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A CLASSIFICATION THEOREM FOR CERTAIN ACTIONS OF SL$(2,\mathbb{R})$ ON C^*-ALGEBRAS

ANDREW J. DEAN

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Abstract. It is shown that two C^*-dynamical systems of the form $(K\otimes A, \mathrm{SL}(2,\mathbb{R}), \mathrm{Ad}\,U\otimes id)$, where U is a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ that decomposes as a finite direct sum of non-trivial irreducible representations whose multiplicities have greatest common denominator 1, and A is a simple, unital C^*-algebra with real rank zero and cancellation, are equivariantly isomorphic if, and only if, the two representations are unitarily equivalent. As a corollary, a classification result for certain inductive limit type actions of $\mathrm{SL}(2,\mathbb{R})$ on stable UHF algebras is given.

Résumé. It est montré que deux systèmes C^*-dynamiques de la forme $(K\otimes A, \mathrm{SL}(2,\mathbb{R}), \mathrm{Ad}\,U\otimes id)$ où U est une representation unitaire de $\mathrm{SL}(2,\mathbb{R})$, qui décompose comme une somme directe et finie des representations non-triviales et irréductibles dont les multiplicités ont 1 comme le dénominateur commun et le plus grand, et A est un C^*-algèbre simple, avec l'unité et avec rang réel zéro et annullation, sont isomorphe équivariantement si et seulement si les deux representations sont équivalentes unitairement. Comme un corollaire, un résultat classification pour quelques actions du type de la limite inductive de $\mathrm{SL}(2,\mathbb{R})$ sur les algèbres d'UHF stables est aussi donné.

1. **Introduction.** There is already a long history of classification results for inductive limit type actions of compact groups on C^*-algebras (*cf.* [2], [7], [8], [9], and [10]). For locally compact, non-compact groups the subject is at a much earlier state. AF flows were classified in [3] and [4], and in [5] certain inductive limit type actions of the Euclidean motion group on stable UHF algebras were classified.

The theorem that is the subject of this paper is an analogue of Theorem 4.2 in [5] in which we replace the infinite dimensional irreducible representations of E with representations of $\mathrm{SL}(2,\mathbb{R})$. What is different about the proof is that we replace the functions $g \mapsto \|e_n\alpha_g(e_n)\|$ used in [5] with a set of coefficients derived from the associated representation of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$.

We collect some facts about the representation theory of $\mathrm{SL}(2,\mathbb{R})$ in an appendix for the reader's convenience. Throughout this paper $\mathfrak{H}$ is a separable, infinite dimensional Hilbert space, $K(\mathfrak{H})$ is the algebra of compact operators on $\mathfrak{H}$, and $B(\mathfrak{H})$ is the algebra of bounded linear operators on $\mathfrak{H}$. We will write $M(A)$ for the multiplier algebra of the C^*-algebra A.

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2. Actions of $\mathrm{SL}(2,\mathbb{R})$**.** Our main result is the following:

THEOREM 2.1. *Let A and B be two simple, unital C^*-algebras with real rank zero and cancellation, and let $U \cong n_1W_1 \oplus \cdots \oplus n_kW_k$ and $V \cong m_1T_1 \oplus \cdots \oplus m_lT_l$, where the W_i's and T_j's are distinct non-trivial irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$, $n_i, m_j \in \mathbb{N}$ for $1 \le i \le k$, $1 \le j \le l$, and $\gcd(n_1, \ldots, n_k) = 1 = \gcd(m_1, \ldots, m_l)$. Suppose that $(K(\mathfrak{H}) \otimes A, \mathrm{Ad}\, V \otimes id) \cong (K(\mathfrak{H}) \otimes B, \mathrm{Ad}\, U \otimes id)$. Then $A \cong B$ and $U \cong V$.*

REMARK. A C^*-algebra A has *cancellation of projections* if whenever p, q, r, s are projections with $p \simeq q$, $r \perp p$, $s \perp q$, and $p + r \simeq q + s$, it follows that $r \simeq s$, where $\simeq$ denotes Murray–von Neumann equivalence. A has *cancellation* if $M_n(A)$ has cancellation of projections for all $n \in \mathbb{N}$. For a C^*-algebra with real rank zero, cancellation is equivalent to having stable rank one (*cf.* [1]).

