ for each $2 \leq j \leq n-1$ by Lemma 2.7. Define $q(z, \lambda)$ to be the image by B of $(z+\lambda)^n$ as in the previous two lemmas. According to our assumptions thus far, $q(z, \lambda)$ has degree $(n-1)k_1$ in z and is therefore non-constant for every $\lambda \neq 0$. And in addition, $q(0, \lambda) = \lambda^n + \gamma_n$.

To begin we argue that $\gamma_n \notin \mathbb{T}$. If $\gamma_n \in \mathbb{T}$, let λ_0 denote an nth root of $-\gamma_n$. Then $\lambda_0 \in \mathbb{T}$ and $q(0, \lambda_0) = 0$, contradicting the fact that the values of z for which $q(z, \lambda_0) = 0$ lie on $\mathbb{T}$. Thus $\gamma_n \notin \mathbb{T}$.

Suppose next that $|\gamma_n| < 1$. In this case, we may repeat the argument from the proof of Lemma 2.7, as follows. Choose any $\lambda_0 \in \mathbb{T}$ and corresponding $z_0 \in \mathbb{T}$ such that $q(z_0, \lambda_0) = 0$. For $1 \leq j \leq n$, define $r_j(t)$ to be a continuous parameterization of the n values of λ for which $q(tz_0, \lambda) = 0$. The only difference is that now each $r_j(0)$ is an nth root of γ_n instead of being zero. But the essential fact that each $r_j(0)$ lies in the open unit disk remains unchanged, and from this it follows as before that each $r_j(1)$ lies on the closed unit disk and that the centroid of the $r_j(1)$ lies on $\mathbb{T}$. Therefore $r_1(1) = \cdots = r_n(1) = \lambda_0$, as before. However, in the present case

$$q(z_0, \lambda) = \lambda^n + n\varphi_1(z_0)\lambda^{n-1} + \cdots + n\varphi_1^{n-1}(z_0)\lambda + \gamma_n$$

and

$$q(z_0, \lambda) = (\lambda - \lambda_0)^n = \lambda^n + -n\lambda_n\lambda^{n-1} + \cdots + n(-\lambda_0)^{n-1} + (-\lambda_0)^n.$$

Comparing constant terms from these two expressions yields that $\gamma_n = (-\lambda_0)^n$, which is a contradiction, since $\gamma_n \in \mathbb{D}$ while $(-\lambda_0)^n \in \mathbb{T}$. Therefore $|\gamma_n| \not< 1$.

The remaining possibility is $|\gamma_n| > 1$. In this case consider the polynomial

$$(z^{n-1} + \lambda)(z + \lambda) = z^n + \lambda z^{n-1} + \lambda z + \lambda^2$$

and let $r(z, \lambda)$ denote its image by B, so that

$$\begin{aligned} r(z, \lambda) &= \gamma_n + \lambda\varphi_1^{n-1}(z) + \lambda\varphi_1(z) + \lambda^2 \\ &= \gamma_n + \lambda^2 + \lambda\varphi_1(z)\big(\varphi_1^{n-2}(z) + 1\big). \end{aligned}$$

If $\lambda \in \mathbb{T}$ then the original polynomial is in $\mathcal{S}(\Omega)$ and so its image $r(z, \lambda)$, being non-constant of degree $(n-1)k_1$, is zero only if $z \in \mathbb{T}$ by Lemma 2.3. Choose λ_0 such that

$$\lambda_0^2 = \frac{\gamma_n}{|\gamma_n|} \in \mathbb{T}.$$

Then $\lambda_0 \in \mathbb{T}$ and $|\gamma_n + \lambda_0^2| = |\gamma_n| + 1 > 2$. Meanwhile if $z \in \mathbb{T}$, then

$$\left|\lambda\varphi_1(z)\big(\varphi_1^{n-2}(z)+1\big)\right| = \left|\varphi_1^{n-2}(z)+1\right| \le 2.$$

Therefore $r(z,\lambda_0) \neq 0$ if $z \in \mathbb{T}$. This is a contradiction since $r(z,\lambda_0)$ is non-constant with respect to z and so has at least one zero; the case $|\gamma_n| > 1$ is thereby ruled out.

