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A NON-VANISHING THEOREM ON DIRICHLET SERIES

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Abstract. The non-vanishing property of certain Dirichlet series is a fundamental problem in analytic number theory. In this paper, we provide a non-vanishing theorem, which is a generalization of Ogg's result. We apply our theorem to get applications on distributions of eigenvalues of Hecke eigenforms and recover the non-vanishing theorem for the L-functions of cuspidal representations.

Résumé. La propriété non nulle de certaines séries de Dirichlet est un problème fondamental dans la théorie analytique des nombres. Dans cet article, nous fournissons un théorème non-non-vanishing, qui est une généralisation du résultat d'Ogg. Nous appliquons notre théorème pour obtenir des applications sur des distributions des valeurs propres des opérateurs de Hecke et nous récupèrous théorème non nulle pour les L-fonctions des représentations cuspidales.

1. **Introduction.** Let f be a Hecke eigenform with the Nebentypus ω and $L(s,f)$ the L-function of f. Then $L(s,f)$ can be written as an Euler product

$$L(s,f)=\prod_{p\in\mathcal{P}\setminus\mathcal{P}(f)}(1-\alpha_p\cdot p^{(k-1)/2}p^{-s})^{-1}(1-\beta_p\cdot p^{(k-1)/2}p^{-s})^{-1}\prod_{p\in\mathcal{P}(f)}l_p(s)^{-1},$$

where $l_p(s)$ are polynomials in p^{-s} with $l_p(0)\neq 0$, $\mathcal{P}$ is the set of all rational primes, and $\mathcal{P}(f)$ is a finite subset of $\mathcal{P}$. By the work of Deligne [2], we know $|\alpha_p|=|\beta_p|=1$. Thus, we can write α_p and β_p as polar forms

$$\alpha_p=e^{i\theta_p},\quad \beta_p=e^{i\psi_p},\quad 0\le\theta_p,\psi_p<2\pi,$$

Let $S_f=\{\theta_p,\psi_p\}_{p\in\mathcal{P}\setminus\mathcal{P}(f)}$. The question is how S_f distributes on $[0,2\pi]$.

This question has many classical origins. For instance, it is related to how often a quadratic form is a prime in a certain region (see [5]) and the distribution of primes in quadratic progressions (see [9]).

If f comes from a non-CM elliptic curve, which corresponds to the trivial ω and weight 2, Sato and Tate independently conjecture the law of distributions for S_f. For the case that ω is non-trivial, we have the following theorem proved in [8].

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THEOREM 1. ([8, Theorem 7]) *Let ω be a non-trivial primitive Dirichlet character and f a Hecke eigenform with the Nebentypus ω. Let π be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ *attached to f. Under the assumption that for all positive integers m, the L-functions $L(s, \mathrm{Sym}^m(\pi))$ and $L(s, \mathrm{Sym}^m(\pi) \otimes \omega)$ have analytic continuation for $\mathrm{Re}(s) \geq 1$, and are non-vanishing on the line $\mathrm{Re}(s) = 1$, we have that $S_f = \{\theta_p, \psi_p\}_{p \in \mathcal{P} \setminus \mathcal{P}(f)}$ is uniformly distributed.*

The purpose of this paper is to remove the non-vanishing condition from the theorem above. Following Ogg's idea (see [14] and [15]), we prove a general theorem for the non-vanishing property of Dirichlet series and provide some applications.

2. A non-vanishing theorem.

The following proposition is about holomorphy and non-vanishing properties for Dirichlet series with non-negative coefficients.

