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BIEXTENSIONS AND 1-MOTIVES

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Résumé. Soit S un schéma. On définit la notion de biextension de 1-motifs par des 1-motifs. De plus, si $\mathcal{M}(S)$ désigne la catégorie Tannakienne engendrée par les 1-motifs sur S (en un sense géométrique), on définit les morphismes de $\mathcal{M}(S)$ du produit tensoriel de deux 1-motifs $M_1 \otimes M_2$ vers un 1-motif M_3, comme étant la classe d'isomorphismes des biextensions (M_1, M_2) par M_3. En généralisant cette définition, on obtient, modulo isogénies, la notion de morphisme de $\mathcal{M}(S)$ d'un produit tensoriel fini de 1-motifs vers un autre 1-motif.

Introduction. Let S be a scheme. Let G_i (for $i = 1, 2, 3$) be an extension of an abelian S-scheme A_i by an S-torus $Y_i(1)$. We first observe that in the topos $\mathbf{T}_{\mathrm{fppf}}$ the category of biextensions of (G_1, G_2) by G_3 is equivalent to the category of biextensions of the underlying abelian S-schemes (A_1, A_2) by the underlying S-torus $Y_3(1)$ (Theorem 1.1):

$$\mathbf{Biext}(G_1, G_2; G_3) \cong \mathbf{Biext}(A_1, A_2; Y_3(1)).$$

Let $\mathcal{M}(S)$ be the conjectural Tannakian category generated by 1-motives over S in a geometrical sense: objects are sub-quotients of sums of finite tensor products of 1-motives and their duals. If $S = \mathrm{Spec}(k)$ with k a field of characteristic 0 embeddable in $\mathbb{C}$, identifying 1-motives with their mixed realizations, we can identify $\mathcal{M}(k)$ with a Tannakian sub-category of an "appropriate" Tannakian category $\mathcal{MR}(k)$ of mixed realizations ([J90, I 2.1] or [D89, 1.10]). However this identification furnishes no new concrete information about the geometrical description of objects and morphisms of $\mathcal{M}(k)$. It would be great to have a description of the Tannakian category $\mathcal{M}(S)$ like the one given in [DG] for the Tannakian category of mixed Tate motives, but it seems that at the moment we don't have enough mathematical results for such a construction. Deligne pointed out to the author that if k is a field of characteristic 0, Brylinski [Br] has constructed a Tannakian category generated by 1-motives in term of (absolute) Hodge cycles.

We use biextensions in order to propose as *a candidate* for the morphisms of $\mathcal{M}(S)$ from the tensor product of two 1-motives $M_1 \otimes M_2$ to a third one M_3, the isomorphism classes of biextensions of (M_1, M_2) by M_3:

$$\mathrm{Hom}_{\mathcal{M}(S)}(M_1 \otimes M_2, M_3) = \mathrm{Biext}^1(M_1, M_2; M_3). \tag{0.1}$$

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If $S = \mathrm{Spec}(k)$ with k a field of characteristic 0, our definition is compatible with the corresponding notion of morphisms in the category $\mathcal{MR}(k)$ of mixed realizations:

$$\mathrm{Biext}^1(M_1, M_2; M_3) = \mathrm{Hom}_{\mathcal{MR}(k)}(\mathrm{T}(M_1) \otimes \mathrm{T}(M_2), \mathrm{T}(M_3)).$$

Following Deligne's philosophy of motives described in [D89, 1.11], our definition (0.1) furnishes *the geometrical origin* of the morphisms of $\mathcal{MR}(k)$ from the tensor product of the realizations of two 1-motives to the realization of another 1-motive. For example, if M is a 1-motive over k, the valuation map $ev_M \colon M \otimes M^\vee \to \mathbb{Z}(0)$ of M (which expresses the duality between M and its dual $M^\vee$ as objects of $\mathcal{M}(k)$) is the twist by $\mathbb{Z}(-1)$ of the Poincaré biextension $\mathcal{P}_M$ of M, which we see as morphism $M \otimes M^* \to \mathbb{Z}(1)$ of $\mathcal{M}(k)$. Therefore $ev_M = \mathcal{P}_M \otimes \mathbb{Z}(-1) \colon M \otimes M^\vee \to \mathbb{Z}(0)$ is the geometrical origin of the corresponding morphism $\mathrm{T}(M) \otimes \mathrm{T}(M^\vee) \to \mathrm{T}(\mathbb{Z}(0))$ in $\mathcal{MR}(k)$ which can therefore be called a *motivic morphism*.

