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HIDDEN STRUCTURE OF THE LIE ALGEBRA OF SYMMETRIES

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ABSTRACT. For any dynamical system, a hidden structure of its Lie algebra of symmetries is disclosed. The structure is based on a new infinite series of the canonically defined Lie subalgebras and on their commutator relations.

RÉSUMÉ. Pour n'importe quel système dynamique, une structure cachée de son algèbre de Lie de symétries est révélée. Cette structure découle d'une nouvelle série de sous-algèbres de Lie canoniquement définies et de leurs relations de commutateurs.

The Lie algebra A of symmetries of a dynamical system

$$\dot{x}^i = V^i(x^1, \dots, x^N) \tag{1}$$

on a smooth manifold M^N consists of all vector fields U on M^N that commute with V: $[V,U] = 0$ [1], [2]. The Lie algebra A is infinite-dimensional for any system (1) that has a non-trivial first integral $a(x)$ since for any smooth function $F(z)$ there is a symmetry $U = F(a(x))V$. In [3], [4], we defined conformal symmetries of a dynamical system (1) as vector fields U_c satisfying the equation

$$[V, U_c] = a(x)V, \tag{2}$$

where $a(x)$ is a first integral of system (1). The conformal symmetries U_c of (2) form a Lie algebra $A_c \supset A$ and transform trajectories of system (1) into other reparametrized trajectories. Both Lie algebras A_c and A are modules over the ring R of first integrals of system (1).

In this paper, we disclose hidden algebraic structures in the Lie algebra of symmetries A and in the ring of first integrals R which are based on some properties of the Lie derivative operator L_V. An action of the Lie derivative L_V on a vector field X and on a smooth function $f(x)$ on the manifold M^N has the form [2]: $L_V X = [V, X]$, $L_V f(x) = V(f(x))$.

THEOREM 1. *For any dynamical system* (1)*, the Lie algebra of symmetries A has a flag structure*

$$A \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_\omega, \tag{3}$$

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where Lie subalgebras A_p are ideals in A. The inclusions

$$[A_p, A_q] \subset A_{p+q} \tag{4}$$

hold. The Lie subalgebras A_p are modules over the ring R of first integrals.

PROOF. Let A_p be the linear subspace of symmetries $U_p \in A$ that satisfy the equation $U_p = L_V^p X_p = \big[V, [V, \cdots [V, X_p] \cdots\big]$, where X_p is some smooth vector field on M^N. For any symmetry $U_{p+1} \in A_{p+1}$ we have $U_{p+1} = L_V^{p+1} X_{p+1} = L_V^p (L_V X_{p+1})$. Hence $A_p \supset A_{p+1}$ and we define $A_\omega = \bigcap_p A_p$.

For any two symmetries $U_p = L_V^p X_p \in A_p$ and $U_q = L_V^q X_q \in A_q$, we have

$$[U_p, U_q] = \frac{p!\, q!}{(p+q)!} L_V^{p+q} [X_p, X_q]. \tag{5}$$

Indeed, equality (5) follows from the generalized Leibnitz formula

$$L_V^n [X, Y] = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} [L_V^k X, L_V^{n-k} Y] \tag{6}$$

for $n = p+q$ and the equations $L_V^{p+1} X_p = 0$ and $L_V^{q+1} X_q = 0$. The equalities (5) prove the inclusions (4). Analogously for any symmetry $U \in A$ we find $[U, U_p] = L_V^p [U, X_p]$, hence all subalgebras A_p are ideals in A. For any first integral $a(x)$, the equation $L_V a(x) = 0$ and the Leibnitz formula yield $a(x) U_p = L_V^p (a(x) X_p)$. Hence $R \cdot A_p \subset A_p$ or all Lie subalgebras A_p are R-modules. ■

REMARK 1. Vector fields Z_k satisfying equations $\big[V, [V, \cdots [V, Z_k] \cdots\big] = L_V^k Z_k = 0$ were first introduced by Fuchssteiner in [5] and named symmetries of order k. For $k = 2$, the vector fields Z satisfying equation $[V, [V, Z]] = 0$ were named *mastersymmetries* [5]. The vector fields Z_k do not form a Lie algebra in general.

