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Fermionic Realization of Two-Parameter Quantum Affine Algebra $U_{r,s}(C_l^{(1)})$

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Abstract. We construct a Fock space representation and the action of the two-parameter quantum algebra $U_{r,s}(\mathfrak{gl}_\infty)$ using extended Young diagrams. In particular, we obtain an integrable representation of the two-parameter quantum affine algebra of type $C_n^{(1)}$ which is a two-parameter generalization of Kang-Misra-Miwa's realization.

Résumé. Nous construisons une représentation sur un espace de Fock de l'algèbre quantique à deux paramètres $U_{r,s}(\mathfrak{gl}_\infty)$ en utilisant les diagrammes de Young prolongés. En particulier, on obtient une représentation intégrable de l'algèbre quantique affine à deux paramètres de type $C_n^{(1)}$ qui est une généralization à deux paramètres de la réalization de Kang-Misra-Miwa.

1. **Introduction** Quantum groups, introduced independently by Drinfeld [6] and Jimbo [15], are deformations of the universal enveloping algebras of the Kac-Moody Lie algebras. Among the most important classes of quantum groups, quantum affine algebras have a rich representation theory and broad applications in mathematics and physics. In particular they are expected to provide the mathematical foundation for q-conformal field theory.

Two-parameter quantum groups associated to $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ were studied in [3–5] by Benkart and Witherspoon (see also earlier work by Takeuchi [21]). Other classical types and some exceptional types of two-parameter quantum groups and their representations have been investigated in [1, 2, 10] (see references therein). The two-parameter quantum affine algebras were introduced in [11] and their Drinfeld realization and vertex operator representations were also known with help of Lyndon bases for type A. More recently these structures have been generalized to all classical untwisted types in [12, 13], which are analogs of the basic representations of the quantum affine algebras [7]. The latter build upon a certain quantization of the so-called bosonic fields. From the other angle

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aimed toward a categorification, [17] provided a group-theoretic realization of two-parameter quantum toroidal algebras using finite subgroups of $SL_2(\mathbb{C})$ via the McKay correspondence.

It is well known that quantum affine algebras also admit fermionic realizations [9, 18–20] that have played an important role in integrable systems and representation theory. In [16] such a fermionic realization of two-parameter quantum affine algebra was constructed for type A using Young diagrams. The combinatorial model gives rise to a natural interpretation of the deforming parameters r and s. In this paper, we construct a fermionic realization of the two-parameter quantum affine algebra of type C along the same lines. We have followed a slightly different presentation from [16] to use the approach of Kang-Misra-Miwa [18]. We expect that this model will also work for other 2-parameter twisted quantum affine algebras.

2. The Fock Space of $U_{r,s}(gl(\infty))$ In this section, we first define the two-parameter quantum algebra $U_{r,s}(gl(\infty))$, and obtain an irreducible integrable representation using extended Young diagrams.

Let $\{\epsilon_i, | i \in \mathbb{Z}\}$ be an orthonormal basis of a Euclidean space E with an inner product $(\,,)$. Let $\{\alpha_i | i \in \mathbb{Z}\}$ be the simple roots of the affine Lie algebra $\mathfrak{g}$.

We assume that the ground field $\mathbb{K}$ is the field $\mathbb{Q}(r, s)$ of rational functions in r, s. Similar to the definition of $U_{r,s}(gl_n)$ (cf. [3]), we define $U_{r,s}(gl(\infty))$ as follows.

DEFINITION 2.1. Let $U_{r,s}(gl(\infty))$ be the unital associative algebra over $\mathbb{K}$ generated by the elements $e_i^\infty, f_i^\infty, \omega_i^\infty, {\omega'}_i^\infty$ for $i \in \mathbb{Z}$ satisfying the following defining relations:

(R1) $(\omega_i^\infty)^{\pm 1}, ({\omega'}_j^\infty)^{\pm 1}$ all commute with each another and

$$\omega_i^\infty(\omega_i^\infty)^{-1} = {\omega'}_j^\infty({\omega'}_j^\infty)^{-1} = 1,$$

(R2) $$\omega_i^\infty e_j^\infty = r^{(\varepsilon_i,\, \alpha_j)}\, e_j^\infty \omega_i^\infty \qquad \text{and} \qquad \omega_i^\infty f_j^\infty = r^{-(\varepsilon_i,\, \alpha_j)}\, f_j^\infty \omega_i^\infty,$$

(R3) $${\omega'}_i^\infty e_j^\infty = s^{(\varepsilon_i,\, \alpha_j)}\, e_j^\infty {\omega'}_i^\infty \qquad \text{and} \qquad {\omega'}_i^\infty f_j^\infty = s^{-(\varepsilon_i,\, \alpha_j)}\, f_j^\infty {\omega'}_i^\infty,$$

(R4) $$[\, e_i^\infty,\, f_j^\infty\,] = \frac{\delta_{ij}}{r-s}(\omega_i^\infty {\omega'}_{i+1}^\infty - \omega_{i+1}^\infty {\omega'}_i^\infty),$$

(R5) $$[\, e_i^\infty,\, e_j^\infty\,] = [\, f_i^\infty,\, f_j^\infty\,] = 0 \quad \text{if} \quad |i-j| > 1,$$

(R6) $$(e_i^\infty)^2 e_{i+1}^\infty - (r+s)\, e_i^\infty e_{i+1}^\infty e_i^\infty + rs\, e_{i+1}^\infty (e_i^\infty)^2 = 0,$$

$$e_i^\infty (e_{i+1}^\infty)^2 - (r+s)\, e_{i+1}^\infty e_i^\infty e_{i+1}^\infty + rs\, (e_{i+1}^\infty)^2 e_i^\infty = 0,$$

(R7) $$(f_i^\infty)^2 f_{i+1}^\infty - (r^{-1}+s^{-1})\, f_i^\infty f_{i+1}^\infty f_i^\infty + (rs)^{-1}\, f_{i+1}^\infty (f_i^\infty)^2 = 0,$$

$$f_i^\infty (f_{i+1}^\infty)^2 - (r^{-1}+s^{-1})\, f_{i+1}^\infty f_i^\infty f_{i+1}^\infty + (rs)^{-1} (f_{i+1}^\infty)^2 f_i^\infty = 0.$$

Now we construct a Fock space representation for the two-parameter quantum algebra $U_{r,s}(\mathfrak{gl}_\infty)$, which generalizes the fermionic representation of the usual quantum algebra given in [18].