LEMMA 2. *Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ for $\mathrm{Re}(s) \gg 0$ and $g(s) = \sum_{n=1}^{\infty} b_n/n^s$ with $b_n \geq 0$ for $\mathrm{Re}(s) \gg 0$. Suppose that $f(s) = \exp(g(s))$ for $\mathrm{Re}(s) \gg 0$ and $f(s)$ has analytic continuation up to $\mathrm{Re}(s) = \sigma$. Then $g(s)$ is holomorphic in the region $\mathrm{Re}(s) \geq \sigma$. As a consequence, $f(s)$ does not have zeros in the region $\mathrm{Re}(s) \geq \sigma$.*

REMARK. Note that we do not assume that the convergence of the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ for $\mathrm{Re}(s) \geq \sigma$.

PROOF. For $\mathrm{Re}(s) \gg 0$, we have

$$(*) \qquad f(s) = \sum_{n=1}^{\infty} a_n/p^s = \exp(g(s)) = 1 + g(s) + \frac{g(s)^2}{2!} + \frac{g(s)^3}{3!} + \cdots ,$$

and $g(s) = \sum_{n=1}^{\infty} b_n/n^s$ with $b_n \geq 0$. By comparing the coefficients of n^{-s} in $(*)$, we have for all $n \geq 1$,

$$a_n \geq b_n \geq 0.$$

In particular, $f(s)$ has non-negative coefficients. By a classical theorem of Landau (see [11, Chapter 1, Exercise 5]), the abscissa of convergence of a Dirichlet series with non-negative coefficients is determined by its *first* real singularity. Since $f(s)$ is holomorphic for $\mathrm{Re}(s) \geq \sigma$, $f(s)$ has no singularity occurring in the same region. Therefore, we can conclude that the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ converges up to $\mathrm{Re}(s) = \sigma$, and then $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ for $\mathrm{Re}(s) \geq \sigma$. At the same time, since $a_n \geq b_n \geq 0$, the Dirichlet series $\sum_{n=1}^{\infty} b_n/n^s$ is convergent up to $\mathrm{Re}(s) = \sigma$ as well. As a consequence, $g(s) = \sum_{n=1}^{\infty} b_n/n^s$ is holomorphic for $\mathrm{Re}(s) \geq \sigma$ and $f(s) = \exp(g(s))$ for $\mathrm{Re}(s) \geq \sigma$. Thus, $f(s)$ does not have zeros in the region $\mathrm{Re}(s) \geq \sigma$. ■

Let $c(s)=\sum_{n=1}^{\infty} c_n/n^s$ and $d(s)=\sum_{n=1}^{\infty} d_n/n^s$ be two Dirichlet series. Define the convolution $c(s)\otimes d(s)$ of $c(s)$ and $d(s)$ as follows:

$$c(s)\otimes d(s)=\sum_{n=1}^{\infty}\frac{c_n\cdot d_n}{n^s}.$$

Also, define the conjugate $\overline{c}(s)$ of $c(s)$ by $\overline{c}(s)=\overline{c(\overline{s})}$, *i.e.*, $\overline{c}(s)=\sum_{n=1}^{\infty}\frac{\overline{c_n}}{n^s}$. Note that the Dirichlet series $c(s)\otimes\overline{c}(s)$ has positive coefficients. Following the idea of Ogg, we prove the following non-vanishing theorem which can be viewed as a generalization of Ogg's work in [14] and [15], and Narayanan's result in [13].

THEOREM 3. *Let $f(s)=\sum_{n=1}^{\infty} a_n/n^s$ be a Dirichlet series which converges absolutely for* $\mathrm{Re}(s)\gg 1$, *and $g(s)=\sum_{p\in\mathcal{P}}^{\infty}\sum_{k=1}^{\infty} b_{p^k}/p^{ks}$, where $\mathcal{P}$ is the set of rational primes. Suppose that $f(s)=\exp(g(s))$ for* $\mathrm{Re}(s)\gg 1$ *and has analytic continuation up to the region* $\mathrm{Re}(s)\geq 1/2$. *We also assume that $f(s)\otimes\overline{f}(s)$ has analytic continuation up to the region* $\mathrm{Re}(s)\geq 1/2$ *except a possible simple pole at $s=1$. Then $f(s)$ has no zero in the region* $\mathrm{Re}(s)\geq 1$. *In particular, $f(s)$ is non-vanishing on the line* $\mathrm{Re}(s)=1$.

