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RADIAL DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS AND SECTIONS OF THEIR POWER SERIES

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ABSTRACT. For an entire function f with non-negative Maclaurin coefficients, a region is obtained which is defined in terms of Hayman's function $b(r) = r(rf'(r)/f(r))'$, and which is free of all zeros of f and those of all its sections. The new region defined improves on previous results. In particular, it is shown that when $\limsup_{n\to\infty} b(r) = A^2/4$, $A > 0$, then the zeros $r_n \exp(i\theta_n)$ of f satisfy the inequality, $\liminf_{n\to\infty} |\theta_n| \geq 4\sin^{-1}(1/A\sqrt{2})$, which is very close to being optimal.

RÉSUMÉ. Etant donnée une function entière f avec des coéfficients positifs, on trouve une région définie en termes de la fonction $b(r) = r(rf'(r)/f(r))'$ de Hayman, dépourvue des zéros de f et de ceux de toutes ses sections. Particulièrement, on démontre qu'au cas où $\limsup_{n\to\infty} b(r) = A^2/4$, $A > 0$, les zéros $r_n \exp(i\theta_n)$ de f satisfont l'inégalité $\liminf_{n\to\infty} |\theta_n| \geq 4\sin^{-1}(1/A\sqrt{2})$, qui est presque optimale.

1. Introduction.

Let f be an entire function. Following Hayman [3], put

$$b(r) = b(r; f) = ra'(r), \quad a(r) = a(r; f) = \frac{d\log M(r)}{d\log r},$$

where $M(r) = M(r; f) = \sup_{|z|=r} |f(z)|$ is the maximum modulus of f.

If f is represented by its Maclaurin series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then the sections and tails of f are given, respectively, by $s_n(z) = \sum_{k=0}^{n} a_k z^k$, $t_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$,and a question of great interest is to find best possible results connecting the location of zeros of f or s_n or t_n, with the growth of f.

In one direction, there are numerous results starting from information about location of zeros of the sections s_n and obtaining information about the growth of f as measured by that of $\log M(r)$. See [4], [8], [7], and the references thereof. Recently [8], some interesting new results on the growth of f were obtained starting from information about the zeros of the tails t_n. In the other direction, recent work has revealed interesting connections between the growth of $b(r)$ and quite a few important properties of f. Thus, $b(r)$ is connected with the gap structure of the Maclaurin series of f [6], the geometry of zeros of sections of f [1], the location of zeros of f, ratios of successive zeros, and the convexity of its coefficients [2]. A particularly elegant result is the following [2]:

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THEOREM A. *If f is entire with positive Maclaurin coefficients and $\limsup_{r\to\infty} b(r) < 0.5$, then all but a finite number of zeros of f are simple, real, and negative. Furthermore, the constant 0.5 is best possible.*

It is rather remarkable that such precise information about the location of zeros can be obtained from an asymptotic relation. For example, the function f may be given by an integral, or by some precise information about its coefficients. If it is possible to derive from these an asymptotic expansion of $\log f(r)$, then careful (since asymptotic expansions cannot, in general, be differentiated) use of analyticity could give an expansion of $b(r)$ from which information about the radial distribution of the zeros may be obtained. If, instead, we have pointwise knowledge of $b(r)$, then we have the following very general result [1]:

THEOREM B. *Let f be an entire function with positive coefficients a_k and put $b_k = a_{k-1}/a_k$. Let G be the region in the plane given in polar coordinates by*

$$G = \left\{ re^{i\theta} : b(r) < \frac{1}{2(1-\cos\theta)} \right\}.$$

Then G is free of all zeros of f.

If $b_k \geq b_{k-1}$, then G is free of all zeros of all sections s_n.

For example, if $f(z) = e^z$, then $b_k = k, b(r) = r$, and $G = \{re^{i\theta} : 2r < 2r\cos\theta + 1\}$. Thus, the parabolic region $\{(x, y) : y^2 < x + 1/4\}$ is free of all zeros of all partial sums of the exponential function. See [7] for a sharper result on the location of zeros of sections of the exponential function.

If, for example, $b(r) < Kr^\lambda$, $0 < \lambda < 1$, then it follows from Theorem B that the region $\{x + iy : |y| < 1/kx^{1-\lambda/2}, x > 0\}$ is free of all zeros of f.

The region G contains a sector for bounded $b(r)$, in which case f is of order zero. It contains a parabolic region if $b(r)$ grows like r, in which case f is of exponential type. As the growth of $b(r)$ increases, the region G closes towards the positive real axis, although it never collapses completely.

The purpose of this paper is to improve on Theorem B by obtaining a region larger than G and free of zeros of f and its sections. As a corollary, we obtain an extension of Theorem A for the case when $\limsup_{r\to\infty} b(r) < \infty$.