From the above theorem we get the following corollary:

COROLLARY 2.2. *Let $\{A_n, \varphi_{nm}\}$ and $\{B_n, \psi_{nm}\}$ be inductive systems of C^*-algebras with $A_n \cong B_n \cong K(\mathfrak{H})$ for all $n \in \mathbb{N}$, and φ_{nm} and ψ_{nm} proper maps for all $n, m \in \mathbb{N}$. Let α_n and β_n be actions of $\mathrm{SL}(2,\mathbb{R})$ on A_n and B_n respectively such that the maps φ_{nm} and ψ_{nm} are all equivariant. Let (A, α) and (B, β) denote the inductive limit C^*-dynamical systems. Suppose that $(A_1, \alpha_1) \cong (K(\mathfrak{H}), \mathrm{Ad}\, V)$ and $(B_1, \beta_1) \cong (K(\mathfrak{H}), \mathrm{Ad}\, U)$, where $U \cong n_1W_1 \oplus \cdots \oplus n_kW_k$ and $V \cong m_1T_1 \oplus \cdots \oplus m_lT_l$, the W_i's and T_j's being distinct irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$, and $n_i, m_j \in \mathbb{N}$ for $1 \le i \le k$, $1 \le j \le l$ with $\gcd(n_1, \ldots, n_k) = 1 = \gcd(m_1, \ldots, m_l)$. Suppose that $(A, \alpha) \cong (B, \beta)$. Then there exists a unique UHF algebra M such that $(A, \alpha) \cong (A_1, \alpha_1) \otimes (M, id) \cong (B_1, \beta_1) \otimes (M, id)$, and $U \cong V$.*

PROOF OF COROLLARY 2.2. Let $\{(A_n, \alpha_n), \varphi_{nm}\}$ and $\{(B_n, \beta_n), \psi_{nm}\}$ be inductive systems of C^*-dynamical systems as in the statement of the corollary. Consider the map $\varphi_{21}\colon (A_1, \alpha_1) \to (A_2, \alpha_2)$. The multiplier algebra $M(A_2)$ is isomorphic to $B(\mathfrak{H})$, and since the map φ_{21} is proper, for some integer k we have that $\varphi_{21}(A_1)' \cap M(A_2) \cong M_k$ and A_2 is the subalgebra of $M(A_2)$ generated by the products ax with $a \in \varphi_{21}(A_1)$ and $x \in \varphi_{21}(A_1)' \cap M(A_2) \cong M_k$. The action of $\mathrm{SL}(2,\mathbb{R})$ on A_2 extends to an action of $\mathrm{SL}(2,\mathbb{R})$, now only strictly continuous, on $M(A_2)$. Under this extended action, $\varphi_{21}(A_1)$ is invariant, and hence so is the relative commutant, $\varphi_{21}(A_1)' \cap M(A_2) \cong M_k$. Consider the C^*-dynamical system $(\varphi_{21}(A_1)' \cap M(A_2), \alpha_2)$. The unique tracial state on $\varphi_{21}(A_1)' \cap M(A_2)$ is invariant under the action, so the GNS construction with this state gives a faithful covariant representation of this C^*-dynamical system on a finite dimensional Hilbert space. Since there are no non-trivial finite dimensional unitary representations of $\mathrm{SL}(2,\mathbb{R})$, we see that the action of $\mathrm{SL}(2,\mathbb{R})$ on $\varphi_{21}(A_1)' \cap M(A_2)$ must be trivial. Hence there is a covariant isomorphism of (A_2, α_2) with $(A_1 \otimes M_k, \alpha_1 \otimes id)$ that carries φ_{21} to the map $a \mapsto a \otimes 1$. Applying this argument to the maps $\varphi_{l+1,l}$ in turn, we see that for some UHF algebra M we have $(A, \alpha) \cong (A_1, \alpha_1) \otimes (M, id)$

and similarly for (B,β). The corollary now follows immediately from Theorem 2.1.