PROOF. Let

$$F(s):=f(s)\otimes\overline{f}(s)=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^s}.$$

Since $F(s)$ have non-negative coefficients with the only singularity at $s=1$, by Landau's theorem (see the proof of Lemma 2, or [11, Chapter 1, Exercise 5]), $F(s)$ converges absolutely for $\mathrm{Re}(s)>1$. By Schwarz's inequality

$$\Big(\sum_{n=1}^{\infty}\frac{|a_n|}{n^\sigma}\Big)\leq\Big(\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^\sigma}\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{\infty}\frac{1}{n^\sigma}\Big)^{\frac{1}{2}}.$$

We conclude that $f(s)$ converges absolutely for $\sigma=\mathrm{Re}(s)>1$. By taking logarithm of $f(s)$, we see that $g(s)$ also converges absolutely for $\mathrm{Re}(s)>1$ as well and $f(s)$ has at most a simple pole or zero at $s=1$. Therefore, $f(s)$ has no zero in the region $\mathrm{Re}(s)>1$.

To prove that $f(s)$ is non-vanishing on the line $\mathrm{Re}(s)=1$, we assume that $f(s)$ has a zero at $1+it_0$. Let

$$h(s)=f(s+it_0)=\sum_{n=1}^{\infty}\frac{a_n\cdot n^{-it_0}}{n^s}=\exp\Big(\sum_{k=1}^{\infty}\frac{b_{p^k}p^{-ikt_0}}{p^{ks}}\Big).$$

Then $h(s)$ satisfies all conditions for $f(s)$. Therefore, without loss of generality, we can assume that $f(s)$ has a zero at $s=1$, so does $\overline{f}(s)$.

Now we consider the Dirichlet series

$$H(s)=\zeta(s)\cdot f(s)\cdot\overline{f}(s)\cdot(f(s)\otimes\overline{f}(s))=\exp\Big(\sum_{p\in\mathcal{P}}\sum_{k=1}^{\infty}\frac{|1+b_{p^k}|^2}{p^{ks}}\Big),$$

which has non-negative coefficients. Since the poles of $\zeta(s)$ and $f(s)\otimes\overline{f}(s)$ at $s=1$ has been canceled by zeros of $f(s)$ and $\overline{f}(s)$ at $s=1$, $H(s)$ is holomorphic for $\mathrm{Re}(s)\geq 1/2$. Note that $H(s)$ has zeros from $\zeta(s)$ on the line $\mathrm{Re}(s)=1/2$. However, by Lemma 2, $H(s)$ has no zero for $\mathrm{Re}(s)\geq 1/2$; we get a contradiction. Thus, $f(s)$ has no zero on $\mathrm{Re}(s)=1$ and it completes the proof of this theorem. ∎

3. Main theorem.

In this section, we provide a refinement of Theorem 1 (see [8, Theorem 7]). More precisely, we remove the non-vanishing condition in Theorem 1. In fact, we prove a more general theorem of that type.

Let $f(s)$ be a Dirichlet series of the form

$$f(s)=\prod_{p\in\mathcal{P}\backslash\mathcal{P}(f)}\Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1},$$

which converges absolutely for $\mathrm{Re}(s)\gg 0$, where $\mathcal{P}$ is the set of rational primes and $\mathcal{P}(f)$ is a finite subset of $\mathcal{P}$. For all positive integer n, define

$$f_n(s)=\prod_{p\in\mathcal{P}\backslash\mathcal{P}(f)}\Big(\prod_{j=0}^{n}\Big(1-\frac{\alpha_p^{n-j}\beta_p^{j}}{p^s}\Big)^{-1}\Big).$$

The following lemma is well-known (see [1, Section 2.4]).

