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EXACT CONTROLLABILITY OF THE WAVE EQUATION IN FRACTIONAL ORDER SPACES

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Abstract. We define norm estimates for the trace of solution of the wave equation with initial conditions in irregular Sobolev spaces of fractional order. Then exact controllability results are deduced.

Résumé. On établit des estimations de normes pour la trace de la solution de l'équation des ondes avec des données initiales dans des espaces de Sobolev non réguliers et à puissances fractionnaires. On déduit les résultats de contrôlabilité exacte correspondants.

1. **Introduction.** Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$ of class C^2 and T a positive number. Let $\Sigma =]0,T[\times\partial\Omega$ and $\nu(x)$ be the outward unit normal vector to $x \in \Gamma$.

For any fixed $x^0 \in \mathbb{R}^2$ with $x^0 = (x_1^0, x_2^0)$, we shall use the following notations:

$$m(x) = x - x^0 \quad (x \in \mathbb{R}^2),$$
$$\Gamma_0 = \{x \in \Gamma : m(x)\cdot\nu(x) > 0\},$$
$$\Gamma_0^* = \{x \in \Gamma : m(x)\cdot\nu(x) < 0\},$$

and

$$\Sigma_0 =]0,T[\times\Gamma_0,$$
$$\Sigma_0^* =]0,T[\times\Gamma_0^*.$$

We introduce the constants $R(x^0) = \max_{x\in\overline{\Omega}}(\sum_{k=1}^{k=2}(x - x_k^0)^2)^{1/2}$ and $T_0 = 2R(x^0)$.

Consider the problem

$$\text{(E)} \qquad \begin{cases} u'' - \Delta u = 0 \text{ in }]0,T[\times\Omega, \\ \gamma u = 0 \text{ on } \Gamma, \\ u(0) = u_0 \text{ in } \Omega, \\ u'(0) = u_1 \text{ in } \Omega, \end{cases}$$

where γ is the trace operator.

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Recall that for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, this problem has a unique solution, whose energy is

$$E_0 = \frac{1}{2}(\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)}).$$

We have [1] the direct inequality

$$\int_0^T \int_{\Gamma_0} \Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma\, dt \leq c_0(T) E_0 \tag{1}$$

and, for any $T > T_0$, the inverse inequality

$$\int_0^T \int_{\Gamma_0} \Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma\, dt \geq c_1(T) E_0 \tag{2}$$

where $c_0(T)$, $c_1(T)$ are positive constants. In this report, we generalize, for nonregular initial states (u_0, u_1), estimation (2) of $\frac{\partial u}{\partial \nu}$ in $L^2(0,T; L^2(\Gamma_0))$ into an estimate in $L^2(0,T; H^{-\theta/2}(\Gamma_0))$, where $\theta \in [0,1]$. Consequently, we give, by HUM [1], the corresponding exact controllability result for initial conditions in Sobolev spaces of fractional order.

2. Main results.

2.1. Norms estimations. Let $\theta \in [0,1]$. For $(\varphi_0, \varphi_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ we set

$$\|(\varphi_0, \varphi_1)\|^2_{F_\theta} = \int_0^T \Big\|\gamma \frac{\partial \varphi}{\partial \nu}\Big\|^2_{H^{-\theta/2}(\Gamma_0)} dt$$

where φ is a solution of the homogeneous wave equation (E) corresponding to the initial state (φ_0, φ_1).

We denote by F_θ the completion of $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ with $\|\cdot\|^2_{F_\theta}$ norm.

THEOREM 2.1. *There exists a positive constant C_T such that, for any $\theta \in [0,1]$ and $T \geq T_0$, the solution φ of the homogeneous wave equation* (E) *with initial conditions $(\varphi_0, \varphi_1) \in F_\theta$ satisfy the following inequality:*

$$\int_0^T \Big\|\gamma \frac{\partial \varphi}{\partial \nu}\Big\|^2_{H^{-\theta/2}(\Gamma_0)} dt \geq C_T \{\|\varphi_0\|^2_{H_0^{1-\theta}(\Omega)} + \|\varphi_1\|^2_{H^{-\theta}(\Omega)}\} \tag{3}$$

where C_T is a positive constant.

To prove the theorem, we shall use the following lemma.