Proof of Theorem 2.1. We deal first with the case where there is just one irreducible representation of SL$(2,\mathbb{R})$ involved. We refer the reader to the appendix for our notation and facts about SL$(2,\mathbb{R})$ and its irreducible unitary representations.

Let W be a non-trivial irreducible unitary representation of SL$(2,\mathbb{R})$, and consider the C^*-dynamical system $(K(\mathfrak{H}), \operatorname{Ad} W)$. Let ANK be the Iwasawa decomposition of SL$(2,\mathbb{R})$, and let $K(\mathfrak{H})^{\mathrm{K}}$ denote the fixed point subalgebra of the restriction of the action by $\operatorname{Ad} W$ of SL$(2,\mathbb{R})$ to K. Then $K(\mathfrak{H})^{\mathrm{K}} \cong c_0$, and the minimal central projections of $K(\mathfrak{H})^{\mathrm{K}}$ are given by $P_n = \langle e_n \,|\, \cdot \rangle e_n$, where the $e_n s$ are unit eigenvectors of H with $d_W(H)e_n = ine_n$. The projections P_n are in the domain of the representation δ_W of the Lie algebra SL$(2,\mathbb{R})_{\mathbb{C}}$ in $K(\mathfrak{H})$. We have:

$$\begin{aligned}\delta_W(E^-)P_n &= [d_W(E^-), P_n] \\ &= \langle d_W(E^+)e_n \,|\, \cdot \rangle e_n + \langle e_n \,|\, \cdot \rangle d_W(E^-)e_n, \\ \delta_W(E^+)P_n &= \langle d_W(E^-)e_n \,|\, \cdot \rangle e_n + \langle e_n \,|\, \cdot \rangle d_W(E^+)e_n,\end{aligned}$$

and

$$\delta_W(H)P_n = 0,$$

where we have used that $d_W(E^-)^* = -d_W(E^+)$ and $d_W(E^+)^* = -d_W(E^-)$. Define constants $g_n^+(W)$ and $g_n^-(W)$ by

$$g_n^+(W) = \|P_n \delta_W(E^-)(P_n)\| = \|d_W(E^+)e_n\|$$

and

$$g_n^-(W) = \|P_n \delta_W(E^+)(P_n)\| = \|d_W(E^-)e_n\|.$$

The set $\{P_n\}$ of minimal central projections of $K(\mathfrak{H})^{\mathrm{K}}$ may be given a linear ordering using the fact that if v is any partial isometry with $vv^* = p_n$ and $v^*v = p_{n+2k}$, then v is an eigenoperator for $\delta_W(H)$ with eigenvalue $-2ki$. We define the order relation $\sim$ on $\{P_n\}$ by $p \sim q$ if, and only if, there exists a partial isometry v with $vv^* = p$, $v^*v = q$, and v is an eigenoperator for $\delta_W(H)$ with eigenvalue in $\{-2ik \mid k = 0,1,2,\dots\}$. Then $(\{P_n\},\sim)$ is isomorphic as an ordered set to one of $\mathbb{Z}$, $\mathbb{Z}^+$, or $-\mathbb{Z}^+$ with the usual ordering, depending on the representation W. The sets of real numbers $\{g_n^+(W)\}$ and $\{g_n^-(W)\}$, and the ordered set $(\{P_n\},\sim)$, along with the natural labelling of the numbers by the ordered set, are then an equivariant isomorphism invariant of the C^*-dynamical system $(K(\mathfrak{H}), \operatorname{Ad} W)$, among such systems. We shall show that we can recover the unitary equivalence class of W from this information.

