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THE RIESZ INTERPOLATION PROPERTY FOR $K_0(A) \oplus K_1(A)$

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ABSTRACT. We show that if A is a C^*-algebra of real rank zero and stable rank one, then the Riesz interpolation property holds in the ordered group $K_0(A) \oplus K_1(A)$.

RÉSUMÉ. Nous montrons que si A est une C^*-algèbre de rang réel zéro et de rang stable égal à un, donc la propriété d'interpolation de Riesz est valable dans le groupe ordonné $K_0(A) \oplus K_1(A)$.

1. **Introduction.** The order structure on $K_0(A)$ played a major role in Elliott's classification [E2] of AF algebras, and also of course in more recent classification results, and the fact that $K_0(A)$ has the Riesz interpolation property when A is AF played a major role in the Effros–Handelman–Shen Theorem [EHS], which characterizes the K_0-groups of AF algebras. Later Dadarlat and Nemethi [DN] and Elliott [E3] introduced an order structure on $K_0(A) \oplus K_1(A)$, and this was important in shape theory and in classification theory for non-simple C^*-algebras.

S. Eilers suggested that I try to prove that $K_0(A) \oplus K_1(A)$ has the Riesz interpolation property when A has real rank zero and stable rank one. Elliott [E3] had proved this for the inductive limits arising in classification theory, and in [E3, Theorem 3.2] had stated the full result with an incorrect proof. Eilers and Elliott [EE] prove the result in some additional cases. The main ingredients used in the proof presented below are different from those used in [EE]. I am grateful to S. Eilers for suggesting the question and providing me with a preliminary version of [EE].

2. **Notations and preliminaries.** We use mainly standard notations. Thus, $\sim$ denotes Murray–von Neumann equivalence of projections in a C*-algebra, $p \lesssim q$ means $p \sim r \leq q$ for some r, $[p]$ denotes the class in $K_0(A)$ of a projection p in $A \otimes \mathcal{K}$, and $\mathcal{K}$ is the algebra of compact operators on a separable infinite dimensional Hilbert space. When no confusion will arise, the same letter ι is used to denote all inclusion maps.

Next we state, for ease of reference, several needed results, some of which are partly folklore and probably all of which are already known. Minimal indications of proof are given in a few cases.

2.1. (Cf. [Bl, 6.5.1]) If $\mathrm{tsr}(A) = 1$, then by cancellation of projections, to each α in $K_0(A)_+$ corresponds a unique Murray–von Neumann equivalence class $[p]$,

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for a projection p in $A \otimes \mathcal{K}$. If I is the (closed, two-sided) ideal of A such that $I \otimes \mathcal{K}$ is the ideal generated by p, then, in particular, I is an invariant of the class α.

2.2. By Rieffel [R2], C^*-algebras of stable rank one have K_1-surjectivity. Thus for α in $K_0(A)_+$ and β in $K_1(A)$, $\alpha \oplus \beta \geq 0$ if and only if β is in $\iota_*(K_1(I))$, where I is the ideal of α as in 2.1 and ι is the inclusion of I into A (see also 2.10 below).

2.3. ([LR]) If $\mathrm{tsr}(A) = 1$, then $\iota_* \colon K_0(I) \to K_0(A)$ is one-to-one for each ideal I of A.

2.4. ([LR]) If $\mathrm{RR}(A) = 0$, then $\iota_* \colon K_1(I) \to K_1(A)$ is one-to-one for each ideal I of A.

2.5. If I and J are ideals of an arbitrary C^*-algebra A, then there is a cyclic six-term exact sequence:

$$\xrightarrow{\partial_0} K_1(I \cap J) \longrightarrow K_1(I) \oplus K_1(J) \longrightarrow K_1(I + J) \xrightarrow{\partial_1}$$
$$\xrightarrow{\partial_1} K_0(I \cap J) \longrightarrow K_0(I) \oplus K_0(J) \longrightarrow K_0(I + J) \xrightarrow{\partial_0}$$

This follows easily from various standard cyclic six-term exact sequences. All the maps shown, except for ∂_i, come from inclusion maps.

2.6. If $\mathrm{tsr}(A) = 1$, then the natural map from $K_1(I) \oplus K_1(J)$ to $K_1(I + J)$ is surjective for all ideals I and J of A.