Let $\mathcal{G}$ be any Lie algebra and $A(x) \subset \mathcal{G}$ be the stabilizer subalgebra for $x \in \mathcal{G}$: all elements $y \in A(x)$ satisfy the equation $\mathrm{ad}_x y = [x, y] = 0$.

THEOREM 2. *The stabilizer subalgebra $A(x) \subset \mathcal{G}$ has the flag structure*

$$A(x) \supseteq A_1(x) \supseteq A_2(x) \supseteq \cdots \tag{7}$$

where the Lie subalgebras $A_p(x)$ are ideals in $A(x)$. The commutator relations

$$[A_p(x), A_q(x)] \subset A_{p+q}(x) \tag{8}$$

hold for any p and q.

Proof. We define a subspace $A_p(x) \subset A(x)$ of elements $y_p \in A(x)$ that satisfy the equation $y_p = \mathrm{ad}_x^p f_p$ for some $f_p \in \mathcal{G}$. It is evident that $A_{p+1}(x) \subseteq A_p(x)$. Since the operator ad_x^p satisfies the generalized Leibnitz formula (6), we find $\mathrm{ad}_x^{p+q}[f_p, f_q] = (p+q)!\,(p!\,q!)^{-1}[y_p, y_q]$ and the inclusions (8) follow. ∎

Remark 2. The inclusions (4), (8) imply several consequences. For example, all quotient Lie algebras A_p/A_{p+q} are nilpotent for $p, q \geq 1$ and are abelian for $q \leq p$. Hence if for some N the Lie algebra $A_N = 0$ then all Lie algebras $A_1, \dots, A_{N-1}$ are nilpotent and for $q \geq [(N+1)/2]$ the Lie algebras A_q are abelian. If Lie algebra A_1 contains a simple Lie algebra G then $G \subset A_p$ for all p and flag (3) is infinite.

Theorem 3. *The ring of first integrals R has a flag structure*

$$R \supseteq R_1 \supseteq R_2 \supseteq \cdots \supseteq R_\omega, \tag{9}$$

where all subrings R_k are ideals in R. The inclusions

$$R_k \cdot R_\ell \subset R_{k+\ell} \tag{10}$$

hold. The Lie subalgebras A_k (3) define differentiations of the rings R_ℓ that satisfy the relations

$$A(R_\ell) \subset R_\ell, \quad A_k(R) \subset R_k, \quad A_k(R_\ell) \subset R_{k+\ell}. \tag{11}$$

The subrings $R_\ell \subset R$ define the R_ℓ-module structures in A_k that satisfy the relations

$$R_\ell \cdot A \subset A_\ell, \quad R_\ell \cdot A_k \subset A_{k+\ell}. \tag{12}$$

Proof. Let $R_k \subset R$ be a subspace of first integrals $a_k(x)$ that have the form $a_k(x) = L_V^k f_k(x)$, where $f_k(x)$ is some smooth function on M^N. For any first integral $a_{k+1} \in R_{k+1}$ we have $a_{k+1} = L_V^{k+1} f_{k+1} = L_V^k(L_V f_{k+1})$. Hence $R_k \supset R_{k+1}$ and we define $R_\omega = \bigcap_k R_k$.

Let $U_k = L_V^k X_k \in A_k$ and $a_\ell = L_V^\ell f_\ell \in R_\ell$, and $U \in A$ be any symmetry. Applying the generalized Leibnitz formula

$$L_V^n(X(f)) = \sum_{p=0}^{n} \frac{n!}{p!\,(n-p)!}(L_V^p X)(L_V^{n-p} f)$$

for $n = k + \ell$ and using the equations $L_V^{k+1} X_k = 0$ and $L_V^{\ell+1} f_\ell = 0$, we derive

$$U_k(a_\ell) = (L_V^k X_k)(L_V^\ell f_\ell) = \frac{k!\,\ell!}{(k+\ell)!} L_V^{k+\ell}(X_k(f_\ell)), \quad U(a_\ell) = L_V^\ell(U(f_\ell)).$$

Hence the inclusions (11) follow. Analogously, for $a \in R$, $a_k = L_V^k f_k \in R_k$ and $a_\ell = L_V^\ell f_\ell \in R_\ell$ the Leibnitz formula implies

$$aa_k = L_V^k(af_k), \quad a_k a_\ell = \frac{k!\,\ell!}{(k+\ell)!} L_V^{k+\ell}(f_k f_\ell).$$

Hence the subrings R_k are ideals in R and the inclusions (10) hold.