We shall show first how one can tell from the invariant whether or not W has a highest or lowest weight vector, and, if it has one, what the weight of that vector is. W has a highest weight vector if, and only if, for one of the eigenvectors of $d_W(H)$, say e_n, we have $d_W(E^+)e_n = 0$ and $d_W(E^-)e_n \neq 0$. This is the case if, and only if, for some P_n we have $g_n^+(W) = 0$ and $g_n^-(W) \neq 0$. Telling whether there is a lowest weight vector is similar. To tell what the weight of the highest, or lowest, weight vector is we use $-(g_n^-(W))^2 = 4n$ in the case of a highest weight vector, and $(g_n^+(W))^2 = 4n$ in the case of a lowest weight vector.

Next, suppose that there are no lowest or highest weight vectors. This is the case if, and only if, $(\{P_n\}, \sim) \cong \mathbb{Z}$, and to tell which representation W is we have to recover the parity and the eigenvalue of the Casimir operator. Using the notation of the appendix, we have $g_n^+(W) = \sqrt{-c_{n+2}}$ and $g_n^-(W) = \sqrt{-c_n}$. The relation $c_n - c_{n+2} = 4n$ allows us to tell the parity in a simple way: The parity is even if, and only if, there are two successive coefficients (successive in the order on $\{P_n\}$), say $g_k^-(W)$ and $g_{k+2}^-(W)$, for which $g_k^-(W) = g_{k+2}^-(W)$. If this is the case, the lower of the two is the 0-th coefficient, so $c_0 = -(g_0^-(W))^2$ gives us the eigenvalue of the Casimir operator. In the case of odd parity, the relation $c_n - c_{n+2} = 4n$ can again be used to isolate c_1, and $c_1 - 1$ is the eigenvalue of the Casimir operator in that case. Taking two successive coefficients, say $g_n^-(W)$ and $g_{n+2}^-(W)$, one gets $4n = c_n - c_{n+2} = (g_{n+2}^-(W))^2 - (g_n^-(W))^2$, so we can identify what n is. We then count $(n-1)/2$ places down to find $g_1^-(W)$ and use $c_1 = -(g_1^-(W))^2$.

From the discussion in the appendix it follows that we have recovered enough information about W from our invariants to determine its unitary equivalence class. Of course, two actions $\operatorname{Ad} W$ and $\operatorname{Ad} V$ on K are equivariantly isomorphic if, and only if, up to unitary equivalence, they differ by tensoring with a scalar representation, and $\mathrm{SL}(2,\mathbb{R})$ has no non-trivial scalar representations, so we knew already that the unitary equivalence class of W was determined by the equivariant isomorphism class of $(K(\mathfrak{H}), \operatorname{Ad} W)$. The point is that our methods extend to the case of a tensor product $K(\mathfrak{H}) \otimes A$. Let A be a simple unital C^*-algebra, with $1 \neq 0$, and consider the C^*-dynamical system $(K(\mathfrak{H}) \otimes A, \operatorname{Ad} W \otimes id)$, where W is an irreducible representation of $\mathrm{SL}(2,\mathbb{R})$. The fixed point subalgebra $(K(\mathfrak{H}) \otimes A)^{\mathrm{K}}$ of the restriction of the action to K is isomorphic to $c_0 \otimes A$, and the set of minimal central projections may be ordered in the same way as above. We also define the coefficients g_n^+ and g_n^- as before, and the arguments above, essentially unchanged, allow us to identify the unitary equivalence class of W. We recover A by noting that $A \cong p(K(\mathfrak{H}) \otimes A)p$ where p is any minimal central projection of $(K(\mathfrak{H}) \otimes A)^{\mathrm{K}}$. This proves the theorem for the special case of just one irreducible representation.