LEMMA 2.1. *Let u be a solution of the homogenous wave equation* (E) *with initial conditions $u_0 \in H^2 \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Then we have the following inequality:*

$$\int_0^T \Big\|\gamma \frac{\partial u''}{\partial \nu}\Big\|^2_{H^{-1/2}(\Gamma_0)} dt \geq C_T E_1 \tag{4}$$

where $E_1 = \frac{1}{2}(\|\Delta u_0\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)})$ and C_T a positive constant.

Proof of Lemma 2.1. If $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, setting $v = u'$; $v(0) := u_1$; $v'(0) := \Delta u_0$, one has a solution v of (E) with initial conditions u_1 and Δu_0. Applying inequality (2), we obtain

$$\int_0^T \int_{\Gamma_0} (\sigma - x^0)\nu(\sigma)\Big(\frac{\partial u'}{\partial \nu}\Big)^2 \, d\sigma \, dt \geq c(T)E_1, \tag{5}$$

where $c(T)$ is a positive constant.

For $\varepsilon \geq 0$ sufficiently small (such as $T - 2\varepsilon - T_0 > 0$), we define a function $\varphi_\varepsilon \in D(\mathbb{R})$ as follows:

(i) $0 \leq \varphi_\varepsilon \leq 1$,
(ii) $\text{supp}(\varphi_\varepsilon) \subset]0, T[$,
(iii) $\varphi_{\varepsilon|[\varepsilon, T-\varepsilon]} = 1$.

Taking into account

$$\int_0^T \varphi_\varepsilon(t) \int_{\Gamma_0} (\sigma - x^0)\nu(\sigma)\Big(\frac{\partial u'}{\partial \nu}\Big)^2 \, d\sigma \, dt \geq \int_\varepsilon^{T-\varepsilon} \int_{\Gamma_0} (\sigma - x^0)\nu(\sigma)\Big(\frac{\partial u'}{\partial \nu}\Big)^2 \, d\sigma \, dt,$$

then multiplying the left-hand side of (5) by φ_ε and integrating we obtain

$$-\int_\varepsilon^{T-\varepsilon} \int_{\Gamma_0} \Big(\varphi_\varepsilon'(t) m(\sigma)\nu(\sigma) \frac{\partial u}{\partial \nu}\frac{\partial u'}{\partial \nu} + \varphi_\varepsilon(t) m(\sigma)\nu(\sigma)\frac{\partial u}{\partial \nu}\frac{\partial u''}{\partial \nu}\Big) \, d\sigma \, dt \geq c(T)E_1. \tag{6}$$

But

$$-\int_0^T \int_{\Gamma_0} \varphi_\varepsilon'(t) m(\sigma)\nu(\sigma) \frac{\partial u}{\partial \nu}\frac{\partial u'}{\partial \nu} \, d\sigma \, dt = -\frac{1}{2}\int_0^T \int_{\Gamma_0} \varphi_\varepsilon'(t) m(\sigma)\nu(\sigma) \frac{d}{dt}\Big(\frac{\partial u}{\partial \nu}\Big)^2 \, d\sigma \, dt.$$

Thus (6) can be rewritten as

$$\frac{1}{2}\int_0^T \int_{\Gamma_0} \varphi_\varepsilon''(t) m(\sigma)\nu(\sigma)\Big(\frac{\partial u}{\partial \nu}\Big)^2 \, d\sigma \, dt - \int_0^T \int_{\Gamma_0} \varphi_\varepsilon(t) m(\sigma)\nu(\sigma)\frac{\partial u}{\partial \nu}\frac{\partial u''}{\partial \nu} \, d\sigma \, dt \geq c(T)E_1.$$

Using (1) we have

$$\frac{1}{2}\int_0^T \int_{\Gamma_0} \varphi_\varepsilon''(t) m(\sigma)\nu(\sigma)\Big(\frac{\partial u}{\partial \nu}\Big)^2 \, d\sigma \, dt \leq c_0 E_0.$$

In the same way, by duality , we have

$$\int_0^T\int_{\Gamma_0}\varphi_\varepsilon(m(\sigma)\nu(\sigma))\frac{\partial u}{\partial\nu}\frac{\partial u''}{\partial\nu}\,d\sigma\,dt\le c_2\int_0^T\Big\|\gamma\frac{\partial u}{\partial\nu}\Big\|_{H^{1/2}(\Gamma_0)}\Big\|\gamma\frac{\partial u''}{\partial\nu}\Big\|_{H^{-1/2}(\Gamma_0)}\,dt.$$

where c_2 is a positive constant. Thus

$$c_0E_0+c_2\int_0^T\Big\|\gamma\frac{\partial u}{\partial\nu}\Big\|_{H^{1/2}(\Gamma_0)}\Big\|\gamma\frac{\partial u''}{\partial\nu}\Big\|_{H^{-1/2}(\Gamma_0)}\,dt\ge c(T)E_1.$$