This follows from 2.3 and 2.5, together with the fact that the property of having stable rank one passes to ideals ([R2, Theorem 4.4]).

2.7. If $\mathrm{RR}(A) = 0$, then $\iota_{1*}(K_1(I_1)) \cap \iota_{2*}(K_1(I_2)) = \iota_*(K_1(I_1 \cap I_2))$, for all ideals I_1 and I_2 of A.

Note that if $\iota_{1*}(\alpha_1) = \iota_{2*}(\alpha_2)$ in $K_1(A)$, then by 2.4 also $j_{1*}(\alpha_1) = j_{2*}(\alpha_2)$ in $K_1(I_1 + I_2)$. Then apply 2.5.

2.8. If $\mathrm{tsr}(A) = 1$ and $\mathrm{RR}(A) = 0$, then $K_0(A)$ has the Riesz interpolation property.

The Riesz decomposition property for equivalence classes of projections was proved by Zhang [Zh], assuming only $\mathrm{RR}(A) = 0$. The stable rank hypothesis allows this to be re-stated in terms of the K_0-group [E3]. (Of course, Riesz decomposition and Riesz interpolation are equivalent in ordered groups.)

2.9. Let $\pi \colon A \to A/I$ be the quotient map where A has real rank zero and I is an ideal. Then of course every projection in A/I lifts to A. Also, as eAe has real rank zero for any projection e, a pair of mutually orthogonal projections in A/I can be lifted to orthogonal projections. And if u is a partial isometry in A/I and p, q are projections in A such that $\pi(p) \geq uu^*$ and $\pi(q) \geq u^*u$, then, as by [BP] qIq has real rank zero and in particular has an approximate unit consisting of projections, u can be lifted to a partial isometry in pAq. (See proof of Lemma 2.6 of [E1].)

2.10. (Cf. [Ex], [R1, Example 6.7]) Inclusions of full hereditary C^*-subalgebras induce isomorphisms of K-groups. Thus if B is a hereditary C^*-subalgebra of $A \otimes \mathcal{K}$ which generates the ideal $I \otimes \mathcal{K}$, then we can identify $K_i(B)$ with $K_i(I)$ in a consistent way.

2.11. If A has real rank zero, then any x in A can be approximated arbitrarily closely by an element x' that has closed range. In fact x' can be taken as xp for a suitable projection p in $(x^*xAx^*x)^-$. Thus if x is in qAr for some projections q and r, the partial isometry from the polar decomposition of x' will be in qAr.

3. Riesz interpolation for $K_0(A) \oplus K_1(A)$.

LEMMA 3.1. *Suppose* $\mathrm{RR}(A) = 0$, $\mathrm{tsr}(A) = 1$, $\alpha_1, \ldots, \alpha_n \in K_1(A)$, $I_1, \ldots, I_n$ *are ideals of* A *and*

$$\alpha_i \equiv \alpha_j \text{ modulo } \iota_*(K_1(I_i + I_j)), \quad \text{for each } i, j. \tag{1}$$

Then there is β *in* $K_1(A)$ *such that*

$$\beta \equiv \alpha_i \text{ modulo } \iota_*(K_1(I_i)), \quad \text{for each } i. \tag{2}$$

PROOF. Use induction on n. The case $n = 2$ follows directly from 2.6. Now assume $n > 2$, and the result is valid for $n - 1$. By 2.6, there is γ_i in $K_1(A)$, $i = 1, \ldots, n-1$, such that

$$\gamma_i \equiv \alpha_i \text{ modulo } \iota_*(K_1(I_i)) \quad \text{and} \quad \gamma_i \equiv \alpha_n \text{ modulo } \iota_*(K_1(I_n)). \tag{3}$$

Apply the induction hypothesis to $\gamma_1, \ldots, \gamma_{n-1}$ and $I_1 \cap I_n, \ldots, I_{n-1} \cap I_n$. Note that 2.7 implies that (1) is satisfied for the γ's. ■

REMARK. A symmetrical proof shows that the same result is valid for K_0.