Lemma 4. *Let $f(s)$ be a Dirichlet series of the form*

$$f(s)=\prod_{p\in\mathcal{P}\backslash\mathcal{P}(f)}\Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1},$$

which converges absolutely for $\mathrm{Re}(s)\gg 0$, where $\mathcal{P}$ is the set of rational primes and $\mathcal{P}(f)$ is a finite subset of $\mathcal{P}$. If there exists a positive number A such that for all positive integers n, $f_n(s)$ have analytic continuation to $\mathrm{Re}(s)\geq A$, then

$$|\alpha_p|=|\beta_p|=1.$$

The following lemma is about the decomposition of Dirichlet series.

Lemma 5. *Let $f(s)$ be defined as in Lemma 4. Assume that there exists a primitive Dirichlet character ω such that for all prime $p\in\mathcal{P}\backslash\mathcal{P}(f)$,*

$$\alpha_p\beta_p=\omega(p),\quad \text{and}\quad |\alpha_p|=|\beta_p|=1.$$

Then for all positive integers n,

$$f_n(s)\otimes\overline{f}_n(s)=\prod_{j=0}^{n}f_{2j}\otimes\omega^{-j},$$

where $f_0(s)=\zeta(s)$ and ω^0 is the trivial character.

Proof. Observe that

$$\begin{aligned}
\overline{f}_n(s) &= \prod_{p\in\mathcal{P}\setminus\mathcal{P}(f)} \Big(\prod_{j=0}^{n}\Big(1-\frac{\overline{\alpha_p}^{n-j}\overline{\beta_p}^{j}}{p^s}\Big)^{-1}\Big) \quad \text{(by definition)}\\
&= \prod_{p\in\mathcal{P}\setminus\mathcal{P}(f)} \Big(\prod_{j=0}^{n}\Big(1-\frac{\alpha_p^{-(n-j)}\beta_p^{-j}}{p^s}\Big)^{-1}\Big) \quad (\text{since } |\alpha_p|=|\beta_p|=1)\\
&= \prod_{p\in\mathcal{P}\setminus\mathcal{P}(f)} \Big(\prod_{j=0}^{n}\Big(1-\frac{\alpha_p^{n-(n-j)}\beta_p^{n-j}}{p^s}\cdot\omega(p)^{-n}\Big)^{-1}\Big) \quad (\text{since } \alpha_p\beta_p=\omega(p))\\
&= \prod_{p\in\mathcal{P}\setminus\mathcal{P}(f)} \Big(\prod_{j=0}^{n}\Big(1-\frac{(\alpha_p^{j}\beta_p^{n-j})\omega(p)^{-n}}{p^s}\Big)^{-1}\Big)\\
&= f_n(s)\otimes\omega^{-n}.
\end{aligned}$$

By a well-known fact of $\mathfrak{sl}_2$ representation theory (see [3, Exercise 11.11]), for all positive integers n and m with $n \geq m \geq 1$, we have

$$f_n(s)\otimes f_m(s) = \prod_{j=0}^{m} f_{n-m+2j}\otimes\omega^{m-j}.$$

Hence,

$$f_n(s)\otimes\overline{f}_n(s) = f_n(s)\otimes f_n\otimes\omega^{-n} = \prod_{j=0}^{n} f_{2j}\otimes\omega^{n-j}\otimes\omega^{-n} = \prod_{j=0}^{n} f_{2j}\otimes\omega^{-j}.$$

This completes the proof of this lemma. ∎

Let $f(s)$ be defined as in Lemma 4 with an extra condition $|\alpha_p| = |\beta_p| = 1$ for all $p \in \mathcal{P}\setminus\mathcal{P}(f)$. Thus, we can write

$$\alpha_p = e^{i\theta_p}, \quad \text{and} \quad \beta_p = e^{i\psi_p},$$

where $0 \leq \theta_p, \psi_p < 2\pi$. Define

$$S_f = \{\theta_p, \psi_p\}_{p\in\mathcal{P}\setminus\mathcal{P}(f)},$$

the set of angles arisen from f.