Indeed, an application of Young's inequality gives, for all $\delta>0$,

$$c_0E_0+\frac{c_2}{2}\left(\frac{1}{\delta}\int_0^T\Big\|\gamma\frac{\partial u''}{\partial\nu}\Big\|^2_{H^{-1/2}(\Gamma_0)}\,dt+\delta\int_0^T\Big\|\gamma\frac{\partial u}{\partial\nu}\Big\|^2_{H^{1/2}(\Gamma_0)}\,dt\right)\ge c(T)E_1.$$

By using the estimate $\|\gamma\frac{\partial u}{\partial\nu}\|_{H^{1/2}(\Gamma_0)}\le c\|u\|_{H^2(\Omega)}$ (c a positive constant), one has

$$c_0E_0+\frac{c_2}{2}\left(\int_0^T\frac{1}{\delta}\Big\|\gamma\frac{\partial u''}{\partial\nu}\Big\|^2_{H^{-1/2}(\Gamma_0)}\,dt+\delta c^2\int_0^T\|u\|^2_{H^2(\Omega)}\,dt\right)\ge c(T)E_1.$$

For a suitable choice of δ, there exists a positive constant C_T such that

$$c_0E_0+c_3\int_0^T\Big\|\gamma\frac{\partial u''}{\partial\nu}\Big\|^2_{H^{-1/2}(\Gamma_0)}\,dt\ge C_TE_1. \tag{7}$$

To finish the proof of the lemma, it is enough to show the existence of a constant α such that $E_0\le\alpha\|\gamma\frac{\partial u"}{\partial\nu}\|^2_{H^{-1/2}(\Gamma_0)}$.

By contradiction, we suppose that there exists a sequence $(\varphi_n)_n$ solutions of

$$\begin{cases}\varphi_n''-\Delta\varphi_n=0\ \text{in}\ \ ]0,T[\times\Omega\\ \varphi_n(0)=\varphi_{0,n},\varphi_n'(0)=\varphi_{1,n}\end{cases}$$

where $(\varphi_{0,n},\varphi_{1,n})\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, satisfying

$$\|\nabla\varphi_{0,n}\|^2_{(L^2(\Omega))^2}+\|\varphi_{1,n}\|^2_{L^2(\Omega)}=1\quad\text{and}\quad\Big\|\gamma\frac{\partial\varphi_n^{''}}{\partial\nu}\Big\|_{H^{-1/2}(\Gamma_0)}\longrightarrow 0.$$

Using inequality (5), we deduce that $(\varphi_{0,n},\varphi_{1,n})$ is bounded in $(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$. There exists a subsequence $(\varphi_{0,n},\varphi_{1,n})$ which converges weakly in $(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\to(\tilde\varphi_0,\tilde\varphi_1)$ and thus, by compactness, strongly in $H_0^1(\Omega)\times L^2(\Omega)$.

Calling $\tilde\varphi$ the solution corresponding to the initial data $(\tilde\varphi_0,\tilde\varphi_1)$, we have

$$\|\tilde\varphi_0\|^2_{H_0^1(\Omega)}+\|\tilde\varphi_1\|^2_{L^2(\Omega)}=1\quad\text{and}\quad\Big\|\gamma\frac{\partial\tilde\varphi^{''}}{\partial\nu}\Big\|_{H^{-1/2}(\Gamma_0)}=\Big\|\gamma\frac{\partial\Delta\tilde\varphi^{''}}{\partial\nu}\Big\|_{H^{-1/2}(\Gamma_0)}=0,$$

which is impossible by a classical unicity result.

PROOF OF THEOREM 2.1. Let $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and φ be the corresponding solution of (E).

Let us introduce the function defined by

$$u(t) = \int_0^t \int_0^s \varphi(\tau)\, d\tau\, ds - ty - z$$

where y and z belong to $H_0^1(\Omega)$ and verify $-\Delta y = \varphi_1$ and $-\Delta z = \varphi_0$, respectively.

By applying the preceding lemma to u, one has

$$\int_0^T \Big\| \gamma \frac{\partial \varphi}{\partial \nu} \Big\|^2_{H^{-1/2}(\Gamma_0)} dt \geq C_T \{ \|\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{H^{-1}(\Omega)} \}. \tag{8}$$

Inequalities (2) and (8) give Theorem 2.1 by interpolation.