We can extend definition (0.1) to a finite tensor product of 1-motives because a morphism from a finite tensor product $\bigotimes_1^l M_j$ of 1-motives to a 1-motive M involves only the quotient $\bigotimes_1^l M_j / \mathrm{W}_{-3}(\bigotimes_1^l M_j)$ of the mixed motive $\bigotimes_1^l M_j$ (Theorem 3.3).

A special case of definition (0.1) was already used in the computation of the unipotent radical of the Lie algebra of the motivic Galois group of a 1-motive defined over a field k of characteristic 0 (*cf.* [B03, (1.3.1)]).

In this paper S is a scheme.

1. Biextensions of extensions of abelian schemes by tori.

Let $\mathbf{T}_{\mathrm{fppf}}$ be the topos associated to the site of locally of finite presentation S-schemes, endowed with the fppf topology.

Theorem 1.1. *Let G_i (for $i = 1, 2, 3$) be an extension of an abelian S-scheme A_i by an S-torus $Y_i(1)$. We have the following equivalence of categories:*

$$\mathbf{Biext}(G_1, G_2; G_3) \cong \mathbf{Biext}(A_1, A_2; Y_3(1)).$$

Proof. We will prove the following equivalences of categories:

$$\begin{aligned} \mathbf{Biext}(G_1, G_2; Y_3(1)) &\cong \mathbf{Biext}(A_1, A_2; Y_3(1)) \\ \mathbf{Biext}(G_1, G_2; G_3) &\cong \mathbf{Biext}(G_1, G_2; Y_3(1)) \end{aligned} \tag{1.1}$$

By [SGA3, Exposé X, Corollary 4.5], we can suppose that tori are split (if necessary we localize over S for the étale topology). So, we can assume that $Y_3(1)$ is $\mathbb{G}_m^{\mathrm{rk}\, Y_3}$. Since the categories $\mathbf{Biext}(G_1, G_2; \mathbb{G}_m)$ and $\mathbf{Biext}(A_1, A_2; \mathbb{G}_m)$ are additive in the variable $\mathbb{G}_m$ (*cf.* [SGA7, I Exposé VII (2.4.2)]), it suffices to prove that

$$\mathbf{Biext}(G_1, G_2; \mathbb{G}_m) \cong \mathbf{Biext}(A_1, A_2; \mathbb{G}_m)$$

and this is done in [SGA7, Exposé VIII (3.6.1)].

According to [SGA7, Exposé VII 3.6.5 and (3.7.4)], from the exact sequence $0 \to Y_3(1) \to G_3 \to A_3 \to 0$, we have the long exact sequence

$$0 \to \mathrm{Biext}^0(G_1, G_2; Y_3(1)) \to \mathrm{Biext}^0(10.1.10)(G_1, G_2; G_3) \to \mathrm{Biext}^0(G_1, G_2; A_3)$$

$$\to \mathrm{Biext}^1(G_1, G_2; Y_3(1)) \to \mathrm{Biext}^1(G_1, G_2; G_3) \to \mathrm{Biext}^1(G_1, G_2; A_3) \to \cdots .$$

Hence, in order to prove the second equivalence of categories of (1.1), it is enough to show that

$$\mathrm{Biext}^0(G_1, G_2; A_3) = \mathrm{Biext}^1(G_1, G_2; A_3) = 0,$$

and this will be done in [B05]. ■

2. Biextensions of 1-motives by 1-motives.

Deligne [D75, (10.1.10)] defines a 1-*motive* $M = [X \xrightarrow{u} G]$ over S as

(1) an S-group scheme X which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module;
(2) an extension G of an abelian S-scheme A by an S-torus $Y(1)$, with cocharacter group Y;
(3) a morphism $u\colon X \to G$ of S-group schemes.

