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CONVERGENCE OF ITERATES OF TYPICAL NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. Let K be a bounded, closed and convex subset of a Banach space X. We show that the iterates of a typical element (in the sense of Baire category) of a class of nonexpansive mappings which take K to X converge uniformly on K to the unique fixed point of this typical element.

RÉSUMÉ. Soit K un sous-ensemble borné, fermé et convexe d'un espace de Banach X. Nous démontrons que les itérés d'un élément typique (au sens des catégories de Baire) d'une classe d'applications non-expansives de K dans X convergent uniformément sur K vers l'unique point fixe de cet élément typique.

1. **Introduction and preliminaries.** Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, bounded, closed and convex subset of X. Denote by $\mathcal{M}_{ne}$ the set of all mappings $A\colon K \to X$ which satisfy

$$\|Ax - Ay\| \leq \|x - y\| \quad \text{for all } x, y \in K.$$

For each $A, B \in \mathcal{M}_{ne}$, set

$$d(A, B) = \sup\{||Ax - Bx|| : x \in K\}. \tag{1.1}$$

It is clear that $(\mathcal{M}_{ne}, d)$ is a complete metric space. Denote by $\mathcal{M}_0$ the set consisting of all $A \in \mathcal{M}_{ne}$ such that

$$\inf\{||x - Ax|| : x \in K\} = 0. \tag{1.2}$$

In other words, $\mathcal{M}_0$ consists of all those nonexpansive mappings taking K to X which have approximate fixed points. Clearly, $\mathcal{M}_0$ is a closed subset of $\mathcal{M}_{ne}$.

Every nonexpansive self-mapping of K belongs to $\mathcal{M}_0$. In order to exhibit two classes of nonself-mappings of K that are also contained in $\mathcal{M}_0$, we first recall that if $x \in K$, then the inward set $I_K(x)$ of X with respect to K is defined by

$$I_K(x) := \{z \in X : z = x + \alpha(y - x) \text{ for some } y \in K \text{ and } \alpha \geq 0\}.$$

A mapping $A\colon K \to X$ is said to be weakly inward if Ax belongs to the closure of $I_K(x)$ for each $x \in K$. Consider now a weakly inward mapping $A \in \mathcal{M}_{ne}$.

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Fix a point $z \in K$ and $t \in [0,1)$ and let the mapping $S\colon K \to X$ be defined by $Sx = tAx + (1-t)z$, $x \in K$. This strict contraction is also weakly inward and therefore has a unique fixed point $x_t \in K$ by Theorem 2.4 in [4]. Since $||x_t - Ax_t|| \to 0$ as $t \to 1^-$, we see that $A \in \mathcal{M}_0$.

If K has a nonempty interior $\mathrm{int}(K)$ and a nonexpansive mapping $A\colon K \to X$ satisfies the Leray–Schauder condition with respect to $w \in \mathrm{int}(K)$, that is, $Ay - w \neq m(y-w)$ for all y in the boundary of K and $m > 1$, then it also belongs to $\mathcal{M}_0$. This is because the strict contraction $S\colon K \to X$ defined by $Sx = tAx + (1-t)w$, $x \in K$, also satisfies the Leray–Schauder condition with respect to $w \in \mathrm{int}(K)$ and therefore has a unique fixed point [3].

Set

$$\rho(K) = \sup\{||z|| : z \in K\}. \tag{1.3}$$

Our purpose in this note is to show that the iterates of a typical element (in the sense of Baire category) of $\mathcal{M}_0$ converge uniformly on K to the unique fixed point of this typical element. As a matter of fact, we are able to establish a more refined result, involving the notion of porosity which we now recall.

Let (Y, λ) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous in (Y, λ) if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y, λ) if it is a countable union of porous subsets in (Y, λ).

Since porous sets are obviously nowhere dense, all σ-porous sets are of the first Baire category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are also of Lebesgue measure zero.

To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B(z, s) \subset B(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0,1)$ is a constant which depends only on E.

We are now ready to formulate our result. Its proof will be given in the next section.

THEOREM 1.1. *There exists a set $\mathcal{F} \subset (\mathcal{M}_0, d)$ such that its complement $\mathcal{M}_0 \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}_0, d)$ and each $B \in \mathcal{F}$ has the following properties:*

(1) *there exists a unique point $x_B \in K$ such that $Bx_B = x_B$;*
(2) *for each $\epsilon > 0$, there exist $\delta > 0$, a natural number q, and a neighborhood $\mathcal{U}$ of B in $(\mathcal{M}_{ne}, d)$ such that:*
   (a) *if $C \in \mathcal{U}$, $y \in K$, and $||y - Cy|| \leq \delta$, then $||y - x_B|| \leq \epsilon$;*

(b) *if* $C \in \mathcal{U}$, $\{x_i\}_{i=0}^q \subset K$, *and* $Cx_i = x_{i+1}$, $i = 0, \dots, q-1$, *then* $||x_q - x_B|| \le \epsilon$.