LEMMA 3.2. *Let* A *be a* C^**-algebra of real rank zero.*

(a) *If* $\alpha \in K_1(A)$*, then there is a projection* p *in* A *such that* pAp *has no (non-trivial) abelian quotients and* $\alpha \in \iota_*(K_1(pAp))$*. In particular* $\alpha \in \iota_*(K_1(I))$*, where* I *is the ideal generated by* p*.*
(b) *If* J *is any ideal such that the primitive ideal space of* J *is compact, then* J *is generated by a projection.*

PROOF. (a) Let J be the closed commutator ideal of A. Then A/J is a commutative C^*-algebra of real rank zero, and hence $K_1(A/J) = 0$. Therefore

$\alpha = \iota_*(\beta)$ for some β in $K_1(J)$. Note also that if I is any ideal of J, then I has no abelian quotients.

Let $(p_j)_{j\in D}$ be a net of projections which is a (not necessarily increasing) approximate identity for J, and let I_j be the ideal generated by p_j. Then $\{I_j : j \in D\}$ is directed upward. In fact, for any j_1, j_2, we have for sufficiently large j that $\|(1-p_j)p_{j_1}\|$, $\|(1-p_j)p_{j_2}\| < 1$. For such j, $p_{j_1}, p_{j_2} \lesssim p_j$, and hence $I_{j_1}, I_{j_2} \subset I_j$. Let $\tilde{D}$ be D with a modified order relation:

$$j_1 \le j_2 \text{ in } \tilde{D} \iff j_1 \le j_2 \text{ in } D \text{ and } I_{j_1} \subset I_{j_2}. \tag{4}$$

Then

$$J = \varinjlim_{j\in\tilde{D}} I_j, \text{ and hence } K_1(J) = \varinjlim_{j\in\tilde{D}} K_1(I_j).$$

It follows that β is in $\iota_*(K_1(I_{j_0}))$ for suitable j_0.

Now let (q_k) be a net of projections which is an approximate identity for I_{j_0} and such that $q_k \ge p_{j_0}$ for each k. We know that I_{j_0} has no abelian quotients and claim that $q_k I_{j_0} q_k = q_k A q_k$ has no abelian quotients for k sufficiently large. To see this, let X be the maximal ideal space of the abelianization of $p_{j_0} I_{j_0} p_{j_0}$. Thus X is a compact Hausdorff space which may be regarded as a subset of $\hat{I}_{j_0}$. (It is a closed subset of $\hat{I}_{j_0}$.) If π_x is an irreducible representation of I_{j_0} corresponding to x in X, then $\dim \pi_x > 1$. It follows that $\operatorname{rank}(\pi_x(q_k)) > 1$ for sufficiently large k. Since $U_k = \{x \in X : \operatorname{rank}(\pi_x(q_k)) > 1\}$ is open, there are $k_1, \ldots, k_n$ such that $X \subset \bigcup_1^n U_{k_i}$. For k sufficiently large, $q_{k_i} \lesssim q_k$ for $i = 1, \ldots, n$. Thus for such a k, $q_k I_{j_0} q_k$ has no abelian quotients.

(b) Let (p_j) be an approximate identity of projections for J, and use the previous notations. Since $\{\operatorname{prim}(I_j)\}$ is an open cover for $\operatorname{prim} J$, and since $\{\operatorname{prim}(I_j)\}$ is directed upward, it follows that $\operatorname{prim} I_j = \operatorname{prim} J$ for some j. Hence $I_j = J$. ■

LEMMA 3.3. *If A is a C^*-algebra of real rank zero, and if p and q are projections generating the same ideal I, then there is a projection r in A such that $r \lesssim p$, $r \lesssim q$, and r generates I as an ideal.*

PROOF. Set

$$E = \{(s,t) : s \text{ and } t \text{ are projections, } s \le p,\ t \le q, \text{ and } s \sim t\}.$$

For each (s,t) in E, consider the ideal $I_{(s,t)}$ generated by s (or, equivalently, by t). Let us first show that $I = \bigvee_{(s,t)\in E} I_{(s,t)}$. If this is false, then there is a proper ideal J of I such that $s, t \in J$ for all $(s,t) \in E$. Let $\pi \colon I \to I/J$ denote the quotient map. Then $\pi(p)(I/J)\pi(q)$ is non-trivial. Thus by 2.11 there is a non-zero partial isometry $\overline{u}$ in $\pi(p)(I/J)\pi(q)$, and by 2.9 $\overline{u}$ can be lifted to a partial isometry u in pIq. This is a contradiction, since it implies $(uu^*, u^*u) \in E$ and $u \in J$.