The weight filtration W_* on $M = [X \xrightarrow{u} G]$ is $\mathrm{W}_i(M) = M$ for each $i \geq 0$, $\mathrm{W}_{-1}(M) = [0 \to G]$, $\mathrm{W}_{-2}(M) = [0 \to Y(1)]$, and $\mathrm{W}_j(M) = 0$ for each $j \leq -3$. Let $M_i = [X_i \xrightarrow{u_i} G_i]$ (for $i = 1, 2, 3$) be a 1-motive over S.

Definition 2.1. A *biextension* $\mathcal{B}, \Psi_1, \Psi_2, \Psi, \lambda$ *of* (M_1, M_2) *by* M_3 consists of the following:

(1) A biextension of $\mathcal{B}$ of (G_1, G_2) by G_3.
(2) A trivialization (= biaddictive section) Ψ_1 (resp. Ψ_2) of the biextension $(u_1, \mathrm{id}_{G_2})^*\mathcal{B}$ (resp. $(\mathrm{id}_{G_1}, u_2)^*\mathcal{B}$) of (X_1, G_2) by G_3 (resp. (G_1, X_2) by G_3) obtained as pull-back of the biextension $\mathcal{B}$ via (u_1, id_{G_2}) (resp. (id_{G_1}, u_2)). These two trivializations Ψ_1 and Ψ_2 have to coincide over (X_1, X_2), *i.e.*,

$$(u_1, \mathrm{id}_{X_2})^*\Psi_2 = \Psi = (\mathrm{id}_{X_1}, u_2)^*\Psi_1$$

with Ψ a trivialization of the biextension $(u_1, u_2)^*\mathcal{B}$ of (X_1, X_2) by G_3 obtained as pull-back by (u_1, u_2) of the biextension $\mathcal{B}$.
(3) A morphism $\lambda\colon X_1 \times X_2 \to X_3$ of S-group schemes such that $u_3 \circ \lambda\colon X_1 \times X_2 \to G_3$ is compatible with the trivialization Ψ of the biextension $(u_1, u_2)^*\mathcal{B}$ of (X_1, X_2) by G_3.

Let $M_i = [X_i \xrightarrow{u_i} G_i]$ and $M'_i = [X'_i \xrightarrow{u'_i} G'_i]$ (for $i = 1, 2, 3$) be 1-motives over S. Moreover let $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$ be a biextension of (M_1, M_2) by M_3 and let $(\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$ be a biextension of (M'_1, M'_2) by M'_3.

Definition 2.2. A *morphism of biextensions*

$$(F, \Upsilon_1, \Upsilon_2, \Upsilon, g_3)\colon (\mathcal{B}, \Psi_1, \Psi_2, \lambda) \to (\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$$

consists of the following:

(1) A morphism $(F, f_1, f_2, f_3)\colon \mathcal{B} \to \mathcal{B}'$ from the biextension $\mathcal{B}$ to the biextension $\mathcal{B}'$. In particular,

$$f_1\colon G_1 \to G'_1 \quad f_2\colon G_2 \to G'_3 \quad f_3\colon G_3 \to G'_3$$

are morphisms of groups S-schemes.

(2) A morphism of biextensions

$$(\Upsilon_1, g_1, f_2, f_3)\colon (u_1, \mathrm{id}_{G_2})^*\mathcal{B} \to (u'_1, \mathrm{id}_{G'_2})^*\mathcal{B}'$$

compatible with the morphism (F, f_1, f_2, f_3) and with the trivializations Ψ_1 and Ψ'_1, and a morphism of biextensions

$$(\Upsilon_2, f_1, g_2, f_3)\colon (\mathrm{id}_{G_1}, u_2)^*\mathcal{B} \to (\mathrm{id}_{G'_1}, u'_2)^*\mathcal{B}'$$

compatible with the morphism (F, f_1, f_2, f_3) and with the trivializations Ψ_2 and Ψ'_2. In particular

$$g_1\colon X_1 \to X'_1 \quad g_2\colon X_2 \to X'_2$$

are morphisms of groups S-schemes. By pull-back, the two morphisms $(\Upsilon_1, g_1, f_2, f_3)$ and $(\Upsilon_2, f_1, g_2, f_3)$ define a morphism of biextensions $(\Upsilon, g_1, g_2, f_3)\colon (u_1, u_2)^*\mathcal{B} \to (u'_1, u'_2)^*\mathcal{B}'$ compatible with the morphism (F, f_1, f_2, f_3) and with the trivializations Ψ and Ψ'.