2. Zero-free regions.

Throughout this section we retain all the notation in the introduction. Our improvement of Theorem B is given in the following:

THEOREM 1. *Let f be an entire function with positive coefficients a_k, and let $b_k = a_{k-1}/a_k$. Let E be the region in the plane given in polar coordinates by*

$$E = \left\{ re^{i\theta} : b(r) < \frac{1}{4(1-\cos(\theta/2))} \right\}.$$

Then E is free of all zeros of f.

Also, if $b_k \geq b_{k-1}$ then E is free of all zeros of all sections s_n.

Before proceeding with the proof let us note first that if $re^{i\theta} \in G$ then $re^{i\theta} \in E$, since

$$b(r) < \frac{1}{2(1-\cos\theta)} \leq \frac{1+\cos(\theta/2)}{4\sin^2(\theta/2)} = \frac{1}{4(1-\cos(\theta/2))}.$$

Thus E is a larger zero-free region.

Proof. The positivity of the Maclaurin coefficients implies that $\log M(r;f) = \log f(r)$ for $r > 0$, and this makes it possible to compute $a(r)$ and $b(r)$ explicitly. In particular, if n is a non-negative integer, then direct calculation with the relation $f(r) = \exp(M(r))$ yields [1]

$$\sum_{k=0}^{\infty}(k-n)^2 a_k r^k = \{b(r) + (a(r)-n)^2\} f(r).$$

Using this formula together with the inequality $|\sin((k-n)\theta/2)| \leq |k-n|\,|\sin(\theta/2)|$, we get

$$\begin{aligned} f(r) - \operatorname{Re}\{e^{-in\theta} f(re^{i\theta})\} &= \sum_{k=0}^{\infty} 2a_k r^k \sin^2((k-n)\theta/2) \\ &\leq 2\sin^2(\theta/2)\{b(r) + (a(r)-n)^2\} f(r). \end{aligned} \tag{2.1}$$

Now using the fact that

$$\Big(\frac{d}{d\log r} - n - 1\Big)\Big(\frac{d}{d\log r} - n\Big) f(r) = \{b(r) + (a(r-n))(a(r)-n-1)\} f(r)$$

together with the identity $\cos A - \cos B = -2\sin(\frac{A-B}{2})\sin(\frac{A+B}{2})$, and the fact that $m(m-1)$ is non-negative for all integers m, we obtain the following inequalities:

$$\begin{aligned} &|\cos(\theta/2) f(r) - \operatorname{Re}\{e^{-i(n+1/2)\theta} f(re^{i\theta})\}| \\ &\quad \leq 2\sin^2(\theta/2)\{b(r) + (a(r)-n)(a(r)-n-1)\} f(r), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &|\cos(\theta/2) f(r) - \operatorname{Re}\{e^{-i(n-1/2)\theta} f(re^{i\theta})\}| \\ &\quad \leq 2\sin^2(\theta/2)\{b(r) + (a(r)-n)(a(r)-n+1)\} f(r). \end{aligned} \tag{2.3}$$

If we add the inequalities (2.1) and (2.2) and then the inequalities (2.1) and (2.3) we obtain respectively,

$$\begin{aligned} &|(1+\cos(\theta/2)) f(r) - \operatorname{Re}\{(e^{-in\theta} + e^{-i(n+1/2)\theta}) f(re^{i\theta})\}| \\ &\quad \leq 4\sin^2(\theta/2)\{b(r) + (a(r)-n)(a(r)-n-1/2)\} f(r), \end{aligned} \tag{2.4}$$

and again,

$$(2.5) \qquad (1+\cos(\theta/2))f(r) - \mathrm{Re}\{(e^{-in\theta} + e^{-i(n-1/2)\theta})f(re^{i\theta})\}|$$
$$\leq 4\sin^2(\theta/2)\{b(r) + (a(r)-n)(a(r)-n+1/2)\}f(r).$$

Now let $re^{i\theta}$, $r > 0$, $\theta \in [-\pi, \pi]$ be a zero of f. Then $a(r) > 0$, and the greatest integer $[a(r)]$ in $a(r)$ is non-negative. If $[a(r)] \leq a(r) \leq [a(r)] + 1/2$, we choose $n = [a(r)]$ and use (2.4) to obtain

$$(1+\cos(\theta/2)) \leq 4\sin^2(\theta/2)\{b(r) + (a(r)-n)(a(r)-n-1/2) \leq 4b(r)\sin^2(\theta/2).$$

If $[a(r)] + \frac{1}{2} \leq a(r) < a(r) + 1$, we choose $n = [a(r)] + 1$ and use (2.5). It follows that, in all cases, any zero $re^{i\theta}$ of f satisfies the inequality $(1 + \cos(\theta/2)) \leq 4b(r)\sin^2(\theta/2)$ and so lies in the complement of the set E. Hence E is free of all zeros of f. This finishes the first part of Theorem 1. To prove the result on the zeros of the sections of f, let s_p be a section of f with $p \geq 1$, and let $re^{i\theta}$ be a zero of s_p. It follows by the first part of Theorem 1, that

$$(1+\cos(\theta/2)) \leq 4b(r; s_p)\sin^2(\theta/2).$$

The proof is now completed by using the inequality from [2] that $b(r, s_p) \leq b(r; f)$, which is true because $b_{k-1} \leq b_k$.