We begin with the definition of extended Young diagram given in [14].

DEFINITION 2.2. An extended Young diagram Y is a sequence $(y_k)_{k\geq 0}$ such that

(i) $y_k \in \mathbb{Z}, \quad y_k \leq y_{k+1}$ for all k,
(ii) there exists a fixed integer y_∞ such that $y_k = y_\infty$ for $k \gg 0$.

The integer y_∞ is called the charge of Y.

Another way to identify an extended Young diagram is by specifying the fourth quadrant of the xy-plane with sites $\{(i,j) \in \mathbb{Z}\times\mathbb{Z} | i \geq 0,\, j \leq 0\}$. Thus an extended Young diagram $Y = (y_k)_{k\geq 0}$ is an infinite Young diagram drawn on the lattice in the right half plane with sites $\{(i,j) \in \mathbb{Z}\times\mathbb{Z} | i \geq 0,\, j \leq 0\}$, where y_k denotes the "depth" of the k-th column.

Note that if $y_k \neq y_{k+1}$ for some k, then we will have corners in the extended Young diagram $Y = (y_k)_{k\geq o}$. A corner is either "concave" or "convex". A corner located at site (i,j) is called a $d-$diagonal corner (or corner with diagonal number d), where $d = i + j$, more detail please see [14] and [18].

For any fixed integer n, let ϕ_n denote the "empty" diagram $(n, n, n, \cdots)$ of charge n. Let Y_n denote the set of all extended Young diagrams of charge n. The Fock space of charge n

$$\mathscr{F}_n = \bigoplus_{Y\in y_n} \mathbf{Q}(r,s)Y$$

denotes the $\mathbf{Q}(r,s)$-vector space having all $Y \in Y_n$ as base vectors.

The algebra $U_{r,s}(gl(\infty))$ acts on the Fock space as follows:

THEOREM 2.3. *$\mathscr{F}_n$ is an irreducible integrable $U_{r,s}(gl(\infty))$- module under the action defined as follows. For $Y \in y_n$,*

$$\begin{aligned}
e_i^\infty Y &= Y', && \text{if } Y \text{ has an } i\text{-diagonal convex corner then,}\\
&&& Y' \text{ is the same as } Y \text{ except that the } i\text{-diagonal}\\
&&& \text{convex corner is replaced by a concave corner,}\\
&= 0, && \text{otherwise;}\\
f_i^\infty Y &= Y'', && \text{if } Y \text{ has an } i\text{-diagonal concave corner then,}\\
&&& Y'' \text{ is the same as } Y \text{ except that the } i\text{-diagonal}\\
&&& \text{concave corner is replaced by a convex corner,}\\
&= 0, && \text{otherwise;}
\end{aligned}$$

$$\begin{aligned}
\omega_i^{\infty} Y &= s^{-1} Y, && \textit{if } Y \textit{ has an } i\textit{-diagonal concave corner },\\
&= r^{-1}, && \textit{if } Y \textit{ has an } i\textit{-diagonal convex corner },\\
&= Y, && \textit{otherwise};\\
\omega'^{\infty}_i Y &= rY, && \textit{if } Y \textit{ has an } i\textit{-diagonal concave corner },\\
&= s, && \textit{if } Y \textit{ has an } i\textit{-diagonal convex corner },\\
&= Y, && \textit{otherwise}.
\end{aligned}$$

PROOF. It is straight forward to verify the relations $(R1)-(R7)$ for the action on $\mathscr{F}_n$ for all generators. We remark that this is very much the same as in type A situation [16]. □

3. Fock Space Representations of $U_{r,s}(C_l^{(1)})$ Having constructed the Fock space representation of the two-parameter quantum affine algebra $U_{r,s}(gl(\infty))$, we can build the Fock space representation of $U_{r,s}(C_l^{(1)})$ by generalizing the well-known embedding of the latter inside $U_{r,s}(gl(\infty))$. First let us recall the definition of the two-parameter quantum affine algebra $U_{r,s}(C_l^{(1)})$ from [13].

Let $I_0 = \{0, 1, 2, \cdots, n\}$, and (a_{ij}), $i, j \in I_0$ be the Cartan matrix of type $C_l^{(1)}$. We take the normalization $(\alpha_0, \alpha_0) = (\alpha_l, \alpha_l) = 1$ and $(\alpha_i, \alpha_i) = \frac{1}{2}$ for $1 \le i \le l-1$. Let $r_i = r^{\frac{(\alpha_i,\alpha_i)}{2}}$ and $s_i = s^{\frac{(\alpha_i,\alpha_i)}{2}}$. Denote by c the canonical central element of $\mathfrak{g}(C_l^{(1)})$ and let δ_{ij} denote the Kronecker symbol.