Now we consider the general situation. The argument is based on the proof of Theorem 4.2 in [5]. Let A, B, U, V, *etc.*, be as in the statement of the theorem. Consider $(K(\mathfrak{H}) \otimes A, \operatorname{Ad} U \otimes id)$. The action of $\mathrm{SL}(2,\mathbb{R})$ on $K(\mathfrak{H})$ given by $\operatorname{Ad} U$ extends uniquely to an action of $\mathrm{SL}(2,\mathbb{R})$ on $B(\mathfrak{H}) \cong M(K(\mathfrak{H}))$ which is given via the same unitary representation, and $\big(U(\mathrm{SL}(2,\mathbb{R}))\big)' \cong \mathbb{C}^l$, so the fixed point subalgebra $M(K(\mathfrak{H}))^{\mathrm{SL}(2,\mathbb{R})}$ is isomorphic to $\mathbb{C}^l$. Let $Q_1,\dots,Q_l$ denote the minimal central projections of $M(K(\mathfrak{H}))^{\mathrm{SL}(2,\mathbb{R})}$. Since the inclusion $K(\mathfrak{H})\to K(\mathfrak{H})\otimes A$ via $a\mapsto a\otimes 1$ is proper, it extends to an inclusion of $M(K(\mathfrak{H}))$ into $M(K(\mathfrak{H})\otimes A)$, and the extension of the action $\mathrm{Ad}\,U\otimes id$ on $K(\mathfrak{H})\otimes A$ to $M(K(\mathfrak{H})\otimes A)$ is given by the inclusion of $\mathfrak{U}\big(M(K(\mathfrak{H}))\big)$ into $\mathfrak{U}\big(M(K(\mathfrak{H})\otimes A)\big)$. We may now view $Q_1,\dots,Q_l$ as projections contained in $M(K(\mathfrak{H})\otimes A)^{\mathrm{SL}(2,\mathbb{R})}$. Our next step is to observe that $Q_1,\dots,Q_l$ are central in $M(K(\mathfrak{H})\otimes A)^{\mathrm{SL}(2,\mathbb{R})}$.

Represent A faithfully and non-degenerately on a Hilbert space $\mathfrak{K}$. Then $K(\mathfrak{H})\otimes A$ is represented faithfully and non-degenerately on $\mathfrak{H}\otimes\mathfrak{K}$ by the tensor product representation. Call this representation π. Then π extends to a representation of $M(K(\mathfrak{H})\otimes A)$ on $\mathfrak{H}\otimes\mathfrak{K}$ such that $\pi\big(M(K(\mathfrak{H})\otimes A)\big)$ is the set of all operators in $B(\mathfrak{H}\otimes\mathfrak{K})$ that multiply $\pi(K(\mathfrak{H})\otimes A)$ into itself (*cf.* [13]). The composition of π with the homomorphism of $\mathrm{SL}(2,\mathbb{R})$ into $\mathfrak{U}\big(M(K(\mathfrak{H})\otimes A)\big)$ is just $U\otimes 1$, and the image of Q_i is $Q_i\otimes 1$. Since the automorphisms in the action of $\mathrm{SL}(2,\mathbb{R})$ on $M(K(\mathfrak{H})\otimes A)$ are inner for this algebra, we have that $\pi\big(M(K(\mathfrak{H})\otimes A)^{\mathrm{SL}(2,\mathbb{R})}\big)\subseteq\pi\big(U(\mathrm{SL}(2,\mathbb{R}))\otimes 1\big)'$, so it will suffice to see that the Q_i are central in $\pi\big(U(\mathrm{SL}(2,\mathbb{R}))\otimes 1\big)'$. This follows from the fact that the representations $U(\,\cdot\,)Q_i\otimes 1$ are disjoint (*cf.* [6]).