An application of the generalized Leibnitz formula

$$L_V^n(fX) = \sum_{p=0}^{n} \frac{n!}{p!\,(n-p)!}(L_V^p f)(L_V^{n-p} X)$$

for $n = k+\ell$, $U \in A$, $U_k = L_V^k X_k \in A_k$ and $a_\ell = L_V^\ell f_\ell \in R_\ell$ gives the equalities

$$a_\ell \cdot U = L_V^\ell(f_\ell U), \quad a_\ell \cdot U_k = \frac{k!\,\ell!}{(k+\ell)!} L_V^{k+\ell}(f_\ell X_k).$$

These formulae prove the inclusions (12). ■

Remark 3. Let L be a linear operator in a linear space B over any field K. Let $B_{\lambda.0}$ be λ-eigenspace for L. We define $B_{\lambda.k} = B_{\lambda.0} \cap (L-\lambda)^k B$. Hence we obtain the flag structure

$$B_{\lambda.0} \supseteq B_{\lambda.1} \supseteq B_{\lambda.2} \supseteq \cdots . \tag{13}$$

The flag structures (3), (7) and (9) are special cases (for $L = L_V$ and $L = \mathrm{ad}_x$) of the hidden canonical flag structure (13) in the eigenspaces of any linear operator L.

Example 1. Let us show that both flags (3) and (9) can be infinite. Consider a Hamiltonian system

$$\dot p_i = -\frac{\partial H}{\partial q_i}, \quad \dot q_i = \frac{\partial H}{\partial p_i}, \quad H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \Phi(q_i), \tag{14}$$

where the potential $\Phi(q_i)$ is a homogeneous function of degree -2, $\Phi(\lambda q_i) = \lambda^{-2}\Phi(q_i)$. For function $F = p_1 q_1 + \cdots + p_n q_n$, the equations $\dot F = 2H$ and $\dot H = 0$ evidently hold. Let V denotes the vector field (14). We have $H = L_V F/2 \in R_1 \neq 0$ and hence $H^k \in R_k \neq 0$. Evidently $H^k V$ is a nonzero symmetry of system (14) and $H^k V = (2^k k!)^{-1} L_V^k(F^k V) \in A_k \neq 0$ for all integers $k > 0$. Hence both flags (3) and (9) are infinite. For example, this is true for the integrable Calogero-Moser system (14) with potential $\Phi(q_i) = \sum_{i \neq j}(q_i - q_j)^{-2}$.

Remark 4. For any system (1), the same method proves that flag (9) is either infinite or trivial, $R_1 = 0$, and if flag (9) is infinite then flag (3) is either.

Example 2. The Kepler problem has the Hamiltonian function

$$H_1(p,q) = \frac{1}{2m}(p_1^2+p_2^2+p_3^2) - \frac{GMm}{\sqrt{q_1^2+q_2^2+q_3^2}}.$$

Using the Delaunay variables, Poincaré constructed in [1] the action-angle coordinates I_j, φ_j where the Hamiltonian $H_1(p,q)<0$ has the form $H_1(I) = -KI_1^{-2}$, $K = G^2M^2m^3/2$. In this coordinates, the dynamics of the Kepler problem is

$$\dot I_1 = 0, \quad \dot I_2 = 0, \quad \dot I_3 = 0, \quad \dot\varphi_1 = 2KI_1^{-3}, \quad \dot\varphi_2 = 0, \quad \dot\varphi_3 = 0. \tag{15}$$

