C. R. Math. Rep. Acad. Sci. Canada Vol. **27**, (3), 2005 pp. 72–75

SOME ALGEBRAS OF BOUNDED FUNCTIONS ON THE DISC

ALEXANDER J. IZZO

Presented by G. A. Elliott, FRSC

ABSTRACT. Let B be a uniformly closed algebra of functions on the unit circle ∂D between H^∞ and L^∞, and let C_B be the C^*-algebra generated by those Blaschke products that are invertible in B. Let A be the algebra of bounded holomorphic functions on the open unit disc D whose boundary value functions are in C_B. It is shown that if f is a bounded harmonic nonholomorphic function on D whose boundary value function is also in C_B, then the uniformly closed algebra $A[f]$ generated by A and f contains $C(\overline{D})$. This generalizes an earlier result of the author, which in turn contains as special cases a result on the disc algebra due to Čirca and a result on $H^\infty(D)$ due to Axler and Shields.

RÉSUMÉ. Soit B une algèbre uniformément fermée de fonctions sur le cercle unité ∂D entre H^∞ et L^∞, et soit C_B la C^*-algèbre générée par les produits de Blaschke qui soient invertibles dans B. Soit A l'algèbre des fonctions holomorphiques bornées sur le disque unité ouvert D dont les fonctions de valeur à la borne se trouvent dans C_B. Il est démontré que, si f est une fonction non-holomorphique harmonique bornée sur D dont la fonction de valeur à la borne est aussi dans C_B, alors l'algèbre uniformément fermeé $A[f]$ générée par A et f renferme $C(\overline{D})$. Ceci généralise un résultat antérieur de l'auteur qui, à son tour, contient, en tant que cas particuliers, un résultat concernant l'algèbre disc dû à Čirca et un résultat sur $H^\infty(D)$ dû à Axler et Shields.

1. **The theorem.** Let D denote the open unit disc in the plane, let $C(\overline{D})$ denote the algebra of all complex-valued continuous functions on $\overline{D}$, let $A(D)$ denote the disc algebra (the algebra of holomorphic functions on D that extend continuously to $\overline{D}$), and let $H^\infty(D)$ denote the algebra of bounded holomorphic functions on D. A theorem of E. M. Čirca [Č] specialized to the disc asserts that if f is a function in $C(\overline{D})$ and f is harmonic but not holomorphic on D, then the uniformly closed subalgebra $A(D)[f]$ of $C(\overline{D})$ generated by $A(D)$ and f is equal to $C(\overline{D})$. There is an analogous result for $H^\infty(D)$ due to Sheldon Axler and Allen Shields [A–S]: If f is a bounded function on D that is harmonic but not holomorphic, then the uniformly closed subalgebra $H^\infty(D)[f]$ of $L^\infty(D)$ generated by $H^\infty(D)$ and f contains $C(\overline{D})$. Given the similarity between these two theorems it was natural to wonder whether they were in fact both special cases of a single more general theorem.

The most obvious way to try to unite the Čirca and Axler–Shields results would be to show that $A(D)[f] \supset C(\overline{D})$ whenever f is a bounded harmonic nonholomorphic function on D. However, this is false; in fact, it is not even true

Received by the editors on October 6, 2004.
AMS subject classification: 46J10, 46J15, 30H05.

that $A(D)[f,\bar{f}] \supset C(\overline{D})$ whenever $f \in H^\infty(D)$ [I1, Theorem 7.6]. Nevertheless, the author did unite the theorems of Čirca and Axler and Shields into a single concrete theorem by interpolating between them [I2, Theorem 1.1]: If E is a subset of ∂D, if A is the algebra of bounded holomorphic functions on D that extend continuously to $D \cup E$, and if f is a bounded harmonic nonholomorphic function on D that also extends continuously to $D \cup E$, then $A[f] \supset C(\overline{D})$.