Let $R_1,\dots,R_k$ be the projections in $M(K(\mathfrak{H})\otimes B)$ corresponding to $m_1W_1,\dots,m_kW_k$ in the same fashion that $Q_1,\dots,Q_l$ correspond to $n_1T_1,\dots,n_lT_l$. Now assume we have given an equivariant isomorphism $K(\mathfrak{H})\otimes A\cong K(\mathfrak{H})\otimes B\cong D$, so that we may view $R_1,\dots,R_k$ and $Q_1,\dots,Q_l$ as all contained in $M(D)$. Then from above we have $R_iQ_j=Q_jR_i$ for each i and j. assume that for some fixed i and j we have $R_iQ_j\neq 0$, and let F_{ij} denote $(R_iQ_j)D(Q_jR_i)=(R_iDR_i)\cap(Q_jDQ_j)$. Let $\{s_n\}$ denote the set of minimal central projections in $(Q_jDQ_j)^{\mathrm{K}}\cong c_0\otimes M_{n_j}\otimes A$ and let $\{r_n\}$ denote the minimal central projections in $(R_jDR_j)^{\mathrm{K}}\cong c_0\otimes M_{m_i}\otimes B$. As in [5], our next step is to find a projection which is a subprojection of both one of the s_k's and one of the r_n's. Let a be a non-zero positive element in F_{ij}. Then $\int_K(\mathrm{Ad}\,U(k))(a)\,dk$ is a non-zero positive element in $(F_{ij})^K$. For some k, $c=s_kbs_k$ is non-zero, and the hereditary subalgebra of D generated by this element is contained in s_kDs_k. D has real rank zero, so the hereditary subalgebra cDc has an approximate unit consisting of projections (*cf.* [14]), so we may choose a non-zero projection $p\in cDc$. Since s_k acts as a unit for this algebra we have $p\le s_k$. Since F_{ij} is hereditary and $c\le b$, we have $p\in(F_{ij})^{\mathrm{K}}$. For some n, $r_npr_n\neq 0$. The hereditary subalgebra of D generated by r_npr_n is contained in r_nDr_n, and has an approximate unit consisting of projections. Let q be a non-zero projection in this algebra. Then we have $q\le r_n$ and $q\le p\le s_k$.

Now it is easy to see that we have

$$\|q\delta(E^+)(q)\|=g_n^+(W_i)=g_k^+(T_j)$$

and

$$\|q\delta(E^-)(q)\|=g_n^-(W_i)=g_k^-(T_j),$$

where δ denotes the derivation from the action on D.

From the appendix we have that if two successive coefficients agree, then the representations are equivalent, so we see that $W_i \cong T_j$. From this it follows that $k = l$ and, possibly after reordering, $R_1 = Q_1, \ldots, R_k = Q_k$ and $W_1 = T_1, \ldots, W_k = T_k$. From the first part of the theorem it follows that we also have $M_{n_1}(A) \cong M_{m_1}(B), \ldots, M_{n_k}(A) \cong M_{m_k}(B)$.

For $i = 1, \ldots, k$, let f_i be a minimal central projection in $(R_i D R_i)^{\mathrm{K}}$. Let 1_A and 1_B denote the image of the e_{11} projection under the identifications of $K(\mathfrak{H}) \otimes A$ and $K(\mathfrak{H}) \otimes B$ with D respectively. Then we have $[f_i] = n_i[1_A] = m_i[1_B]$ for $i = 1, \ldots, k$ in $K_0(D)$. Since D has cancellation, and all of the $[f_i]$'s are natural number multiples of $[1_A]$, it follows that the subgroup of $K_0(D)$ generated by $[f_1], \ldots, [f_k]$ is order isomorphic to $\mathbb{Z}$, and is contained in the subgroup generated by $[1_A]$. Since $\gcd(n_1, \ldots, n_k) = 1$, it follows that it is actually the whole subgroup generated by $[1_A]$, and that $[1_A]$ is its smallest non-zero positive element. A similar argument shows that its smallest non-zero positive element is also $[1_B]$, so we have $[1_A] = [1_B]$. From cancellation we have $1_A \simeq 1_B$, so we get $A \cong 1_A D 1_A \cong 1_B D 1_B \cong B$. Also, from above we get $n_i = m_i$ for $i = 1, \ldots, k$, so $U \cong V$, which completes the proof.