Now since prim I is compact, there are $(s_1,t_1),\dots,(s_n,t_n)$ in E such that I is generated by $\{s_1,\dots,s_n\}$. We construct recursively projections $r_1,\dots,r_n$ such that $r_j \le p$, $r_j \lesssim q$, and r_k generates the same ideal as $\{s_1,\dots,s_k\}$ for each k. To start we may take $r_1 = s_1$. Now if r_k has been constructed for $k < n$ let J_k denote the ideal generated by r_k and $\pi_k\colon I \to I/J_k$ the quotient map, let $r_k \sim r'_k \le q$, and let u be a partial isometry such that $u^*u = s_{k+1}$ and $uu^* = t_{k+1}$. Since $\pi_k(p) = \pi_k(p - r_k)$ and $\pi_k(q) = \pi_k(q - r'_k)$, there is by 2.9 a partial isometry v in $(q - r'_k)I(p - r_k)$ such that $\pi_k(v) = \pi_k(u) = \pi_k((q - r'_k)u(p - r_k))$. We then can take $r_{k+1} = r_k + v^*v$. Finally, we take $r = r_n$. ∎

Lemma 3.4. *Let A be a C^*-algebra with compact primitive ideal space, real rank zero, and with no abelian quotients. Then there are full projections p, q in A such that $pq = 0$.*

Proof. Set

$$E = \{(s,t) : s \text{ and } t \text{ are projections}, st = 0, \text{ and } s \sim t\}.$$

For (s,t) in E denote by $I(s,t)$ the ideal generated by s. We first show that $A = \bigvee_{(s,t)\in I} I(s,t)$. If this is false, there is a proper ideal J such that $s,t \in J$ for all $(s,t) \in E$. Let $\pi\colon A \to A/J$ be the quotient map. Since A/J is a non-abelian C^*-algebra of real rank zero, there is a projection $\overline{r}$ in A/J which is not central. Then by 2.11 there is a non-zero partial isometry $\overline{u}$ in $(1-\overline{r})(A/J)\overline{r}$, and $\overline{u}$ can be lifted by 2.9 to a partial isometry u in $(1-r)Ar$, where r is a lift of $\overline{r}$. This is a contradiction, since $(u^*u, uu^*) \in E$.

Now since prim A is compact, there are $(s_1,t_1),\dots,(s_n,t_n)$ in E such that A is generated as an ideal by $\{s_1,\dots,s_n\}$. We recursively construct $p_1,\dots,p_n$ such that p_k generates the same ideal as $\{s_1,\dots,s_k\}$ and $p_k \lesssim 1 - p_k$. To start we take $p_1 = s_1$. Now if p_k has been constructed for $k < n$, denote by J_k the ideal generated by p_k and $\pi_k\colon A \to A/J_k$ the quotient map, let $p_k \sim q_k \le 1 - p_k$, and let u be a partial isometry such that $u^*u = s_{k+1}$ and $uu^* = t_{k+1}$. Since the restriction of π_k to $(1 - p_k - q_k)A(1 - p_k - q_k)$ is surjective, there is by 2.9 a partial isometry v in $(1 - p_k - q_k)A(1 - p_k - q_k)$ such that $\pi_k(v) = \pi_k(u) = \pi_k((1-p_k-q_k)u(1-p_k-q_k))$ and $v^*v \cdot vv^* = 0$. Then we can take $p_{k+1} = p_k + v^*v$, to complete the recursion, and finally $p = p_n$. ∎

Theorem 3.5. *If A is a C^*-algebra of real rank zero and stable rank one, then the ordered group $K_0(A) \oplus K_1(A)$ satisfies the Riesz interpolation property.*

Proof. We are given $\gamma_i = \alpha_i \oplus \beta_i$ in $K_0(A) \oplus K_1(A)$, for $i = 1,\dots,4$, such that $\gamma_1,\gamma_2 \le \gamma_3,\gamma_4$. We wish to find δ such that $\gamma_1,\gamma_2 \le \delta \le \gamma_3,\gamma_4$. First apply 2.8 to obtain α in $K_0(A)$ with $\alpha_1,\alpha_2 \le \alpha \le \alpha_3,\alpha_4$. Let I_1, I_2 be the ideals associated to the elements $\alpha - \alpha_1$, $\alpha - \alpha_2$ of $K_0(A)_+$, as in 2.1, and let I_3, I_4 be the ideals for $\alpha_3 - \alpha$, $\alpha_4 - \alpha$. Thus the ideals for $\alpha_3 - \alpha_1$, $\alpha_3 - \alpha_2$, $\alpha_4 - \alpha_1$, $\alpha_4 - \alpha_2$ are $I_1 + I_3$, $I_2 + I_3$, $I_1 + I_4$, and $I_2 + I_4$. Since $\gamma_3 - \gamma_1, \gamma_3 - \gamma_2, \gamma_4 - \gamma_1,$

