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THE BIVARIATE F_1-BETA DISTRIBUTION

SARALEES NADARAJAH AND SAMUEL KOTZ

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Abstract. A new bivariate beta distribution based on the Appell function of the first kind is introduced. Various representations are derived for its product moments, marginal densities, marginal moments, conditional densities and conditional moments.

Résumé. Une nouvelle distribution beta conjointe pour deux variables basée sur la fonction de Appel du premier genre est présentée. Des représentations diverses sont développées pour calculer les moments de produit, de densités marginales, de moments marginaux, de densités conditionnelles ainsi que de moments conditionnels.

1. **Introduction.** There have been very few bivariate beta distributions proposed in the statistics literature, see [6, Chapter 9], [3, Chapter 4], and [8, Chapter 49] for good reviews. The most recent bivariate beta distribution has been proposed by Olkin and Liu [9]. These distributions have attracted useful applications in several areas; for example, in the modeling of the proportions of substances in a mixture, brand shares, *i.e.*, the proportions of brands of some consumer product that are bought by customers (Chatfield, [4]), proportions of the electorate voting for the candidate in a two-candidate election (Hoyer and Mayer, [5]) and the dependence between two soil strength parameters (A-Grivas and Asaoka, [1]). They have also been used extensively as a prior in Bayesian statistics (see, for example, Apostolakis and Moieni [2]).

In this note, we introduce a new bivariate beta distribution and study its properties. The joint pdf of this new distribution is taken to be

$$f(x,y) = \frac{C x^{\beta-1} y^{\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1}}{(1-ux-vy)^{\alpha}} \tag{1}$$

for $0 < x < 1$, $0 < y < 1$, $0 < x + y < 1$, $\alpha > 0$, $\beta > 0$, $\beta' > 0$ and $\gamma > \beta + \beta'$, where C denotes the normalizing constant. Application of equation (3.1.2.6) in [10] shows that one can determine C as

$$\frac{1}{C} = \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta'; \gamma; u, v), \tag{2}$$

where F_1 is the Appell function of the first kind defined by

$$F_1(a, b, b'; c; z, \xi) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{k+l} (b)_k (b')_l z^k \xi^l}{(c)_{k+l} k! l!},$$

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where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. Because of this, we refer to (1) as the F_1-*beta distribution.* Note that if $u=0$ and $v=0$ then (1) reduces to the usual bivariate beta distribution. The derivatives of $\log f$ with respect to x and y are

$$(3) \qquad \frac{\partial \log f}{\partial x} = \frac{\beta-1}{x} - \frac{\gamma-\beta-\beta'-1}{1-x-y} + \frac{\alpha u}{1-ux-vy}$$

and

$$(4) \qquad \frac{\partial \log f}{\partial y} = \frac{\beta'-1}{y} - \frac{\gamma-\beta-\beta'-1}{1-x-y} + \frac{\alpha v}{1-ux-vy},$$

respectively. Setting (3) and (4) to zero, one notes that the critical points of (1) are given by the simultaneous solutions of the two quadratic equations

$$u(\gamma-\beta'-\alpha-2)x^2 + \{\alpha u(1-y) - (\gamma-\beta-\beta'-1)(1-vy) \\ - (\beta-1)(1+u-uy-vy)\}x + (\beta-1)(1-y)(1-vy) = 0$$

and

$$v(\gamma-\beta-\alpha-2)y^2 + \{\alpha v(1-x) - (\gamma-\beta-\beta'-1)(1-ux) \\ - (\beta'-1)(1+v-ux-vx)\}y + (\beta'-1)(1-x)(1-ux) = 0.$$

Thus, (1) can exhibit up to four critical points. Figures 1 and 2 below illustrate some possible shapes of (1).

In the rest of this note, we derive various representations for the product moments, marginal densities, marginal moments, conditional densities and conditional moments associated with (1).

2. **Product moments.**

Theorems 1 and 2 derive two representations for the product moments of (1). The first is expressed in terms of the Appell function of the first kind while the second representation is an infinite series of Gauss hypergeometric functions.

THEOREM 1. *The product moment of X and Y associated with* (1) *is given by*

$$(5) \quad E(X^mY^n) = \frac{C\Gamma(m+\beta)\Gamma(n+\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(m+n+\gamma)} \times \\ F_1(\alpha, m+\beta, n+\beta'; m+n+\gamma; u, v)$$

for any real $m>0$ and $n>0$.

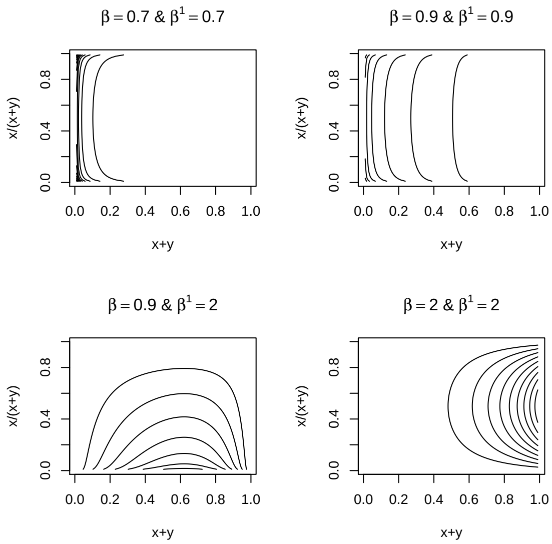

Figure 1: Plots of the pdf of (1) for $\gamma = 5$, $\alpha = 2$, $u = 0.5$, $v = 0.5$ and selected values of (β, β').