3. Appendix. In this appendix we collect some facts about $\mathrm{SL}(2,\mathbb{R})$ and its irreducible unitary representations, and establish our notation. The reader is referred to [11] and [12] for full discussions of these topics.

Every element $g \in \mathrm{SL}(2,\mathbb{R})$ can be uniquely expressed in the form $g = ank$ with $a \in \left\{ \left(\begin{smallmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{smallmatrix}\right) \;\middle|\; \alpha \in \mathbb{R} \right\} = A$, $n \in \left\{ \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; r \in \mathbb{R} \right\} = N$, and $k \in \left\{ \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \;\middle|\; \theta \in \mathbb{R} \right\} = K$. The decomposition $G \cong ANK$ is continuous and is called the Iwasawa decomposition. K is a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$.

We shall use the following basis for $\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}$, the complexified Lie algebra of $\mathrm{SL}(2,\mathbb{R})$:

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

We then have the fundamental relation

$$E^+E^- - E^-E^+ = -4iH.$$

If W is a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ on $\mathfrak{H}$, we write d_W for the associated representation of $\mathfrak{sl}(2,\mathbb{R})$, and also for its extension to $\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}$. We write δ_W for the representation of $\mathfrak{sl}(2,\mathbb{R})$ by $*$-derivations on $K(\mathfrak{H})$ associated with $\mathrm{Ad}\, W$, and we also write δ_W for the extension of this representation to $\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}$.

For the remainder we assume further that W is a non-trivial irreducible representation. One may then choose an orthonormal basis $\{e_n\}$ for $\mathfrak{H}$, where n varies over a subset of $\mathbb{Z}$ as described below, and negative real coefficients c_n for each e_n in the basis $\{e_n\}$ such that the following relations hold:

$$d_W(E^+)e_n = \sqrt{-c_{n+2}}e_{n+2}$$
$$d_W(E^-)e_n = -\sqrt{-c_n}e_{n-2}$$
$$d_W(H)e_n = ine_n.$$

From these and the relation above we then have

$$c_n - c_{n+2} = 4n.$$

From this relation it follows that if we know two successive coefficients we may recover n and all of the other coefficients. The subsets of $\mathbb{Z}$ over which n may vary in the above are:

(1) the set of all odd integers;
(2) the set of all even integers;
(3) $\{(k + 2p) \mid p \in \mathbb{N}\}$ where $k \in \mathbb{N}$, $k > 0$;
(4) $\{-(l + 2p) \mid p \in \mathbb{N}\}$ where $l \in \mathbb{N}$, $l > 0$.

(The set of numbers n appearing may be identified with the set of eigenvalues of $-id_W(H)$. The set of eigenvalues of $(i/2)d_W(H)$ is what is called the set of K-weights, $M(W)$, in [12].)

In case (3) above, the integer k, the smallest one appearing, is called the lowest weight, and the corresponding vector e_k is called a lowest weight vector. Highest weight and highest weight vector are defined similarly in case (4). Two representations with lowest (resp. highest) weight vectors are unitarily equivalent if and only if the lowest (resp. highest) weights are the same.

The operator $\omega = 2id_W(H) - d_W(H)^2 + d_W(E^+)d_W(E^-)$ is a scalar operator called the Casimir operator. For a representation in case (1), its eigenvalue is $c_1 - 1$, and for a representation in case (2), it is c_0. Two representations in case (1) (resp. case (2)) are unitarily equivalent if and only if the Casimir operator has the same eigenvalue in both of them.

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Department of Mathematical Sciences
Lakehead University
955 Oliver Road
Thunder Bay, Ontario
P7B 5E1
email: andrew.dean@lakeheadu.ca