In the present paper, we extend this result to a larger class of algebras. Throughout the paper, L^∞ will denote $L^\infty(\partial D)$, and H^∞ will denote the subalgebra of L^∞ consisting of those functions that arise as boundary values of functions in $H^\infty(D)$. In addition, B will always be an arbitrary, but fixed, uniformly closed algebra on the unit circle ∂D between H^∞ and L^∞, and C_B will be the C^*-algebra (*i.e.*, uniformly closed self-adjoint unital algebra) generated by those Blaschke products that are invertible in B. (We assume that B is strictly larger than H^∞ so that C_B is nontrivial.) The algebras $H^\infty \cap C_B$ were considered by Alice Chang and Donald Marshall [C–M] as generalizations of both the disc algebra and H^∞. We will prove the following result.

THEOREM 1.1. *Let A be the algebra of bounded holomorphic functions on D whose boundary value functions are in C_B. Let f be a bounded harmonic nonholomorphic function on D whose boundary value function is also in C_B. Then $A[f] \supset C(\overline{D})$.*

For E a subset of ∂D, denote by L^∞_E the set of L^∞ functions on ∂D that are continuous on E. Then $H^\infty + L^\infty_E$ is a uniformly closed algebra and $L^\infty_E = C_{(H^\infty + L^\infty_E)}$ [G, Exercise IX.15(c)]. Thus the earlier result of the author mentioned above is the special case of our new theorem when $B = H^\infty + L^\infty_E$. There is also a stronger form of the earlier result in which the function f is allowed to have discontinuities on a small subset of E [I2, Theorem 1.2]. That stronger result is not contained in the new theorem and will not be considered in this paper.

Considering the algebras C_B in the setting of [I2] was suggested to me by Donald Marshall, and I would like to thank him for the suggestion. The work was carried out while I was a visitor at Brown University. I would like to thank the Department of Mathematics for its hospitality.

2. The proof.

The proof of Theorem 1.1 is similar to the proof of the earlier special case [I2, Theorem 1.1] mentioned in the introduction. In particular the theorem will be obtained as a consequence a characterization, due to the author, of the uniformly closed algebras of bounded continuous functions on a bounded open set $\Omega \subset \mathbb{C}$ that contain $C(\overline{\Omega})$. Before giving the statement of the characterization we discuss some terminology and notation to be used throughout the paper.

By a uniformly closed algebra of bounded continuous functions on a space Σ, we mean a supremum-norm closed subalgebra of the algebra $C_b(\Sigma)$ of all bounded continuous complex-valued functions on Σ. That is, by a uniformly closed algebra, we mean an algebra of complex-valued functions that forms a Banach algebra under the supremum-norm. The maximal ideal space of such an algebra A will be denoted by $\mathcal{M}_A$. If A is such an algebra and f is a bounded continuous complex-valued function on Σ, we will denote by $A[f]$ the uniformly closed algebra generated by A and f. Also $r\colon \mathcal{M}_{A[f]} \to \mathcal{M}_A$ will denote the map sending each multiplicative linear functional on $A[f]$ to its restriction to A. We will be concerned almost exclusively with uniformly closed algebras of bounded continuous functions on D that contain the disc algebra. The Gelfand transform of a function f in A will be denoted by $\hat{f}$. The Gelfand transform of the identity function z will be denoted by π or π_A.

Here now is the statement of the author's characterization mentioned above [I1, Theorem 3.1].

THEOREM 2.1. *Let Ω be a bounded open set in the plane, and let A be a uniformly closed algebra of bounded continuous functions on Ω that contains $A(\Omega)$. Then $A \supset C(\overline{\Omega})$ if and only if both of the following conditions hold:*

(i) *$\pi\colon \mathcal{M}_A \to \overline{\Omega}$ is one-to-one over Ω, and*
(ii) *for almost every point a in Ω there is a function f in A that is differentiable at a and such that $(\partial f/\partial \bar{z})(a) \neq 0$.*

We will also use the following lemma [I1, Lemma 6.1].