Proof. One can write

$$E(X^m Y^n) = C \int_0^1 \int_0^{1-x} \frac{x^{m+\beta-1} y^{n+\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1}}{(1-ux-vy)^\alpha} \, dy \, dx. \tag{6}$$

The result of the theorem follows by applying equation (3.1.2.6) in [10] to calculate the integral in (6). ■

Theorem 2. *The product moment of X and Y associated with* (1) *is given by*

$$E(X^m Y^n) = CB(n+\beta', \gamma-\beta-\beta') \sum_{k=0}^{\infty} \frac{(n+\beta')_k (\alpha)_k v^k}{(n+\gamma-\beta)_k k!} \times$$
$$B(m+\beta, n+k+\gamma-\beta) \; {}_2F_1(m+\beta, \alpha+k; m+n+k+\gamma; u) \tag{7}$$

for any real $m > 0$ and $n > 0$, where ${}_2F_1$ is the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}.$$

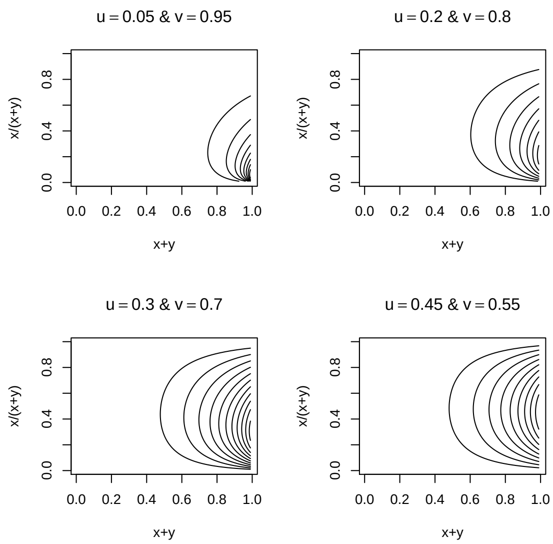

Figure 2: Plots of the pdf of (1) for $\beta = 2$, $\beta' = 2$, $\gamma = 5$, $\alpha = 2$ and selected values of (u, v).

PROOF. Consider the integral with respect to y in (6). Applying equation (2.2.6.15) in [10], one can reduce (6) to

$$E(X^m Y^n) = CB(n+\beta', \gamma-\beta-\beta') \int_0^1 \frac{x^{m+\beta-1}(1-x)^{n+\gamma-\beta-1}}{(1-ux)^\alpha} \times {}_2F_1\left(n+\beta', \alpha; n+\gamma-\beta; \frac{(1-x)v}{1-ux}\right) dx \tag{8}$$

Using the definition of the Gauss hypergeometric function, (8) can be rewritten as

$$\begin{aligned} E(X^m Y^n) &= CB(n+\beta', \gamma-\beta-\beta') \int_0^1 \frac{x^{m+\beta-1}(1-x)^{n+\gamma-\beta-1}}{(1-ux)^\alpha} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(n+\beta')_k (\alpha)_k (1-x)^k v^k}{(n+\gamma-\beta)_k k!\,(1-ux)^k} \, dx \\ &= CB(n+\beta', \gamma-\beta-\beta') \sum_{k=0}^{\infty} \frac{(n+\beta')_k (\alpha)_k v^k}{(n+\gamma-\beta)_k k!} \\ &\quad \times \int_0^1 \frac{x^{m+\beta-1}(1-x)^{n+k+\gamma-\beta-1}}{(1-ux)^{\alpha+k}} \, dx. \end{aligned} \tag{9}$$

The result in (7) follows by another application of equation (2.2.6.15) in [10] to calculate the integral in (9). ■

3. Marginal pdfs and moments. Theorems 3 and 4 derive the marginal pdfs and marginal moments of (1). Expressions for the pdfs involve the Gauss hypergeometric function while the moments are expressed in terms of the Appell function.

THEOREM 3. *If X and Y have the joint pdf (1) then the marginal pdfs are given by*

(10)
$$f_X(x) = CB(\beta', \gamma-\beta-\beta')x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-ux)^{-\alpha}\ {}_2F_1\left(\beta', \alpha; \gamma-\beta; \frac{(1-x)v}{1-ux}\right)$$

and

(11)
$$f_Y(y) = CB(\beta, \gamma-\beta-\beta')y^{\beta'-1}(1-y)^{\gamma-\beta'-1}(1-vy)^{-\alpha}\ {}_2F_1\left(\beta, \alpha; \gamma-\beta'; \frac{(1-y)u}{1-vy}\right)$$

for $0 < x < 1$ and $0 < y < 1$.