$\gamma_4-\gamma_2 \geq 0$, we see from 2.2 that $\beta_3 \equiv \beta_1$ modulo $\iota_*(K_1(I_1+I_3))$, *etc.* We would like to find β in $K_1(A)$ such that $\beta \equiv \beta_i$ modulo $\iota_*(K_1(I_i))$. Unfortunately, two of the six conditions in (1) of Lemma 3.1 are missing, and so we will have to modify α before using 3.1.

Since $\beta_1 \equiv \beta_3 \mod \iota_*(K_1(I_1+I_3))$ and $\beta_2 \equiv \beta_3 \mod \iota_*(K_1(I_2+I_3))$, then $\beta_1 \equiv \beta_2 \mod \iota_*(K_1(I_1+I_2+I_3))$. Similarly, $\beta_1 \equiv \beta_2 \mod \iota_*(K_1(I_1+I_2+I_4))$. With the help of 2.7 we see in this way that

$$\begin{aligned} &\beta_1 \equiv \beta_2 \text{ modulo } \iota_*\big(K_1(I_1+I_2+(I_3\cap I_4))\big), \quad \text{and} \\ &\beta_3 \equiv \beta_4 \text{ modulo } \iota_*\big(K_1(I_3+I_4+(I_1\cap I_2))\big). \end{aligned} \tag{5}$$

Now 2.6 implies that $\beta_1-\beta_2 = \beta'+\beta''$ where $\beta' \in \iota_*(K_1(I_1+I_2))$ and $\beta'' \in \iota_*(K_1(I_3\cap I_4))$. Let p be a projection in $A\otimes\mathcal{K}$ such that $[p]=\alpha_3-\alpha$, and apply Lemma 3.2 (a) to β'' and $A_1 = p((I_3\cap I_4)\otimes\mathcal{K})p$. Note that p is full in $I_3\otimes\mathcal{K}$, and hence A_1 is full in $(I_3\cap I_4)\otimes\mathcal{K}$. Hence by 2.10 we may consider β'' as an element of $K_1(A_1)$. We obtain a projection p_1 generating an ideal $J_1\otimes\mathcal{K}$, where $J_1 \subset I_3\cap I_4$, $p_1 \leq p$, and $\beta'' \in \iota_*(K_1(J_1))$. Next, let q be a projection in $A\otimes\mathcal{K}$ such that $[q]=\alpha_4-\alpha$, and apply Lemma 3.2 (b) to $A_1' = q(J_1\otimes\mathcal{K})q$. (Note that prim J_1 is compact.) We obtain a projection $q_1 \leq q$ which generates A_1' as an ideal.

Now Lemma 3.3 gives us a full projection r_1 in $J_1\otimes\mathcal{K}$ such that $r_1 \lesssim p_1$ and $r_1 \lesssim q_1$. Thus if $A_2 = r_1(A\otimes\mathcal{K})r_1 = r_1(J_1\otimes\mathcal{K})r_1$, we may by 2.10 and 2.4 consider β'' as an element of $K_1(A_2)$. Applying 3.2 (a) again, we obtain an ideal $J_2 \subset J_1$ such that $J_2\otimes\mathcal{K}$ is generated by r_2 for some projection $r_2 \leq r_1$, $\beta'' \in \iota_*(K_1(J_2))$, and $r_2(J_2\otimes\mathcal{K})r_2$ has no abelian quotients. Finally, Lemma 3.4 gives a projection $r \leq r_2$ such that both r and r_2-r are full in $J_2\otimes\mathcal{K}$.