DEFINITION 3.1. *The* two-parameter quantum affine algebra $U_{r,s}(\mathrm{C}_n^{(1)})$ *is the unital associative algebra over* $\mathbb{K}$ *generated by the elements* $e_j,\ f_j,\ \omega_j^{\pm 1},\ \omega_j'^{\pm 1}$ $(j \in I_0)$, $\gamma^{\pm\frac{1}{2}}$, $\gamma'^{\pm\frac{1}{2}}$, $D^{\pm 1}, D'^{\pm 1}$, *satisfying the following relations:*

$(\hat{C}1)$ $\gamma^{\pm\frac{1}{2}}, \gamma'^{\pm\frac{1}{2}}$ *are central with* $\gamma = \omega_\delta$, $\gamma' = \omega'_\delta$, $\gamma\gamma' = (rs)^c$, *such that* $\omega_i\,\omega_i^{-1} = \omega_i'\,\omega_i'^{-1} = 1 = DD^{-1} = D'D'^{-1}$, *and*

$$\begin{aligned}
[\,\omega_i^{\pm 1}, \omega_j^{\pm 1}\,] &= [\,\omega_i^{\pm 1}, D^{\pm 1}\,] = [\,\omega_j'^{\pm 1}, D^{\pm 1}\,] = [\,\omega_i^{\pm 1}, D'^{\pm 1}\,] = 0\\
&= [\,\omega_i^{\pm 1}, \omega_j'^{\pm 1}\,] = [\,\omega_j'^{\pm 1}, D'^{\pm 1}\,] = [D'^{\pm 1}, D^{\pm 1}] = [\,\omega_i'^{\pm 1}, \omega_j'^{\pm 1}\,].
\end{aligned}$$

$(\hat{C}2)$ *For* $0 \leqslant i, j \leqslant l$,

$$\begin{aligned}
D\, e_i\, D^{-1} &= r_i^{\delta_{0i}}\, e_i, & D\, f_i\, D^{-1} &= r_i^{-\delta_{0i}}\, f_i,\\
\omega_j e_i \omega_j^{-1} &= \langle i, j\rangle\, e_i, & \omega_j f_i \omega_j^{-1} &= \langle j, i\rangle^{-1}\, f_i.
\end{aligned}$$

$(\hat{C}3)$ *For* $0 \leqslant i, j \leqslant l$,

$$\begin{aligned}
D'\, e_i\, D'^{-1} &= s_i^{\delta_{0i}}\, e_i, & D'\, f_i\, D'^{-1} &= s_i^{-\delta_{0i}}\, f_i,\\
\omega_j' e_i \omega_j'^{-1} &= \langle i, j\rangle^{-1}\, e_i, & \omega_j' f_i \omega_j'^{-1} &= \langle j, i\rangle\, f_i.
\end{aligned}$$

($\hat{C}4$) *For* $0 \leqslant i,\, j \leqslant l$,

$$[\, e_i, f_j\,] = \frac{\delta_{ij}}{r_i - s_i}(\omega_i - \omega_i').$$

($\hat{C}5$) *For all* $1 \leqslant i \neq j \leqslant l$ *but* $(i,j) \notin \{(0,l),(l,0)\}$ *such that* $a_{ij} = 0$,

$$[e_i, e_j] = [f_i, f_j] = 0,$$

$$e_l e_0 = rs\, e_0 e_l, \qquad f_0 f_l = rs\, f_l f_0.$$

($\hat{C}6$) *For* $1 \leqslant i \leqslant l-2$, *the* (r,s)*-Serre relations for* $e_i's$:

$$\begin{gathered} e_0^2 e_1 - (r{+}s)\, e_0 e_1 e_0 + rs\, e_1 e_0^2 = 0,\\ e_i^2 e_{i+1} - (r_i{+}s_i)\, e_i e_{i+1} e_i + (r_i s_i)\, e_{i+1} e_i^2 = 0,\\ e_{i+1}^2 e_i - (r_{i+1}^{-1}{+}s_{i+1}^{-1})\, e_{i+1} e_i e_{i+1} + (r_{i+1}^{-1} s_{i+1}^{-1})\, e_i e_{i+1}^2 = 0,\\ e_l^2 e_{l-1} - (r^{-1}{+}s^{-1})\, e_l e_{l-1} e_l + (r^{-1}s^{-1})\, e_{l-1} e_l^2 = 0,\\ e_{l-1}^3 e_l - (r{+}(rs)^{\frac12}{+}s)\, e_{l-1}^2 e_l e_{l-1}\\ +(rs)^{\frac12}\, (r{+}(rs)^{\frac12}{+}s)\, e_{l-1} e_l e_{l-1}^2 - (rs)^{\frac32}\, e_l e_{l-1}^3 = 0\\ e_1^3 e_0 - (r^{-1}{+}(rs)^{-\frac12}{+}s^{-1})\, e_1^2 e_0 e_1\\ +(rs)^{-\frac12}\, (r^{-1}{+}(rs)^{-\frac12}{+}s^{-1})\, e_1 e_0 e_1^2 - (rs)^{-\frac32}\, e_0 e_1^3 = 0. \end{gathered}$$

($\hat{C}7$) *For* $1 \leqslant i \leqslant l-2$, *the* (r,s)*-Serre relations for* $f_i's$ *are obtained from* ($\hat{C}6$) *by replacing* e_i *for* f_i *and* r, s *by* r^{-1}, s^{-1} *respectively.*

In the above $\langle i,\, j\rangle$ *are the matrix entries of the two-parameter quantum Cartan matrix for type* $\mathrm{C}_l^{(1)}$:

$$\begin{pmatrix} rs^{-1} & r^{-1} & 1 & \cdots & 1 & rs\\ s & r^{\frac12}s^{-\frac12} & r^{-\frac12} & \cdots & 1 & 1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & 1 & 1 & \cdots & r^{\frac12}s^{-\frac12} & r^{-1}\\ (rs)^{-1} & 1 & 1 & \cdots & s & rs^{-1} \end{pmatrix}$$

We now describe the integrable representation of the two-parameter quantum affine algebra $U_{r,s}(C_l^{(1)})$. We start with the folding map

$$\pi: \qquad \{0,\, 1, \cdots, 2l-1\} \to \{0,\, 1, \cdots, l\}$$

where $\pi(0) = 0, \pi(l) = l$ and $\pi(i) = \pi(2l-i) = i$ for $1 \leq i \leq l-1$.Extend π to a map from $\mathbb{Z}$ into $\{0,\, 1, \cdots, l\}$ by periodicity $2l$.