Although analogous results for the closed subspace of $(\mathcal{M}_0, d)$ comprising all nonexpansive self-mappings of K were established by De Blasi and Myjak in [1] and [2], Theorem 1.1 seems to be the first generic result dealing with nonself-mappings. In this connection see also the related papers [5] and [6]. Additional information regarding various generic aspects of (metric) fixed point theory can be found, for instance, in [7] and [8].

2. Proof of Theorem 1.1.

We begin with a simple lemma.

Denote by E the set of all $A \in \mathcal{M}_{ne}$ for which there exists $x \in K$ satisfying $Ax = x$. That is, E consists of all those nonexpansive mappings $A\colon K \to X$ which have a fixed point.

Lemma 2.1. *E is an everywhere dense subset of $(\mathcal{M}_0, d)$.*

Proof. Let $A \in \mathcal{M}_0$ and $\epsilon > 0$. By (1.2), there exists $\bar{x} \in K$ such that

$$||\bar{x} - A\bar{x}|| < \epsilon/2.$$

Define

$$By = Ay + \bar{x} - A\bar{x}, \quad y \in K. \tag{2.1}$$

Clearly, $B \in \mathcal{M}_{ne}$ and $B\bar{x} = \bar{x}$. Thus $B \in E$. It is easy to see that $d(A, B) = ||\bar{x} - A\bar{x}|| < \epsilon$. This completes the proof of Lemma 2.1. ∎

For each natural number n, denote by $\mathcal{F}_n$ the set of all those mappings $A \in \mathcal{M}_0$ which have the following property:

(P1) There exist a natural number q, $x_* \in K$, $\delta > 0$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_{ne}$ such that:

- (i) if $B \in \mathcal{U}$ and if $z \in K$ satisfies $||z - Bz|| \le \delta$, then $||z - x_*|| \le 1/n$;
- (ii) if $B \in \mathcal{U}$ and if $\{x_i\}_{i=0}^q \subset K$ satisfies $x_{i+1} = Bx_i$, $i = 0, \dots, q-1$, then $||x_q - x_*|| \le 1/n$.

Set

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n.$$

We intend to prove that $\mathcal{M}_0 \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}_0, d)$. To meet this goal, it is sufficient to show that for each natural number n, the set $\mathcal{M}_0 \setminus \mathcal{F}_n$ is a porous subset of $(\mathcal{M}_0, d)$.

Indeed, let n be a natural number. Choose a positive number

$$\alpha \le 2^{-11}\big(\rho(K) + 1\big)^{-1} n^{-1}. \tag{2.2}$$

Let

$$A \in \mathcal{M}_0 \quad \text{and} \quad r \in (0, 1]. \tag{2.3}$$

By Lemma 2.1, there are $A_0 \in E$ and $x_* \in K$ such that

$$d(A_0, A) < r/8 \quad \text{and} \quad A_0 x_* = x_*. \tag{2.4}$$

Set

$$\gamma = \big[32\big(\rho(K) + 1\big)\big]^{-1} r \tag{2.5}$$

and

$$\delta = (4n)^{-1}\gamma - 2\alpha r. \tag{2.6}$$

By (2.6), (2.5) and (2.2),

$$\delta > 0. \tag{2.7}$$

Now choose an integer $q \geq 4$ such that

$$(1 - \gamma)^q 2\big(\rho(K) + 1\big) < (16n)^{-1}. \tag{2.8}$$

Define

$$A_1 y = (1 - \gamma)A_0 y + \gamma x_*, \quad y \in K. \tag{2.9}$$

Clearly, $A_1 \in \mathcal{M}_{ne}$ and

$$A_1 x_* = x_*. \tag{2.10}$$

By (1.1), (2.9), (2.4) and (1.3),

$$\begin{aligned} d(A_1, A_0) &= \sup\{||A_1 y - A_0 y|| : y \in K\} = \sup\{||\gamma A_0 y - \gamma x_*|| : y \in K\} \\ &= \gamma \sup\{||A_0 y - A_0 x_*|| : y \in K\} \\ &\leq \gamma \sup\{||y - x_*|| : y \in K\} \leq 2\gamma\rho(K), \end{aligned}$$

so that

$$d(A_1, A_0) \leq 2\gamma\rho(K). \tag{2.11}$$

By (2.11), (2.4) and (2.5),

$$d(A, A_1) \leq d(A, A_0) + d(A_0, A_1) \leq r/8 + 2\gamma\rho(K) \leq r/4. \tag{2.12}$$

Assume that $B \in \mathcal{M}_{ne}$ satisfies

$$d(B, A_1) \le 2\alpha r. \tag{2.13}$$

Assume further that

$$z \in K \text{ and } ||z - Bz|| \le \delta. \tag{2.14}$$

By (2.10) and (2.9),

$$\begin{aligned} ||A_1 z - x_*|| &= ||A_1 z - A_1 x_*|| \\ &= (1-\gamma)||A_0 z - A_0 x_*|| \le (1-\gamma)||z - x_*||. \end{aligned} \tag{2.15}$$