PROOF. The marginal pdf of X can be written as

$$f_X(x) = Cx^{\beta-1}\int_0^{1-x} \frac{y^{\beta'-1}(1-x-y)^{\gamma-\beta-\beta'-1}}{(1-ux-vy)^{\alpha}}\, dy. \tag{12}$$

The result in (10) follows by applying equation (2.2.6.15) in [10] to calculate the integral in (12). The result in (11) follows similarly. ■

THEOREM 4. *The moments of the marginal pdfs in (10) and (11) are given by*

$$E(X^m) = \frac{C\Gamma(m+\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(m+\gamma)} F_1(\alpha, m+\beta, \beta'; m+\gamma; u, v)$$

and

$$E(Y^n) = \frac{C\Gamma(\beta)\Gamma(n+\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(n+\gamma)} F_1(\alpha, \beta, n+\beta'; n+\gamma; u, v)$$

for any real $m > 0$ and $n > 0$.

PROOF. Set $m = 0$ ($n = 0$) into (5) and simplify. ■

The pdfs in (10) and (11) belong to a generalized beta family. If $u = v = 0$ then both (10) and (11) reduce to standard beta pdfs. Writing (10) and (11) as

$$f_X(x) = CB(\beta', \gamma - \beta - \beta') \sum_{k=0}^{\infty} \frac{(\beta')_k (\alpha)_k v^k}{(\gamma - \beta)_k k!} x^{\beta - 1} (1-x)^{k+\gamma-\beta-1} (1-ux)^{-(\alpha+k)}$$

and

$$f_Y(y) = CB(\beta, \gamma - \beta - \beta') \sum_{k=0}^{\infty} \frac{(\beta)_k (\alpha)_k u^k}{(\gamma - \beta')_k k!} y^{\beta' - 1} (1-y)^{k+\gamma-\beta'-1} (1-vy)^{-(\alpha+k)},$$

respectively, one notes that the marginal pdfs are infinite mixtures of pdfs which belong to Libby and Novick's [7] generalized beta family. Also, if $u = 0$ (respectively, $v = 0$) then (11) (respectively, (10)) reduces to Libby and Novick's [7] generalized beta family.

4. **Conditional pdfs and moments.** Theorems 5 and 6 derive the conditional pdfs and conditional moments of (1). Expressions for both the pdfs and the moments involve the Gauss hypergeometric function.

THEOREM 5. *If X and Y have the joint pdf* (1) *then the conditional pdf of X given $Y = y$ is given by*

$$f_{X|Y}(x \mid y) = \frac{(1-vy)^{\alpha} x^{\beta-1} (1-x-y)^{\gamma-\beta-\beta'-1}}{B(\beta, \gamma-\beta-\beta')(1-y)^{\gamma-\beta'-1} \, {}_2F_1(\beta, \alpha; \gamma-\beta'; \frac{(1-y)u}{1-vy})} \tag{13}$$

for $0 < x < 1$. The conditional pdf of Y given $X = x$ is given by

$$f_{Y|X}(y \mid x) = \frac{(1-ux)^{\alpha} y^{\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1}}{B(\beta', \gamma-\beta-\beta')(1-x)^{\gamma-\beta-1} \, {}_2F_1(\beta', \alpha; \gamma-\beta; \frac{(1-x)v}{1-ux})} \tag{14}$$

for $0 < y < 1$.

PROOF. The proof follows immediately from (1) and Theorem 3. ■

THEOREM 6. *The moments of the conditional pdfs in* (13) *and* (14) *are given by*

$$E(X^m \mid y) = \frac{B(m+\beta, \gamma-\beta-\beta')(1-y)^m (1-vy)^{\alpha}}{B(\beta, \gamma-\beta-\beta') \, {}_2F_1(\beta, \alpha; \gamma-\beta'; \frac{(1-y)u}{1-vy})} \tag{15}$$

and

$$E(Y^n \mid x) = \frac{B(n+\beta', \gamma-\beta-\beta')(1-y)^m (1-ux)^{\alpha}}{B(\beta', \gamma-\beta-\beta') \, {}_2F_1(\beta', \alpha; \gamma-\beta; \frac{(1-x)v}{1-ux})} \tag{16}$$

for any real $m > 0$ and $n > 0$.

Proof. Using (13), one can write

$$E(X^m \mid y) = \frac{(1-vy)^\alpha (1-y)^{1+\beta'-\gamma}}{B(\beta, \gamma-\beta-\beta')\ {}_2F_1(\beta, \alpha; \gamma-\beta'; \frac{(1-y)u}{1-vy})} \times \int_0^{1-y} x^{m+\beta-1}(1-x-y)^{\gamma-\beta-\beta'-1}\, dx. \tag{17}$$

The integral in (17) in a beta-type integral and thus the result in (15). The result in (16) follows similarly. ■

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Department of Statistics
University of Nebraska
Lincoln, NE 68583
USA

Department of Engineering Management
and Systems Engineering
The George Washington University
Washington, DC 20052
USA