For any $Y = (y_k)_{k\geq 0} \in Y_n$ define the operators:

$$t_k Y = r^a Y, \qquad t'_k = s^a Y$$

where $a = |\{p \in \mathbb{Z} | y_k < p \leq n \quad \text{and} \quad \pi(p+k) = 0\}|$ which depends on k.

As we still act on the space $\mathscr{F}_n$, we continue to use the same notation for the new Fock space representation. The following theorem is proved by direct verification (see [16]).

THEOREM 3.2. *For $k = 0, 1, \cdots, l$, the algebra $U_{r,s}(\mathrm{C}_l^{(1)})$ acts on $\mathscr{F}_n$ by the equations:*

$$e_i = \sum_{\substack{j \\ \pi(j)=i}} \Big(\prod_{\substack{k>j \\ \pi(k)=i}} \omega_k^{\infty} \Big)^{(\alpha_i,\, \alpha_i)} e_j^{\infty}, \tag{3.1}$$

$$f_i = \sum_{\substack{j \\ \pi(j)=i}} f_j^{\infty} \Big(\prod_{\substack{k<j \\ \pi(k)=i}} {\omega'}_k^{\infty} \Big)^{(\alpha_i,\, \alpha_i)}, \tag{3.2}$$

$$\omega_i = \Big(\prod_{\substack{j \\ \pi(j)=i}} \omega_j^{\infty} \Big)^{(\alpha_i,\, \alpha_i)}, \tag{3.3}$$

$$\omega'_i = \Big(\prod_{\substack{j \\ \pi(j)=i}} {\omega'}_j^{\infty} \Big)^{(\alpha_i,\, \alpha_i)}, \tag{3.4}$$

$$D = \prod_{k\geq 0} t_k, \qquad D' = \prod_{k\geq 0} t'_k. \tag{3.5}$$

Under the above action $\mathscr{F}_n$ is an integrable $U_{r,s}(\mathrm{C}_l^{(1)})$-module.

PROOF. We proceed in the same way. First we have

$$\omega'_j\, e_i\, {\omega'}_j^{-1} = \Big(\prod_{\substack{k \\ \pi(k)=j}} {\omega'}_k^{\infty} \Big)^{(\alpha_j,\, \alpha_j)} \sum_{\substack{j \\ \pi(j)=i}} \Big(\prod_{\substack{j'>j \\ \pi(j')=i}} \omega_{j'}^{\infty} \Big)^{(\alpha_i,\, \alpha_i)}$$
$$\times\, e_j^{\infty} \Big(\prod_{\substack{k \\ \pi(k)=j}} {\omega'}_k^{\infty} \Big)^{-(\alpha_j,\, \alpha_j)}$$

We do not have to prove anything for $|i-j| \geqslant 2$ due to ${\omega'}_i^{\infty} e_j^{\infty} = e_j^{\infty} {\omega'}_i^{\infty}$. For $i = j$, we have $e_m^{\infty} ({\omega'}_m^{\infty})^{-1} = r^{-1} s\, ({\omega'}_m^{\infty})^{-1} e_m^{\infty}$, which follows from $\langle i, i\rangle^{-1} = r^{-(\alpha_i,\, \alpha_i)} s^{(\alpha_i,\, \alpha_i)}$. For $0 \leqslant i = j-1 \leqslant l-1$, applying

$$e_m^{\infty} ({\omega'}_{m+1}^{\infty})^{-1} = s^{-1} ({\omega'}_{m+1}^{\infty})^{-1} e_m^{\infty}$$

and $\langle i+1,\, i\rangle^{-1} = s^{-(\alpha_{i+1},\,\alpha_{i+1})}$, we arrive at the required relation. Finally, when $1 \leqslant i = j+1 \leqslant l$, we have $e_m^\infty\,({\omega'}_{m-1}^{\infty})^{-1} = r\,({\omega'}_{m-1}^{\infty})^{-1}\,e_m^\infty$ and $\langle i-1,\, i\rangle^{-1} = r^{(\alpha_{i-1},\,\alpha_{i-1})}$, and this implies the conclusion.

For further reference, we need a few useful identities.

Lemma 3.3. *By direct calculations, we get the actions on* $\mathscr{F}_n$*,*

$$\begin{aligned} f_m^\infty\,(\omega_{m'}^\infty)^{-1} &= \langle m, m'\rangle_\infty^{-1}(\omega_{m'}^\infty)^{-1}\,f_m^\infty, \\ e_k^\infty\,(\omega_{m'}^\infty)^{-1} &= \langle k, m'\rangle_\infty(\omega_{m'}^\infty)^{-1}\,e_k^\infty, \\ f_m^\infty\,({\omega'}_{k'}^{\infty})^{-1} &= \langle m, k'\rangle_\infty({\omega'}_{k'}^{\infty})^{-1}\,f_m^\infty. \end{aligned}$$

where $\langle i,\, j\rangle_\infty$ *is defined as follows:*

$$\langle i,\, j\rangle_\infty = \begin{cases} rs^{-1}, & i=j; \\ r^{-1}, & i=j-1; \\ s, & i=j+1; \\ 1, & \text{otherwise.} \end{cases}$$