The following theorem is about the distribution of θ_p and ψ_p in $[0, 2\pi]$.

Theorem 6. *Let $f(s)$ be a Dirichlet series of the form*

$$f(s) = \prod_{p\in\mathcal{P}\setminus\mathcal{P}(f)} \Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1},$$

where $\mathcal{P}$ is the set of rational primes and $\mathcal{P}(f)$ is a finite subset of $\mathcal{P}$. Suppose that $f(s)$ satisfies the following conditions:

(1) *$f(s)$ converges absolutely for* $\mathrm{Re}(s) > 1$*;*
(2) *there exists a non-trivial primitive Dirichlet character ω such that for all primes $p \in \mathcal{P}\backslash\mathcal{P}(f)$, $\alpha_p\beta_p = \omega(p)$;*
(3) *for all positive integers n and integers m(can be negative), the functions $f_n(s) \otimes \omega^m$ have analytic continuation in the region* $\mathrm{Re}(s) \geq 1/2$.

Then for all prime $p \in \mathcal{P}\backslash\mathcal{P}(f)$, we have

$$|\alpha_p| = |\beta_p| = 1.$$

Moreover the set $S_f = \{\theta_p, \psi_p\}_{p\in\mathcal{P}\backslash\mathcal{P}(f)}$ defined above is uniformly distributed.

Proof. As a direct consequence of Lemma 4, we have $|\alpha_p| = |\beta_p| = 1$. It remains to prove that S_f is uniformly distributed. As we can see from Theorem 1 (see [8, Theorem 7]) and its proof, it suffices to show that the functions $f_n(s)$ and $f_n(s) \otimes \omega$ are non-vanishing on the line $\mathrm{Re}(s) = 1$.

To prove the non-vanishing property for $f_n(s)$ and $f_n(s) \otimes \omega$, by Theorem 3, it suffices to prove that $f_n(s) \otimes \overline{f}_n(s)$ has analytic continuation in the region $\mathrm{Re}(s) \geq 1/2$ except a possible simple pole at $s = 1$. By Lemma 5, we have

$$f_n(s) \otimes \overline{f}_n(s) = \prod_{j=0}^{n} f_{2j} \otimes \omega^{-j} = \zeta(s) \cdot \prod_{j=1}^{n} f_{2j} \otimes \omega^{-j}.$$

From our assumptions, the product $\prod_{j=1}^{n} f_{2j} \otimes \omega^{-j}$ is holomorphic in the region $\mathrm{Re}(s) \geq 1/2$. Therefore, $f_n(s) \otimes \overline{f}_n(s)$ is holomorphic in the region $\mathrm{Re}(s) \geq 1/2$ except a possible simple pole at $s = 1$, which comes from $\zeta(s)$. Thus, $f_n(s) \otimes \overline{f}_n(s)$ has analytic continuation for $\mathrm{Re}(s) \geq 1/2$.

Now it follows from Theorem 1 and its proof that S_f is uniformly distributed. This completes the proof of our theorem. ■

4. Applications.

From Theorem 6, we can derive several applications on distributions of eigenvalues of modular forms. Let $\mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$. Let $\pi = \bigotimes_p \pi_p$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with central character ω_π. Given a positive integer n, we can define the n-th symmetric power representation $\mathrm{Sym}^n(\pi)$ of π (see [16] and [17]).