(3) A morphism $g_3\colon X_3 \to X'_3$ of S-group schemes compatible with u_3 and u'_3 (*i.e.*, $u'_3 \circ g_3 = f_3 \circ u_3$) and such that

$$\lambda' \circ (g_1 \times g_2) = g_3 \circ \lambda.$$

Remark 2.3. The pair (g_3, f_3) defines a morphism from M_3 to M'_3. The pairs (g_1, f_1) and (g_2, f_2) define morphisms from M_1 to M'_1 and from M_2 to M'_2 respectively.

2.1. A more useful definition. From now on we will work on the topos $\mathbf{T}_{\mathrm{fppf}}$ associated to the site of locally of finite presentation S-schemes, endowed with the fppf topology.

With Proposition 10.2.14 of [D75], Deligne furnishes a more symmetric description of 1-motives: consider the 7-tuple $(X, Y^\vee, A, A^*, v, v^*, \psi)$ where

- X and $Y^\vee$ are two S-group schemes which are locally for the étale topology constant group schemes defined by finitely generated free $\mathbb{Z}$-modules;
- A and A^* are two abelian S-schemes dual to each other;

- $v\colon X \to A$ and $v^*\colon Y^\vee \to A^*$ are two morphisms of S-group schemes;
- ψ is a trivialization of the pull-back $(v,v^*)^*\mathcal{P}_A$ via (v,v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A,A^*).

To have the data $(X,Y^\vee,A,A^*,v,v^*,\psi)$ is equivalent to having the 1-motive $M=[X \stackrel{u}{\to} G]$: in fact, to have the extension G is the same thing as to have the morphism $v^*\colon Y^\vee \to A^*$ (*cf.* [SGA7, Exposé VIII 3.7], where G corresponds to the biextension $(\mathrm{id}_A,v^*)^*\mathcal{P}_A$ of $(A,Y^\vee)$ by $\mathbb{G}_m$), and to have the morphism $u\colon X \to G$ is equivalent to having the morphism $v\colon X \to A$ and the trivialization ψ of $(v,v^*)^*\mathcal{P}_A$ (the trivialization ψ furnishes the lift $u\colon X \to G$ of the morphism $v\colon X \to A$).

Remark 2.4. The pull-back $(v,v^*)^*\mathcal{P}_A$ by (v,v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A,A^*) is a biextension of $(X,Y^\vee)$ by $\mathbb{G}_m$. According to [SGA3, Exposé X, Corollary 4.5], we can suppose that the character group $Y^\vee$ is constant, *i.e.*, $\mathbb{Z}^{\mathrm{rk}\,Y^\vee}$ (if necessary we localize over S for the étale topology). Moreover, since by [SGA7, Exposé VII (2.4.2)] the category **Biext** is additive in each variable, we have that

$$\mathbf{Biext}(X,Y^\vee;\mathbb{G}_m) \cong \mathbf{Biext}(X,\mathbb{Z};Y(1)).$$

We denote by $((v,v^*)^*\mathcal{P}_A)\otimes Y$ the biextension of $(X,\mathbb{Z})$ by $Y(1)$ corresponding to the biextension $(v,v^*)^*\mathcal{P}_A$ through this equivalence of categories. The trivialization ψ of $(v,v^*)^*\mathcal{P}_A$ defines a trivialization $\psi\otimes Y$ of $((v,v^*)^*\mathcal{P}_A)\otimes Y$, and vice versa.