Now we turn to the relation $(\hat{C}4)$. From definition and Lemma 3.3, it follows that

$$\begin{aligned} & e_i f_j - f_j e_i \\ &= \sum_{\substack{k \\ \pi(k)=i}} \Big(\prod_{\substack{k'>k \\ \pi(k')=i}} \omega_{k'}^\infty\Big)^{(\alpha_i,\,\alpha_i)} e_k^\infty \sum_{\substack{m \\ \pi(m)=j}} f_m^\infty \Big(\prod_{\substack{m'<m \\ \pi(m')=j}} {\omega'}_{m'}^{\infty}\Big)^{(\alpha_j,\,\alpha_j)} \\ &\quad - \sum_{\substack{m \\ \pi(m)=j}} f_m^\infty \Big(\prod_{\substack{m'<m \\ \pi(m')=j}} {\omega'}_{m'}^{\infty}\Big)^{(\alpha_j,\,\alpha_j)} \sum_{\substack{k \\ \pi(k)=i}} \Big(\prod_{\substack{k'>k \\ \pi(k')=i}} \omega_{k'}^\infty\Big)^{(\alpha_i,\,\alpha_i)} e_k^\infty \\ &= \sum_{\substack{k,m \\ \pi(k)=i \\ \pi(m)=j}} \Big[\Big(\prod_{\substack{k'>k \\ \pi(k')=i}} \omega_{k'}^\infty\Big)^{(\alpha_i,\,\alpha_i)} e_k^\infty\, f_m^\infty \Big(\prod_{\substack{m'<m \\ \pi(m')=j}} {\omega'}_{m'}^{\infty}\Big)^{(\alpha_j,\,\alpha_j)} \\ &\quad - f_m^\infty \Big(\prod_{\substack{m'<m \\ \pi(m')=j}} {\omega'}_{m'}^{\infty}\Big)^{(\alpha_j,\,\alpha_j)} \Big(\prod_{\substack{k'>k \\ \pi(k')=i}} \omega_{k'}^\infty\Big)^{(\alpha_i,\,\alpha_i)} e_k^\infty\Big] \end{aligned}$$

$$= \sum_{\substack{k>m\\ \pi(k)=i\\ \pi(m)=j}} \Big(\prod_{\substack{k'>k\\ \pi(k')=i}} \omega^{\infty}_{k'}\Big)^{(\alpha_i,\,\alpha_i)}\Big(\prod_{\substack{m'<m\\ \pi(m')=j}} {\omega'}^{\infty}_{m'}\Big)^{(\alpha_j,\,\alpha_j)} (e^{\infty}_k\, f^{\infty}_m - f^{\infty}_m\, e^{\infty}_k)$$

$$+\delta_{i,j} \sum_{\substack{k\\ \pi(k)=i}} \Big(\prod_{\substack{k'>k\\ \pi(k')=i}} \omega^{\infty}_{k'}\Big)^{(\alpha_i,\,\alpha_i)}\Big(\prod_{\substack{k'<k\\ \pi(k')=j}} {\omega'}^{\infty}_{k'}\Big)^{(\alpha_j,\,\alpha_j)} (e^{\infty}_k\, f^{\infty}_k - f^{\infty}_k\, e^{\infty}_k)$$

$$+ \sum_{\substack{k<m\\ \pi(k)=i\\ \pi(m)=j}} \Big(\prod_{\substack{k'>k\\ \pi(k')=i}} \omega^{\infty}_{k'}\Big)^{(\alpha_i,\,\alpha_i)}\Big(\prod_{\substack{m'<m\\ \pi(m')=j}} {\omega'}^{\infty}_{m'}\Big)^{(\alpha_j,\,\alpha_j)}$$

$$\Big(\sum_{m'<m} \langle k, m'\rangle^{(\alpha_j,\,\alpha_j)}_{\infty} e^{\infty}_k\, f^{\infty}_m - \sum_{k'>k} \langle m, k'\rangle^{(\alpha_i,\,\alpha_i)}_{\infty} f^{\infty}_m\, e^{\infty}_k\Big)$$

Note that if $m = k+1$, then we have $e^{\infty}_k\, f^{\infty}_m = 0 = f^{\infty}_m\, e^{\infty}_k$, and if $m > k+1$, then

$$\sum_{\substack{m'<m\\ \pi(k)=i\\ \pi(m)=j=\pi(m')}} \langle k, m'\rangle^{(\alpha_j,\,\alpha_j)}_{\infty} = \sum_{\substack{k'>k\\ \pi(m)=j\\ \pi(k)=i=\pi(k')}} \langle m, k'\rangle^{(\alpha_i,\,\alpha_i)}_{\infty} \tag{3.6}$$

On $\mathscr{F}_n$, it is clear that

$$e^{\infty}_k\, f^{\infty}_m - f^{\infty}_m\, e^{\infty}_k = \delta_{k,m}\Big(\frac{(\omega^{\infty}_k)^{(\alpha_i,\,\alpha_i)} - {\omega'}^{\infty}_k)^{(\alpha_i,\,\alpha_i)}}{r_i - s_i}\Big) \tag{3.7}$$

Consequently, it follows that on $\mathscr{F}_n$,

$$\begin{aligned} & e_i f_j - f_j e_i \\ = \ & \delta_{i,j}(r_i - s_i)^{-1} \sum_{\substack{k\\ \pi(k)=i}} \Big\{\Big(\prod_{k'\geqslant k} \omega^{\infty}_{k'}\Big)^{(\alpha_i,\,\alpha_i)}\Big(\prod_{k'<k} {\omega'}^{\infty}_{k'}\Big)^{(\alpha_i,\,\alpha_i)} \\ & -\Big(\prod_{k'>k} \omega^{\infty}_{k'}\Big)^{(\alpha_i,\,\alpha_i)}\Big(\prod_{k'\leqslant k} {\omega'}^{\infty}_{k'}\Big)^{(\alpha_i,\,\alpha_i)}\Big\} \\ = \ & \delta_{i,j}(r_i - s_i)^{-1}\Big\{\Big(\prod_{\substack{k\\ \pi(k)=i}} \omega^{\infty}_{k}\Big)^{(\alpha_i,\,\alpha_i)} - \Big(\prod_{\substack{k\\ \pi(k)=i}} {\omega'}^{\infty}_{k}\Big)^{(\alpha_i,\,\alpha_i)}\Big\} \\ = \ & \delta_{i,j}\, \frac{\omega_i - {\omega'}_i}{r_i - s_i}. \end{aligned}$$