LEMMA 2.2. *Let Σ be a topological space and let A be a uniformly closed algebra of bounded continuous functions on Σ that separates points and contains the constants. Suppose that f is a function in the uniform closure of the complex-linear span of $\log|A^{-1}|$. Then the mapping $r\colon \mathcal{M}_{A[f]} \to \mathcal{M}_A$ sending each multiplicative linear functional on $\mathcal{M}_{A[f]}$ to its restriction to A is one-to-one.*

A subalgebra H of L^∞ is called *stable* if H satisfies the following condition: for each $\lambda \in D$ and each function $h \in H$, the function

$$g = \frac{h - h(\lambda)}{z - \lambda}$$

is also in H. The following result is from David Dawson's dissertation [D, Theorem 5.4]. For the reader's convenience we include the short proof.

THEOREM 2.3. *Let S be a C^*-subalgebra of L^∞, and let $H = S \cap H^\infty$. Then*

(a) *if f and g are in H, g^{-1} is in L^∞, and fg^{-1} is in H^∞, then fg^{-1} is in H;*
(b) *if z is in S, then H is stable.*

PROOF. Because S is a C^*-subalgebra of L^∞, a function in S is invertible in S if and only if it is invertible in L^∞. To prove (a), take f and g in H and suppose g^{-1} is in L^∞. Then g^{-1} is in S, so fg^{-1} is also in S. Since fg^{-1} is assumed to be in H^∞, this gives fg^{-1} is in H. Part (b) follows from part (a) by taking $f = h - h(\lambda)$ and $g = z - \lambda$. ∎

By [D, Theorem 2.3] for a stable algebra H the mapping $\pi\colon \mathcal{M}_H \to \overline{D}$ is one-to-one over D. (The proof of this can also be found in [I2, Theorem 2.2].) Combining this with Theorem 2.3, we immediately obtain the following.

COROLLARY 2.4. *Let A be the algebra of bounded holomorphic functions on D whose boundary value functions are in C_B. Then $\pi\colon \mathcal{M}_A \to \overline{D}$ is one-to-one over D.*

Proof of Theorem 1.1 Since the set where the $\bar{z}$-derivative of a nonholomorphic harmonic function vanishes is discrete, to prove the theorem it is enough, by Theorem 2.1, to show that the mapping $\pi_{A[f]}\colon \mathcal{M}_{A[f]} \to \overline{D}$ is one-to-one over D. By Corollary 2.4, the mapping $\pi_A\colon \mathcal{M}_A \to \overline{D}$ is one-to-one over D. Thus, noting that $\pi_{A[f]} = \pi_A \circ r$, we see that it suffices to show that r is injective. By [C–M, Lemma 3.1], $H^\infty \cap C_B$ is a logmodular subalgebra of C_B, that is every real-valued function in C_B is in the uniform closure of $\log|(H^\infty \cap C_B)^{-1}|$. It follows that f is in the uniform closure of the complex-linear span of $\log|A^{-1}|$. Hence r is injective by Lemma 2.2, and the proof is complete.

References

[**A–S**] S. Axler and A. Shields, *Algebras generated by analytic and harmonic functions.* Indiana Univ. Math. J. **36**(1987), 631–638.

[**C–M**] S. Y. Chang and D. E. Marshall, *Some algebras of bounded analytic functions containing the disk algebra.* In: Banach Spaces of Analytic Functions (Proc. Pelczynski Conf., Kent State Univ., Kent, Ohio, 1976), Lecture Notes in Math. **604**, Springer-Verlag, Berlin, 1977, 12–20.

[**Č**] E. M. Čirka, *Approximation by holomorphic functions on smooth manifolds in $\mathbb{C}^n$.* Mat. Sb. **78**(1969), 101–123; English transl., Math. USSR Sb. **7**(1969), 95–114.

[**D**] D. D. Dawson, *Subalgebras of H^∞.* Ph.D. dissertation, Indiana University, 1975.

[**G**] J. B. Garnett, *Bounded Analytic Functions.* Academic Press, New York, 1981.

[**I1**] A. J. Izzo, *Algebras containing bounded holomorphic functions.* Indiana Univ. Math. J. **52**(2003), 1305–1342.

[**I2**] A. J. Izzo, *Algebras generated by holomorphic and harmonic functions on the disc.* Bull. London Math. Soc., to appear.

Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403
USA
email: aizzo@math.bgsu.edu