Corollary 1. *Let f be an entire function with positive Maclaurin coefficients and suppose that*

$$\limsup_{r\to\infty} b(r; f) = \frac{A^2}{4}, \quad A > 0.$$

If $r_n e^{i\theta_n}$, $0 < r_n \leq r_{n+1}$, $\theta_n \in [-\pi, \pi]$, are the zeros of f, then

$$\liminf_{n\to\infty} |\theta_n| \geq 4\sin^{-1}(1/\sqrt{2}A) \geq \frac{2\sqrt{2}}{A}. \qquad \blacksquare$$

When $A = 1$, the corollary gives $\liminf_{n\to\infty} |\theta_n| = \pi$, so that in fact $\liminf_{n\to\infty} |\theta_n| = \pi$, which is sharp. But Theorem A goes far beyond this. The best possible result for general values of A appears to be $\frac{\pi}{A}$, but our corollary falls short of this, giving (roughly) $\frac{0.9\pi}{A}$, and suggesting that the best possible result is that $\liminf_{n\to\infty} |\theta_n| \geq \frac{\pi}{A}$. Compare with a question of Goldberg and Ostrovskii [5].

3. Integer values of the expectation.

For $r > 0$, let X be a random variable whose distribution function is given by $P(X = n) = a_n r^n / f(r)$. Then the expectation and variance are given by [9]

$$E(X_r) = a(r; f), \quad V(X_r) = b(r; f).$$

It is known that, since $a_k \geq 0$, $\liminf_{n\to\infty} b(r_n; f) \geq \frac{1}{4}$ and this result is sharp. We shall show here that, if the expectation is an integer, then the variance must be at least $\frac{1}{2}$.

THEOREM 2. *Let $r_n \exp(i\theta_n)$, $|\theta_n| \le \pi$, be the sequence of zeros of f. If $a(r_n;f) = m$, an integer, then $b(r_n;f) \ge \frac{1}{2}$.*

PROOF. Suppose that $a(r_n;f) = m$, a positive integer, and let p be a positive integer. We use the inequality (2.1) in Theorem 1, but replace the integer n by m. Multiplying (2.1) by p and adding it to (2.2) we obtain:

$$\begin{aligned}&|(p+\cos(\theta/2))f(r) - \mathrm{Re}\{(pe^{-im\theta} + e^{-i(m+1/2)\theta})f(re^{i\theta})\}| \\ &\qquad \le 2(p+1)\sin^2(\theta/2)\{b(r) + (a(r)-m)(a(r)-m-1/p)\}f(r).\end{aligned}$$

Since $f(r_n e^{i\theta_n}) = 0$, $a(r_n;f) = m$, it follows that

$$p \le p + \cos(\theta/2) \le 2(p+1)\sin^2(\theta/2) b(r_n;f) \le 2(p+1) b(r_n;f).$$

Since the integer $\dot{p} > 0$ was otherwise arbitrary, it follows that $b(r_n;f) \ge \frac{1}{2}$.

As an application of this theorem, we have the following result on integer values of series of hyperbolic functions:

If $q \ge 9$, then, for each positive integer n, $\sum_{k=1}^{\infty} \frac{q^n}{q^n+q^k}$ is not an integer.

For the proof, introduce the function $f(z) = \prod_{k=1}^{\infty}(1 + z/q^k)$.

Then f is entire with positive coefficients and $a(r;f) = \sum_{k=1}^{\infty} \frac{r}{r+q^k}$.

By [1] we have

$$\begin{aligned} b(q^n;f) &= \sum_{k=0}^{n-1} \frac{q^k}{(1+q^k)^2} + \sum_{k=1}^{\infty} \frac{q^k}{(1+q^k)^2} \\ &< \frac{1}{4} + 2\sum_{k=1}^{\infty} \frac{q^k}{(1+q^k)^2} < \frac{1}{2}, \quad q > 9, \end{aligned}$$

for every positive integer n, and the above Theorem gives that $\sum_{k=1}^{\infty} \frac{q^n}{q^k+q^n}$ is not an integer for any n. ■

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