It is straightforward to check the relation ($\hat{C}5$),

$$\begin{aligned}
e_l e_0 &= \sum_{\substack{k\\ \pi(k)=l}} \Big(\prod_{\substack{k'>k\\ \pi(k')=l}} \omega_{k'}^{\infty}\Big)^{(\alpha_l,\,\alpha_l)} e_k^{\infty} \sum_{\substack{m\\ \pi(m)=0}} \Big(\prod_{\substack{m'>m\\ \pi(m')=0}} \omega_{m'}^{\infty}\Big)^{(\alpha_0,\,\alpha_0)} e_m^{\infty}\\
&= r \sum_{\substack{k,m\\ \pi(k)=l\\ \pi(m)=0}} \Big(\prod_{\substack{k'>k\\ \pi(k')=l}} \omega_{k'}^{\infty}\Big)^{(\alpha_l,\,\alpha_l)} \Big(\prod_{\substack{m'>m\\ \pi(m')=0}} \omega_{m'}^{\infty}\Big)^{(\alpha_0,\,\alpha_0)} e_k^{\infty} e_m^{\infty}\\
&= rs \sum_{\substack{k,m\\ \pi(k)=l\\ \pi(m)=0}} \Big(\prod_{\substack{m'>m\\ \pi(m')=0}} \omega_{m'}^{\infty}\Big)^{(\alpha_0,\,\alpha_0)} e_m^{\infty} \Big(\prod_{\substack{k'>k\\ \pi(k')=l}} \omega_{k'}^{\infty}\Big)^{(\alpha_l,\,\alpha_l)} e_k^{\infty}\\
&= rs\, e_0 e_l.
\end{aligned}$$

The others relations can be proved similarly.

The last task is to verify the Serre relations ($\hat{C}6$) and ($\hat{C}7$). Here we only check the relation ($\hat{C}6$) as the other relations are proved exactly in the same way.

To show the Serre relations ($\hat{C}6$), let us begin with the following notation to save space.

$$p_j = \prod_{\substack{j'>j\\ \pi(j')=\pi(j)}} \omega_{j'}^{\infty} \tag{3.8}$$

$$p'_j = \prod_{\substack{j'>j\\ \pi(j')=\pi(j)}} {\omega'}_{j'}^{\infty}. \tag{3.9}$$

Let us write $i \gg j$ if $i - j > 2$. The following lemmas can be checked directly.

LEMMA 3.4. *For all j and k, on $\mathscr{F}_n$ then we obtain,*

$$e_k^{\infty}\, e_k^{\infty} = 0, \tag{3.10}$$

$$e_k^{\infty}\, e_j^{\infty}\, e_k^{\infty} = 0. \tag{3.11}$$

LEMMA 3.5. *If $\pi(k) = 0 = \pi(j)$, then it holds*

$$e_j^{\infty}\, p_k = \begin{cases} p_k\, e_j^{\infty}, & \text{for } j \leqslant k;\\ r^{-1}s\, p_k\, e_j^{\infty}, & \text{for } j > k. \end{cases} \tag{3.12}$$

$$e_j^{\infty}\, p'_k = \begin{cases} p'_k\, e_j^{\infty}, & \text{for } j \leqslant k;\\ rs^{-1}\, p'_k\, e_j^{\infty}, & \text{for } j > k. \end{cases} \tag{3.13}$$

LEMMA 3.6. *If $\pi(j)=0, \pi(k)=1$, then it follows that*

$$e_j^\infty\, p_k = \begin{cases} p_k\, e_j^\infty, & \text{for } j \leqslant k; \\ r\, p_k\, e_j^\infty, & \text{for } j = k+1; \\ r^2\, p_k\, e_j^\infty, & \text{for } j \gg k. \end{cases} \tag{3.14}$$

$$e_j^\infty\, p'_k = \begin{cases} p'_k\, e_j^\infty, & \text{for } j \leqslant k; \\ s\, p'_k\, e_j^\infty, & \text{for } j = k+1; \\ s^2\, p'_k\, e_j^\infty, & \text{for } j \gg k. \end{cases} \tag{3.15}$$

LEMMA 3.7. *If $\pi(j)=1, \pi(k)=0$, then we have*

$$e_j^\infty\, p_k = \begin{cases} p_k\, e_j^\infty, & \text{for } j \leqslant k \text{ or } j = k+1; \\ s^{-1}\, p_k\, e_j^\infty, & \text{for } j \gg k. \end{cases} \tag{3.16}$$

$$e_j^\infty\, p'_k = \begin{cases} p'_k\, e_j^\infty, & \text{for } j \leqslant k \text{ or } j = k+1; \\ r^{-1}\, p'_k\, e_j^\infty, & \text{for } j \gg k. \end{cases} \tag{3.17}$$

LEMMA 3.8. *If $\pi(j)=1=\pi(k)$, it is easy to see that*

$$e_j^\infty\, p_k = \begin{cases} p_k\, e_j^\infty, & \text{for } j \leqslant k; \\ r^{-1}s\, p_k\, e_j^\infty, & \text{for } j > k. \end{cases} \tag{3.18}$$

$$e_j^\infty\, p'_k = \begin{cases} p'_k\, e_j^\infty, & \text{for } j \leqslant k; \\ rs^{-1}\, p'_k\, e_j^\infty, & \text{for } j > k. \end{cases} \tag{3.19}$$

We now prove the following Serre relation:

$$e_0^2 e_1 + (r+s) e_0 e_1 e_0 + rs e_1 e_0^2 = 0 \tag{3.20}$$

We first use definition to simply the left hand side (LHS) of (3.20).