Thus the ring of first integrals R consists of arbitrary functions $f(I_j, \varphi_2, \varphi_3)$. Applying the Lemma on Symmetries of [3], we obtain that all symmetries U of system (15) are

$$U = U^2\frac{\partial}{\partial I_2} + U^3\frac{\partial}{\partial I_3} + f^1\frac{\partial}{\partial\varphi_1} + f^2\frac{\partial}{\partial\varphi_2} + f^3\frac{\partial}{\partial\varphi_3}, \tag{16}$$

where U^2, U^3, f^1, f^2, f^3 are arbitrary smooth functions of $I_j, \varphi_2, \varphi_3$. Applying Proposition 10 of [3], we find that system (15) has no symmetries $U_2 = L_V^2 X_2$ (that means $A_2 = 0$) and if $U_1 = L_V X_1$ then vector field X_1 (a mastersymmetry [5]) has the form

$$X_1 = \sum_{i=1}^{3} F^j \frac{\partial}{\partial I_j} + \sum_{i=1}^{3} G^j \frac{\partial}{\partial \varphi_j},$$

where F^j and G^j are arbitrary functions of $I_j, \varphi_2, \varphi_3$. Hence we find

$$U_1 = L_V X_1 = [V, X_1] = 6KF^1 I_1^{-4}\partial/\partial\varphi_1 = 3F^1 I_1^{-1} V. \tag{17}$$

Thus for the Kepler problem at $H_1 < 0$, the formulae (16) and (17) prove that the Lie algebra of symmetries A is a free module of rank 5 over the ring R and the Lie subalgebra $A_1 \subset A$ is a free module of rank 1, and $A_2 = 0$. Since formula (17) has form (2) with first integral $a(x) = 3F^1 I_1^{-1}$, we obtain that for the Kepler problem any mastersymmetry X_1 is a conformal symmetry.

Proposition 1. *Let system* (1) *be any Hamiltonian system*

$$\dot p_i = -\frac{\partial H}{\partial q_i}, \quad \dot q_i = \frac{\partial H}{\partial p_i}, \tag{18}$$

on a symplectic manifold M^{2n} which is integrable in the Liouville sense and has compact invariant submanifolds. Then flag (3) *has the form $A \supset A_1 \supset 0$, $A_2 = 0$, and the Lie subalgebra A_1 is abelian. If the two Lie algebras coincide, $A = A_1$, then system* (18) *is almost everywhere non-degenerate in the Poincaré–Kolmogorov sense* [1], [6].

PROOF. In Theorem 10 of [3], we proved that for any integrable system (18) the equation $L_V^3 X = 0$ implies $L_V^2 X = 0$, where X is a vector field on M^{2n}. Hence any symmetry $U_2 = L_V^2 X_2 \in A_2$ necessarily vanishes and $A_2 = 0$. Since $[A_1, A_1] \subset A_2$, the Lie subalgebra A_1 is abelian.

In Theorem on symmetries [3] we proved that an integrable Hamiltonian system (18) is almost everywhere non-degenerate in the Poincaré–Kolmogorov sense if and only if its Lie algebra of symmetries A is abelian. Since the equality $A = A_1$ implies that A is abelian we obtain that the system (18) is non-degenerate almost everywhere. ■

REMARK 5. Suppose the system (18) is non-degenerate in the Poincaré–Kolmogorov sense [1], [6] almost everywhere. This means that in the action-angle variables I_j, φ_j the system has the form

$$\dot{I}_j = 0, \quad \dot{\varphi}_j = \frac{\partial H(I)}{\partial I_j}, \quad \det\left\| \frac{\partial^2 H(I)}{\partial I_j \partial I_k} \right\| \neq 0. \tag{19}$$

Any symmetry U of system (19) has the form $U = \sum_{k=1}^n S_k(I_j)\partial/\partial\varphi_k$ with arbitrary functions $S_k(I_j)$ and the Lie algebra A is abelian [3]. For any symmetry $U_1 = L_V X_1$, the vector field X_1 is $X_1 = \sum_{j=1}^n f_j(I)\partial/\partial I_j$ [3]. Hence we find

$$U_1 = [V, X_1] = -\sum_{k,j=1}^{n} f_j(I) \frac{\partial^2 H(I)}{\partial I_j \partial I_k} \frac{\partial}{\partial \varphi_k}. \tag{20}$$

Suppose that the Hessian in (19) is degenerate at some N points x_i where its rank is $n - r_i < n$. Then the symmetries U_1 (20) at the points x_i belong to the subspaces of dimensions $n - r_i$. Hence the Lie subalgebra $A_1 \neq A$ and the quotient Lie algebra A/A_1 has dimension $D \geq r_1 + \cdots + r_N$.

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