2.2. Exact controllability. Denote by Δ_Γ the Laplace-Beltrami operator.

The following exact controllability results is a consequence of inequality (3):

THEOREM 2.2. *There is a time T_0 such that if $T > T_0$, then for any given $\theta \in [0,1]$ and initial data $(y_0, y_1) \in (H_0^\theta(\Omega) \times H^{\theta-1}(\Omega))$, there exists $v \in L^2(0,T; H^{-\theta/2}(\Gamma_0))$ such that, if y is the solution of the problem*

$$\begin{cases} y'' - \Delta y = 0 & \text{in }]0,T[\times \Omega, \\ \gamma y = 0 & \text{on } \Gamma_0^*, \\ \gamma y = (-\Delta_\Gamma)^{-\theta/2} v & \text{on } \Gamma_0, \\ y(0) = y_0 & \text{on } \Omega, \\ y'(0) = y_1 & \text{on } \Omega. \end{cases}$$

then, at T, $y(T) = y'(T) = 0$.

PROOF OF THEOREM 2.2. To prove this theorem, one will apply HUM [1]. One begins with $\varphi_0 \in H_0^{1-\theta}(\Omega)$, $\varphi_1 \in H^{-\theta}(\Omega)$.

Let $\varphi \in C(0,T; H_0^{1-\theta}(\Omega)) \cap C^1(0,T; H^{-\theta}(\Omega))$ be the unique solution of problem

$$\begin{cases} \varphi'' - \Delta \varphi = 0 & \text{in }]0,T[\times \Omega, \\ \varphi(0) = \varphi_0 & \text{in } \Omega, \\ \varphi'(0) = \varphi_1 & \text{in } \Omega, \\ \gamma \varphi = 0 & \text{on }]0,T[\times \Gamma. \end{cases}$$

Let ξ be the unique solution defined by transposition of the following equation

$$\begin{cases} \xi'' - \Delta\xi = 0 & \text{in }]0,T[\times\Omega, \\ \xi(T) = 0 & \text{in } \Omega, \\ \xi'(T) = 0 & \text{in } \Omega, \\ \gamma\xi = \eta & \text{on }]0,T[\times\Gamma, \end{cases}$$

where $\eta \in L^2(0,T;H_0^{\theta/2}(\Gamma_0))$. Setting $v = \gamma\frac{\partial}{\partial\nu}\varphi$ and taking $\eta = (-\Delta_\Gamma)^{-\theta/2}v$, we have

$$\xi \in C(0,T;H_0^{\theta}(\Omega)) \cap C^1(0,T;H^{\theta-1}(\Omega)) \quad \text{and} \quad \xi(0) \in H^{\theta}(\Omega), \xi'(0) \in H^{\theta-1}(\Omega).$$

Under the hypothesis of Theorem 2.1, one has $F_\theta \subset H_0^{1-\theta}(\Omega) \times H^{-\theta}(\Omega)$ and thus $H^{\theta-1} \times H_0^{\theta} \subset F'_\theta$. We define

$$\begin{aligned} \bigwedge \colon F_\theta &\longmapsto F'_\theta \\ (\varphi_0,\varphi_1) &\longmapsto (\xi'(0); -\xi(0)). \end{aligned}$$

It is checked that

$$\begin{aligned} \Big\langle \bigwedge(\varphi_0,\varphi_1),(\varphi_0,\varphi_1)\Big\rangle_{F_\theta,F'_\theta} &= \int_\Sigma \langle(-\Delta_\Gamma)^{-\theta/2}v, v\rangle_{H_0^{\theta/2}(\Gamma_0),H^{-\theta/2}(\Gamma_0)}\,d\sigma\,dt \\ &= \|v\|^2_{L^2(0,T;H^{-\theta/2}(\Gamma_0))} \end{aligned}$$

and that $\bigwedge$ is an isomorphism. Thus for any $T > T_0$ and any $(y_0,y_1) \in F'_\theta$, there exists a $(\varphi_0,\varphi_1) \in F_\theta$ such that $\bigwedge(\varphi_0,\varphi_1) = (y_1,-y_0)$.

Let φ and ξ be the functions defined previously with $\xi_0 = y_0$ and $\xi_1 = y_1$.

According to the unicity of solution, one has $y = \xi$. Thus $y(T) = y'(T) = 0$. This proves the theorem.

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