$$
\begin{aligned}
&LHS\\
=&\sum_{\substack{j,k,m\\ \pi(j)=0=\pi(k)\\ \pi(m)=1}}\Big[\Big(\prod_{\substack{j'>j\\ \pi(j')=0}}\omega_{k'}^{\infty}\Big)e_j^{\infty}\Big(\prod_{\substack{k'>k\\ \pi(k')=0}}\omega_{k'}^{\infty}\Big)e_k^{\infty}\Big(\prod_{\substack{m'>m\\ \pi(m')=1}}\omega_{m'}^{\infty}\Big)^{\frac{1}{2}}e_m^{\infty}\\
&-(r+s)\Big(\prod_{\substack{j'>j\\ \pi(j')=0}}\omega_{k'}^{\infty}\Big)e_j^{\infty}\Big(\prod_{\substack{m'>m\\ \pi(m')=1}}\omega_{m'}^{\infty}\Big)^{\frac{1}{2}}e_m^{\infty}\Big(\prod_{\substack{k'>k\\ \pi(k')=0}}\omega_{k'}^{\infty}\Big)e_k^{\infty}\\
&+\Big(\prod_{\substack{m'>m\\ \pi(m')=1}}\omega_{m'}^{\infty}\Big)^{\frac{1}{2}}e_m^{\infty}\Big(\prod_{\substack{j'>j\\ \pi(j')=0}}\omega_{k'}^{\infty}\Big)e_j^{\infty}\Big(\prod_{\substack{k'>k\\ \pi(k')=0}}\omega_{k'}^{\infty}\Big)e_k^{\infty}\Big]\\
=&\Big(\sum_{m\gg j>k}+\sum_{m=j+1\gg k}+\sum_{j\gg m\gg k}+\sum_{j\gg m=k+1}+\sum_{j=m+1>k}+\sum_{j>k\gg m}+\sum_{j>k=m+1}\Big)\\
&\times\{p_je_j^{\infty}\,p_ke_k^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}+p_ke_k^{\infty}\,p_je_j^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}\\
&-(r+s)(p_je_j^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}\,p_ke_k^{\infty}+p_ke_k^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}\,p_me_m^{\infty})\\
&+(rs)(p_m^{\frac{1}{2}}e_m^{\infty}\,p_je_j^{\infty}\,p_ke_k^{\infty}+p_m^{\frac{1}{2}}e_m^{\infty}\,p_ke_k^{\infty}\,p_je_j^{\infty})\}.
\end{aligned}
$$

Using Lemma 3.4 through Lemma 3.8, we would like to show that each summand is actually 0. Taking the second summand for example, we get immediately,

$$
\begin{aligned}
&\sum_{m=j+1\gg k}\{p_je_j^{\infty}\,p_ke_k^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}+p_ke_k^{\infty}\,p_je_j^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}\\
&\qquad-(r+s)(p_je_j^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}\,p_ke_k^{\infty}+p_ke_k^{\infty}\,p_m^{\frac{1}{2}}e_m^{\infty}\,p_je_j^{\infty})\\
&\qquad+(rs)(p_m^{\frac{1}{2}}e_m^{\infty}\,p_je_j^{\infty}\,p_ke_k^{\infty}+p_m^{\frac{1}{2}}e_m^{\infty}\,p_ke_k^{\infty}\,p_je_j^{\infty})\}\\
=&\sum_{m=j+1>k}\{(r^{-1}s+1-r^{-1}(r+s))p_jp_kp_m^{\frac{1}{2}}e_j^{\infty}e_m^{\infty}e_k^{\infty}\\
&\qquad+(-(r+s)s^{-1}+(rs)(r^{-1}s^{-1}+s^{-2}))p_jp_kp_m^{\frac{1}{2}}e_m^{\infty}e_j^{\infty}e_k^{\infty}\}\\
=&\ 0.
\end{aligned}
$$

The other summands are seen as zero by the same method. Consequently Relation (3.20) has been verified.

Next we turn to the relation

$$
\begin{aligned}
&e_1^3e_0-(r^{-1}+(rs)^{-\frac12}+s^{-1})\,e_1^2e_0e_1+(rs)^{-\frac12}\\
&\times(r^{-1}+(rs)^{-\frac12}+s^{-1})\,e_1e_0e_1^2-(rs)^{-\frac32}\,e_0e_1^3=0.
\end{aligned}\tag{3.21}
$$

Note that by definition, the left hand side (LHS) of (3.21) is equal to

$$
\begin{aligned}
LHS \;=\; &\sum_{\substack{i,j,k,m\\ \pi(i)=\pi(j)=\pi(k)=1\\ \pi(m)=0}}\Big[p_i^{\frac12}e_i^\infty\,p_j^{\frac12}e_j^\infty\,p_k^{\frac12}e_k^\infty\,p_me_m^\infty\\
&-(r^{-1}+(rs)^{-\frac12}+s^{-1})\,p_i^{\frac12}e_i^\infty\,p_j^{\frac12}e_j^\infty\,p_me_m^\infty\,p_k^{\frac12}e_k^\infty\\
&+(rs)^{-\frac12}\,(r^{-1}+(rs)^{-\frac12}+s^{-1})\,p_i^{\frac12}e_i^\infty\,p_me_m^\infty\,p_j^{\frac12}e_j^\infty\,p_k^{\frac12}e_k^\infty\\
&-(rs)^{-\frac32}\,p_me_m^\infty\,p_i^{\frac12}e_i^\infty\,p_j^{\frac12}e_j^\infty\,p_k^{\frac12}e_k^\infty\Big]
\end{aligned}
$$