Now we replace α by $\alpha' = \alpha+[r]$. Since $r \lesssim p$ and $r \lesssim q$, it follows that $\alpha' \leq \alpha_3, \alpha_4$. Then we obtain ideals I_1', I_2', I_3', I_4' from α', as above, and (5) still holds for these new ideals. Clearly, for $j=1,2$, $I_j' = I_j + J_2$. Therefore $\beta_1 \equiv \beta_2$ modulo $\iota_*(K_1(I_1'+I_2'))$. Also since $[r_2-r] \leq [p]-[r]$, $[q]-[r]$, and since r is in the ideal generated by r_2-r, we see that $I_3' = I_3$ and $I_4' = I_4$.

Next we perform a similar process starting with $\beta_3-\beta_4$, which is in $\iota_*(K_1(I_3'+I_4')) + \iota_*(K_1(I_1'\cap I_2'))$. This time we replace α' by $\alpha'' \leq \alpha'$, obtaining four new ideals I_1'', I_2'', I_3'', I_4'', where $I_j'' = I_j'$ for $j=1,2$ and I_j' is large enough for $j=3,4$ to yield that $\beta_3 \equiv \beta_4$ modulo $\iota_*(K_1(I_3''+I_4''))$. Since all the hypotheses of Lemma 3.1 are now met, there is β in $K_1(A)$ such that $\beta \equiv \beta_j$ modulo $\iota_*(K_1(I_j''))$ for $j=1,\dots,4$. Then if $\delta = \alpha''\oplus\beta$, we have $\gamma_1,\gamma_2 \leq \delta \leq \gamma_3,\gamma_4$, as desired. ■

4. **Concluding remarks.** Consider the set G_{01} of pairs (p,u) where p is a projection in $A\otimes\mathcal{K}$ and u a unitary in $p(A\otimes\mathcal{K})p$. One can define an equivalence relation $\sim$ on G_{01} which is analogous to Murray–von Neumann equivalence:

$$\begin{aligned} &(p,v)\sim(q,w) \text{ if there is } u \text{ in } A\otimes\mathcal{K} \text{ such that } u^*u=p, \\ &\qquad uu^*=q, \text{ and } uvu^* \text{ is homotopic to } w \text{ in } \mathcal{U}(q(A\otimes\mathcal{K})q). \end{aligned} \tag{6}$$

Also we write $(p, v) \leq (q, w)$ in G_{01} if

$$(7) \quad p \leq q \quad \text{and} \quad w = v + v' \quad \text{for some unitary } v' \text{ in } (q-p)(A \otimes \mathcal{K})(q-p).$$

Then we write $(p, v) \lesssim (q, w)$ to mean $(p, v) \leq (r, u) \sim (q, w)$ for some (r, u) in G_{01}. If $D_{01} = G_{01}/\sim$, then $\lesssim$ induces a possibly improper partial ordering on the semigroup D_{01}. All of this is in agreement with the definitions made by Elliott in [E3, p. 182], though described a little differently.

Since Zhang [Zh] proved Riesz decomposition for equivalence classes of projections, assuming only $\mathrm{RR}(A) = 0$, it is natural to ask whether Riesz decomposition holds in D_{01} in this generality. Or, failing that, is there a property weaker than stable rank one which, together with real rank zero, implies Riesz decomposition in D_{01}? These are only questions, not conjectures. It is still unknown whether real rank zero implies K_1-surjectivity. It is not clear whether K_1-surjectivity is actually related to Riesz interpolation for $K_0(A) \oplus K_1(A)$, but K_1-surjectivity does occur in all proofs known to me.

When A has stable rank one, D_{01} can be identified with $(K_0(A) \oplus K_1(A))_+$. Thus when A also has real rank zero, the Riesz decomposition property holds for D_{01}. This restatement of Theorem 3.5 is more operator theoretic in tone, but it also is more complicated than one might guess. If we wanted to prove the basic result in this notation, we might first apply [Zh] to obtain $p \sim p_1 \oplus p_2$, $p_1 \leq q_1$, $p_2 \leq q_2$. In order to deal with the K_1-components, we would need to consider the ideals J_1 and J_2 generated by p_1 and p_2 and the ideals K_1 and K_2 generated by $q_1 - p_1$ and $q_2 - p_2$. The intersections $J_1 \cap K_2$ and $J_2 \cap K_1$ would then play the same roles as $I_1 \cap I_2$ and $I_3 \cap I_4$ in the proof presented above.

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