Using Theorem 1.1 and the more symmetrical definition of 1-motives, we can now give a more useful definition of a biextension of two 1-motives by a third one:

Proposition 2.5. *Let* $M_i=(X_i,Y_i^\vee,A_i,A_i^*,v_i,v_i^*,\psi_i)$ *(for* $i=1,2,3$*) be a* 1*-motive. A biextension* $(B,\Psi_1',\Psi_2',\Psi',\Lambda)$ *of* (M_1,M_2) *by* M_3 *consists of the following:*

(1) *A biextension of* B *of* (A_1,A_2) *by* $Y_3(1)$*;*
(2) *A trivialization* Ψ_1' *(resp.* Ψ_2'*) of the biextension* $(v_1,\mathrm{id}_{A_2})^*B$ *(resp.* $(\mathrm{id}_{A_1},v_2)^*B$*) of* (X_1,A_2) *by* $Y_3(1)$ *(resp. of* (A_1,X_2) *by* $Y_3(1)$*) obtained as pull-back of the biextension* B *via* (v_1,id_{A_2}) *(resp. via* (id_{A_1},v_2)*). These two trivializations* Ψ_1' *and* Ψ_2' *have to coincide over* (X_1,X_2)*, i.e.,*

$$(v_1,\mathrm{id}_{X_2})^*\Psi_2' = \Psi' = (\mathrm{id}_{X_1},v_2)^*\Psi_1',$$

with Ψ' *a trivialization of the biextension* $(v_1,v_2)^*B$ *of* (X_1,X_2) *by* $Y_3(1)$ *obtained as pull-back of the biextension* B *via* (v_1,v_2)*.*

(3) *A morphism* $\Lambda\colon (v_1,v_2)^*B \to ((v_3,v_3^*)^*\mathcal{P}_{A_3})\otimes Y_3$ *of trivial biextensions, with* $\Lambda_{|Y_3(1)}$ *equal to the identity, such that the following diagram is commutative*

$$
\begin{array}{ccc}
Y_3(1) & = & Y_3(1) \\
\Big| & & \Big| \\
(v_1,v_2)^*B & \longrightarrow & ((v_3,v_3^*)^*\mathcal{P}_{A_3})\otimes Y_3 \\
{\scriptstyle \Psi'}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle \psi_3\otimes Y_3} \\
X_1\times X_2 & \longrightarrow & X_3\times\mathbb{Z}.
\end{array}
$$

3. Morphisms from a finite tensor product of 1-motives to a 1-motive.

Definition 3.1. In the category $\mathcal{M}(S)$, *the morphism* $M_1\otimes M_2 \to M_3$ *from the tensor product of two* 1*-motives to a third* 1*-motive* is an isomorphism class of biextensions of (M_1,M_2) by M_3.

In other words, *the biextensions of two* 1*-motives by a* 1*-motive are the "geometrical interpretation" of the morphisms of* $\mathcal{M}(S)$ *from the tensor product of two* 1*-motives to a* 1*-motive.* The formulas (1.1) shows that biextensions satisfy the main property of morphisms of motives: they respect weights.

Remark 3.2. Definition 2.2 of morphisms of biextensions of 1-motives by 1-motives allows us to define a morphism between the morphisms of $\mathcal{M}(S)$ corresponding to such biextensions. More precisely, let M_i and M_i' (for $i=1,2,3$) be 1-motives over S. If we denote b the morphism $M_1\otimes M_2\to M_3$ corresponding to a biextension $(\mathcal{B},\Psi_1,\Psi_2,\lambda)$ of (M_1,M_2) by M_3 and by b' the morphism $M_1'\otimes M_2'\to M_3'$ corresponding to a biextension $(\mathcal{B}',\Psi_1',\Psi_2',\lambda')$ of (M_1',M_2') by M_3', a morphism $(F,\Upsilon_1,\Upsilon_2,\Upsilon,g_3)\colon(\mathcal{B},\Psi_1,\Psi_2,\lambda)\to(\mathcal{B}',\Psi_1',\Psi_2',\lambda')$ of biextensions defines the vertical arrows of the following diagram of morphisms of $\mathcal{M}(S)$:

$$
\begin{array}{ccc}
M_1\otimes M_2 & \xrightarrow{\ b\ } & M_3 \\
\downarrow & & \downarrow \\
M_1'\otimes M_2' & \xrightarrow{\ b'\ } & M_3'.
\end{array}
$$