Applying Lemma 3.4 through Lemma 3.8, the last relation becomes,

$$
\begin{aligned}
LHS \;=\; &\Big(\sum_{m\gg j>k>i}+\sum_{m=j+1>k>i}+\sum_{j\gg m\gg k\gg i}+\sum_{j\gg m=k+1>i}\\
&+\sum_{j=m+1\gg k>i}+\sum_{j>k\gg m\gg i}+\sum_{j>k\gg m=i+1}+\sum_{j>k=m+1>i}\\
&+\sum_{j>k\gg i\gg m}+\sum_{j>k>i=m+1}\Big)\{\Big(p_i^{\frac12}e_i^\infty\,p_j^{\frac12}e_j^\infty\,p_k^{\frac12}e_k^\infty\,p_me_m^\infty\\
&+p_i^{\frac12}e_i^\infty\,p_k^{\frac12}e_k^\infty\,p_j^{\frac12}e_j^\infty\,p_me_m^\infty+p_j^{\frac12}e_j^\infty\,p_k^{\frac12}e_k^\infty\,p_i^{\frac12}e_i^\infty\,p_me_m^\infty\\
&+p_j^{\frac12}e_j^\infty\,p_i^{\frac12}e_i^\infty\,p_k^{\frac12}e_k^\infty\,p_me_m^\infty+p_k^{\frac12}e_k^\infty\,p_i^{\frac12}e_i^\infty\,p_j^{\frac12}e_j^\infty\,p_me_m^\infty\\
&+p_k^{\frac12}e_k^\infty\,p_j^{\frac12}e_j^\infty\,p_i^{\frac12}e_i^\infty\,p_me_m^\infty\Big)-(r^{-1}+(rs)^{-\frac12}+s^{-1})\\
&\Big(p_i^{\frac12}e_i^\infty\,p_j^{\frac12}e_j^\infty\,p_me_m^\infty\,p_k^{\frac12}e_k^\infty+p_i^{\frac12}e_i^\infty\,p_k^{\frac12}e_k^\infty\,p_me_m^\infty\,p_j^{\frac12}e_j^\infty\\
&+p_j^{\frac12}e_j^\infty\,p_k^{\frac12}e_k^\infty\,p_me_m^\infty\,p_i^{\frac12}e_i^\infty+p_j^{\frac12}e_j^\infty\,p_i^{\frac12}e_i^\infty\,p_me_m^\infty\,p_k^{\frac12}e_k^\infty\\
&+p_k^{\frac12}e_k^\infty\,p_i^{\frac12}e_i^\infty\,p_me_m^\infty\,p_j^{\frac12}e_j^\infty+p_k^{\frac12}e_k^\infty\,p_j^{\frac12}e_j^\infty\,p_me_m^\infty\,p_i^{\frac12}e_i^\infty\Big)\\
&+(rs)^{-\frac12}\,(r^{-1}+(rs)^{-\frac12}+s^{-1})\,\Big(p_i^{\frac12}e_i^\infty\,p_me_m^\infty\,p_j^{\frac12}e_j^\infty\,p_k^{\frac12}e_k^\infty
\end{aligned}
$$

$$
\begin{aligned}
&+p_i^{\frac{1}{2}} e_i^{\infty}\, p_m e_m^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty} + p_j^{\frac{1}{2}} e_j^{\infty}\, p_m e_m^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty} \\
&+p_j^{\frac{1}{2}} e_j^{\infty}\, p_m e_m^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty} + p_k^{\frac{1}{2}} e_k^{\infty}\, p_m e_m^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty} \\
&+p_k^{\frac{1}{2}} e_k^{\infty}\, p_m e_m^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\Big) - (rs)^{-\frac{3}{2}} \Big(p_m e_m^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty} \\
&+p_m e_m^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty} + p_m e_m^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty} \\
&+p_m e_m^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty} + p_m e_m^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty} \\
&+p_m e_m^{\infty}\, p_k^{\frac{1}{2}} e_k^{\infty}\, p_j^{\frac{1}{2}} e_j^{\infty}\, p_i^{\frac{1}{2}} e_i^{\infty}\Big)\Big\} = 0.
\end{aligned}
$$

Every summand of the last relation can be shown to be 0 as before.

Finally we check that the Serre relation involving i and $i+1$ holds in the Fock space. For $1 \leqslant i \leqslant l-2$, we compute that

$$
\begin{aligned}
&e_i^2 e_{i+1} - (r_i+s_i)\, e_i e_{i+1} e_i + (r_i s_i)\, e_{i+1} e_i^2 \\
=& \sum_{\substack{j,k,m \\ \pi(j)=i=\pi(k) \\ \pi(m)=i+1}} \{p_j e_j^{\infty}\, p_k e_k^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty} + p_k e_k^{\infty}\, p_j e_j^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty} \\
& -(r_i+s_i)(p_j e_j^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty}\, p_k e_k^{\infty} + p_k e_k^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty}\, p_j e_j^{\infty}) \\
& +(r_i s_i)(p_m^{\frac{1}{2}} e_m^{\infty}\, p_j e_j^{\infty}\, p_k e_k^{\infty} + p_m^{\frac{1}{2}} e_m^{\infty}\, p_k e_k^{\infty}\, p_j e_j^{\infty})\} \\
=& \Big(\sum_{m\gg j>k} + \sum_{m=j+1\gg k} + \sum_{j\gg m\gg k} + \sum_{j\gg m=k+1} + \sum_{j=m+1>k} \\
& + \sum_{j>k\gg m} + \sum_{j>k=m+1}\Big)\{p_j e_j^{\infty}\, p_k e_k^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty} + p_k e_k^{\infty}\, p_j e_j^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty} \\
& -(r_i+s_i)(p_j e_j^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty}\, p_k e_k^{\infty} + p_k e_k^{\infty}\, p_m^{\frac{1}{2}} e_m^{\infty}\, p_j e_j^{\infty}) \\
& +(r_i s_i)(p_m^{\frac{1}{2}} e_m^{\infty}\, p_j e_j^{\infty}\, p_k e_k^{\infty} + p_m^{\frac{1}{2}} e_m^{\infty}\, p_k e_k^{\infty}\, p_j e_j^{\infty})\} = 0.
\end{aligned}
$$

Therefore we have finished the proof of Theorem 3.2. □

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