Let $\mathcal{P}(\pi)$ be the set of places where π is ramified. One can define the L-function associated to π as follows:

$$L(s, \pi) := \prod_{p\in\mathcal{P}\backslash\mathcal{P}(\pi)} \Big(1 - \frac{\alpha_p}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_p}{p^s}\Big)^{-1} \prod_{p\in\mathcal{P}(\pi)} l_p(s)^{-1},$$

where $l_p(s)$'s are polynomials in p^{-s} with $l_p(0) \neq 0$. The generalized Ramanujan conjecture predicts that if π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, then for all $p \in \mathcal{P}\backslash\mathcal{P}(\pi)$, $|\alpha_p| = |\beta_p| = 1$. If π satisfies the generalized Ramanujan conjecture, similar to the case of Hecke eigenforms, we can define the set

$$S_\pi := \{\theta_p, \psi_p\}_{p\in\mathcal{P}\backslash\mathcal{P}(\pi)}.$$

THEOREM 7. *Let π be a cuspidal representation of* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$*. If for all positive integers n and integers m(can be negative), the L-functions $L(s, \mathrm{Sym}^n(\pi) \otimes \omega^m)$ have analytic continuation for* $\mathrm{Re}(s) \geq 1/2$*, then π satisfies the generalized Ramanujan conjecture and the set $S_\pi = \{\theta_p, \psi_p\}_{p \in \mathcal{P} \setminus \mathcal{P}(\pi)}$ is uniformly distributed.*

PROOF. By our assumptions, $L(s, \pi)$ satisfies all conditions in Theorem 6. Note that the contribution from ramified primes is negligible. Hence the result follows. ∎

Now, let us restrict ourselves on Hecke eigenforms. Let ω be a non-trivial primitive Dirichlet character. From Deligne's work [2], for each $f \in H(\Gamma, \omega)$, we can attach f to a cuspidal representation π_f of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ such that $L(s, f) = L(s, \pi_f)$. Moreover, π_f satisfies the Ramanujan conjecture. Applying Theorem 6 on Hecke eigenforms, we have:

COROLLARY 8. *Let ω be a non-trivial primitive Dirichlet character and f a Hecke eigenform with the Nebentypus ω. Let π_f be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ *attached to f. If for all positive integers n and integers m (can be negative), the L-functions $L(s, \mathrm{Sym}^n(\pi_f) \otimes \omega^m)$ have analytic continuation for* $\mathrm{Re}(s) \geq 1/2$*, then the set $S_f = \{\theta_p, \psi_p\}_{p \in \mathcal{P} \setminus \mathcal{P}(f)}$ is uniformly distributed.*

REMARK.

(1) For the case that ω is trivial in the corollary above, it is a theorem of Ogg in [15]. In the case of elliptic curves without complex multiplication, a better result was obtained by K. Murty in [12] as the condition on the admission of analytic continuation can be relaxed to $\mathrm{Re}(s) \geq 1$.
(2) The assumption of analytic continuation is not too absurd. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ attached to a Hecke eigenform. It is a general belief that for all positive integers n, $\mathrm{Sym}^n(\pi)$ are cuspidal; it is not true for all cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, though (see [7]). By the work of Mœglin and Waldspuerger [10], for any Grössencharacter χ, $L(s, \mathrm{Sym}^n(\pi) \otimes \chi)$ is entire. Therefore, in this case, our assumption holds.

We can also apply Theorem 3 to recover a theorem of Jacquet and Shalika.

COROLLARY 9. ([6, Theorem 1]) *Let π be a cuspidal representation of* GL_n *and $L(s, \pi)$ the L-function associated to π. Then $L(s, \pi) \neq 0$ on the line* $\mathrm{Re}(s)$=1.

PROOF. By [4], we know $L(s, \pi)$ is entire. Therefore, from the work of Mœglin and Waldspuerger [10], the L-function $L(s, \pi \otimes \overline{\pi})$ of the Rankin–Selberg convolution $\pi \otimes \overline{\pi}$ has a simple pole at $s = 1$ and admits analytic continuation up to $\mathrm{Re}(s) \geq 1/2$. Thus, by Theorem 3, we have $L(s, \pi) \neq 0$ for $\mathrm{Re}(s) = 1$. This completes the proof of this corollary. ∎

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