It is clear now why from the data $(F,\Upsilon_1,\Upsilon_2,\Upsilon,g_3)$ we get a morphism of $\mathcal{M}(S)$ from M_3 to M_3' as remarked in 2.3. Moreover since $M_1\otimes\mathbb{Z}(0)$, $M_1'\otimes\mathbb{Z}(0)$, $\mathbb{Z}(0)\otimes M_2$ and $\mathbb{Z}(0)\otimes M_2'$, are sub-1-motives of the motives $M_1\otimes M_2$ and $M_1'\otimes M_2'$, it is clear that from the data $(F,\Upsilon_1,\Upsilon_2,\Upsilon,g_3)$ we get morphisms from M_1 to M_1' and from M_2 to M_2' as remarked in 2.3.

We will denote by $\mathcal{M}^{\mathrm{iso}}(S)$ the Tannakian category generated by the iso-1-motives, *i.e.*, by 1-motives modulo isogenies.

THEOREM 3.3. *Let M and $M_1, \dots, M_l$ be 1-motives over S. In the category $\mathcal{M}^{\mathrm{iso}}(S)$, the morphism $\bigotimes_{j=1}^{l} M_j \to M$ from a finite tensor product of 1-motives to a 1-motive is the sum of copies of isomorphism classes of biextensions of (M_i, M_j) by M for $i, j = 1, \dots, l$ and $i \neq j$. We have that*

$$\mathrm{Hom}_{\mathcal{M}^{\mathrm{iso}}(S)}\Big(\bigotimes_{j=1}^{l} M_j, M\Big) = \sum_{\substack{i,j \in \{1,\dots,l\} \\ i \neq j}} \mathrm{Biext}^1(M_i, M_j; M).$$

PROOF. Because morphisms between motives have to respect weights, the nontrivial components of the morphism $\bigotimes_{j=1}^{l} M_j \to M$ are those of the morphism

$$\bigotimes_{j=1}^{l} M_j \Big/ \mathrm{W}_{-3}\Big(\bigotimes_{j=1}^{l} M_j\Big) \to M.$$

We can write this last morphism explicitly in the following way

$$\sum_{\substack{\iota_1 < \iota_2 \text{ and } \nu_1 < \cdots < \nu_{l-2} \\ \iota_1, \iota_2 \notin \{\nu_1, \dots, \nu_{l-2}\}}} X_{\nu_1} \otimes \cdots \otimes X_{\nu_{l-2}} \otimes (M_{\iota_1} \otimes M_{\iota_2} / \mathrm{W}_{-3}(M_{\iota_1} \otimes M_{\iota_2})) \to M.$$

Since "to tensorize a motive by a motive of weight 0" means to take a certain number of copies of the motive, from Definition 3.1 we get the expected conclusion. ■

REFERENCES

[**B03**] C. Bertolin, *Le radical unipotent du groupe de Galois motivique d'un 1-motif.* Math. Ann. **327**(2003), 585–607.

[**B05**] ———, *Multilinear morphisms between 1-motives*, In preparation, 2005.

[**Br**] J.-L. Brylinski, *1-motifs et formes automorphes (théorie arithmétique des domaines de Siegel).* Publ. Math. Univ. Paris VII **15**(1983), 43–106.

[**D75**] P. Deligne, *Théorie de Hodge. III.* Inst. Hautes Etudes Sci. Publ. Math. **44**(1974), 5–77. (French)

[**D89**] ———, *Le groupe fondamental de la droite projective moins trois points.* Math. Sci. Res. Inst. Publ. **16**(1989), 79–297.

[**DG**] P. Deligne and A. Goncharov, *Groupes fondamentaux motiviques de Tate mixte.* Preprint, 2003.

[**J90**] U. Jannsen, *Mixed motives and algebraic K-theory.* Lecture Notes in Math. **1400**, Springer, Berlin, 1990.

[**SGA3**] SGA3 II: Schémas en groupes, LN 152 (1970).

[**SGA7**] SGA7 I: Groupes de Monodromie en Géométrie Algébrique, LN 288 (1972).

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