It follows that φ_n is non-constant, meaning there is no $n \geq 2$ for which φ_n is constant, and the lemma is proved. □

PROPOSITION 2.3. *If φ_1 is non-constant then $A = M_\psi C_\varphi$, where $\psi \in \mathcal{S}(\Omega)$ and $\varphi(z)$ has the form $\varphi(z) = \gamma z^k$ for some $\gamma \in \mathbb{T}$ and some integer $k \geq 1$.*

PROOF. If φ_1 is non-constant, then by Lemma 2.4 there exist $\gamma \in \mathbb{T}$ and and an integer $k \geq 1$ such that $\varphi_1(z) = \gamma z^k$. Applying Lemmas 2.8 and 2.7, it follows that $\varphi_n(z) = \gamma^n z^{kn}$ for every $n \geq 1$. It follows by linearity that $B = C_\varphi$, where $\varphi(z) = \varphi_1(z) = \gamma z^k$. By definition $B = M_{\frac{1}{\psi_0}} A$, therefore $A = M_{\psi_0} B = M_{\psi_0} C_\varphi$. Since $\psi_0 \in \mathcal{S}(\Omega)$, this proves the proposition, with $\psi = \psi_0$. □

Theorem 1 follows from Propositions 2.1, 2.2 and 2.3. In the case $\psi_0 = 0$, Proposition 2.1 shows that A has rank at most 1. If $\psi_0 \neq 0$ there are two cases. If ψ_1 is a constant multiple of ψ_0, or equivalently, if φ_1 is constant, then Proposition 2.2 shows that A has rank precisely 1. And if ψ_1 is not a constant multiple of ψ_0, equivalently, if φ_1 is non-constant, then Proposition 2.3 shows that A is a product-composition operator of the precise structure described in Theorem 1.

3. Remarks

We discuss briefly the significance of Theorem 1 in the context of the general Pólya-Schur problem; in this connection the following result from the literature is relevant.

THEOREM 2 (From Theorem 2 in [4]). *Suppose $\Omega \subset \mathbb{C}$ is bounded and has non-empty interior. A linear map $A : \mathbb{C}[z] \to \mathbb{C}[z]$ belongs to $\mathscr{S}(\Omega)$ if and only if either:*

1. *there exist a linear functional $\nu : \mathbb{C}[z] \to \mathbb{C}$ and a polynomial $\psi \in \mathcal{S}(\Omega)$ such that $A(f) = \nu(f)\psi$, for all $f \in \mathbb{C}[z]$; or*
2. *there exist $\psi \in \mathcal{S}(\Omega)$ and a non-constant polynomial φ for which $\varphi(\Omega) \subset \Omega$ such that $A = M_\psi C_\varphi$.*

In the case of arbitrary $\Omega \subset \mathbb{C}$, it is straightforward to check that $M_\psi C_\varphi \in \mathscr{S}(\Omega)$ for any $\psi \in \mathcal{S}(\Omega)$ and any polynomial $\varphi \in \mathbb{C}[z]$ such that $\varphi(\Omega) \subset \Omega$. Thus product-composition operators comprise a sub-semigroup of $\mathscr{S}(\Omega)$ for any Ω. Referring to an operator $A \in \mathscr{S}(\Omega)$ as trivial if it has rank at most 1, let us call a semigroup $\mathscr{S}(\Omega)$ *minimal* if all its nontrivial operators are product-composition operators. The content of Theorem 2 is to give a very general sufficient condition on Ω for $\mathscr{S}(\Omega)$ to be minimal.

Importantly, $\mathscr{S}(\Omega)$ is not minimal in the classical case $\Omega = \mathbb{C} \setminus \mathbb{R}$. This is because $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$ contains the differential operator $\frac{d}{dz}$, for example; the latter is nontrivial and noninjective, whereas every nontrivial product-composition operator is injective. Indeed there are many other nontrivial operators in $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$ besides differentiation which cannot be expressed as product-composition operators, giving the semigroup $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$ a qualitatively more complex structure than if it were minimal. (See [5, Ch. 8] for further examples.)

The main result of the present paper, Theorem 1, asserts that $\mathscr{S}(\mathbb{C} \setminus \mathbb{T})$ is minimal, despite the fact that $\Omega = \mathbb{C} \setminus \mathbb{T}$ is unbounded and therefore does not satisfy the hypothesis of Theorem 2. Thus the sufficient condition in Theorem 2 is not necessary, and it remains an open problem to find a precise characterization of regions Ω for which $\mathscr{S}(\Omega)$ is minimal.

Theorem 1 is also significant in that it shows the circle to be qualitatively different from the line with respect to the Pólya-Schur problem, since $\mathscr{S}(\mathbb{C} \setminus \mathbb{T})$ is minimal and $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$ is not. Established non-constructive characterizations of Pólya-Schur semigroups, such as [3, Cor. 4], give no indication of this qualitative difference. By contrast, the explicit nature of Theorem 1 makes the structure of $\mathscr{S}(\mathbb{C} \setminus \mathbb{T})$ completely transparent, and it would be of interest to find a similarly explicit description of a set of generators of $\mathscr{S}(\mathbb{C} \setminus \mathbb{R})$.

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