By (1.1), (2.13) and (2.15),

$$\begin{aligned} ||Bz - z|| &\ge ||A_1 z - z|| - ||Bz - A_1 z|| \\ &\ge ||A_1 z - z|| - d(B, A_1) \ge ||A_1 z - z|| - 2\alpha r \\ &\ge ||z - x_*|| - ||x_* - A_1 z|| - 2\alpha r \\ &\ge ||z - x_*|| - (1-\gamma)||z - x_*|| - 2\alpha r = \gamma ||z - x_*|| - 2\alpha r. \end{aligned}$$

When combined with (2.14) and (2.6), this inequality implies that

$$\delta \ge ||Bz - z|| \ge \gamma ||z - x_*|| - 2\alpha r$$

and

$$||z - x_*|| \le \gamma^{-1}(\delta + 2\alpha r) \le (4n)^{-1}.$$

Thus we have shown that

$$\text{if } z \in K \text{ satisfies } ||z - Bz|| \le \delta, \text{ then } ||z - x_*|| \le (4n)^{-1}. \tag{2.16}$$

Now assume that

$$\{x_i\}_{i=0}^{q} \subset K, \quad Bx_i = x_{i+1}, \quad i = 0, \dots, q-1. \tag{2.17}$$

By (2.17), (1.1), (2.13), (2.9) and (2.4), for $i = 0, \dots, q-1$, there holds

$$\begin{aligned} ||x_{i+1} - x_*|| &= ||Bx_i - x_*|| \le ||Bx_i - A_1 x_i|| + ||A_1 x_i - x_*|| \\ &= ||Bx_i - A_1 x_i|| + ||A_1 x_i - A_1 x_*|| \\ &\le d(B, A_1) + (1-\gamma)||A_0 x_i - A_0 x_*|| \\ &\le 2\alpha r + (1-\gamma)||x_i - x_*||, \end{aligned}$$

that is,

$$||x_{i+1} - x_*|| \le 2\alpha r + (1-\gamma)||x_i - x_*||.$$

In view of this inequality, which is valid for $i = 0, \dots, q-1$, we get

$$\begin{aligned}||x_q - x_*|| &\le 2\alpha r \sum_{i=0}^{q-1} (1-\gamma)^i + (1-\gamma)^q ||x_0 - x_*|| \\ &\le 2\alpha r \gamma^{-1} + (1-\gamma)^q ||x_0 - x_*|| \\ &\le 2\alpha r \gamma^{-1} + 2\rho(K)(1-\gamma)^q.\end{aligned}$$

When combined with (2.5), (2.8) and (2.2), this inequality implies that

$$\begin{aligned}||x_q - x_*|| &\le (1-\gamma)^q 2\rho(K) + 2\alpha\big[32\big(\rho(K)+1\big)\big] \\ &\le (16n)^{-1} + 64\alpha[\rho(K)+1] \le (16n)^{-1} + (32n)^{-1} < (8n)^{-1}.\end{aligned}$$

Thus we have shown that

$$\text{if } \{x_i\}_{i=0}^q \subset K \text{ satisfies (2.17), then } ||x_q - x_*|| \le (8n)^{-1}. \tag{2.18}$$

By (2.18), (2.17) and (2.16), each $C \in \mathcal{M}_0$ which satisfies $d(C, A_1) \le \alpha r$ has property (P1). Therefore

$$\{C \in \mathcal{M}_0 : d(C, A_1) \le \alpha r\} \subset \mathcal{F}_n.$$

When combined with (2.2) and (2.12), this inclusion implies that

$$\{C \in \mathcal{M}_0 : d(C, A_1) \le \alpha r\} \subset \{B \in \mathcal{M}_0 : d(B, A) \le r\} \cap \mathcal{F}_n.$$

This means that $\mathcal{M}_0 \setminus \mathcal{F}_n$ is a porous set in $(\mathcal{M}_0, d)$ for all natural numbers n. Therefore $\mathcal{M}_0 \setminus \mathcal{F}$ is a σ-porous set in $(\mathcal{M}_0, d)$.

Now let $A \in \mathcal{F}$ and $\epsilon > 0$. Choose a natural number

$$n > 8(\min\{1, \epsilon\})^{-1}. \tag{2.19}$$

Since $A \in \mathcal{F}_n$, property (P1) implies that there exist a natural number q_n, a number $\delta_n > 0$, a neighborhood $\mathcal{U}_n$ of A in $\mathcal{M}_{ne}$, and a point $x_n \in K$ such that the following property holds:

(P2) (i) if $B \in \mathcal{U}_n$, $z \in K$, and $||z - Bz|| \le \delta_n$, then $||z - x_n|| \le 1/n$;

(ii) if $B \in \mathcal{U}_n$, $\{z_i\}_{i=0}^{q_n} \subset K$, and $z_{i+1} = Bz_i$, $i = 0, \dots, q_n - 1$, then $||z_{q_n} - x_n|| \le 1/n$.

Since $A \in \mathcal{M}_0$, there exists a sequence $\{y_i\}_{i=1}^\infty \subset K$ such that

$$\lim_{i \to \infty} ||y_i - Ay_i|| = 0. \tag{2.20}$$

Hence there exists a natural number i_0 such that

$$||y_i - Ay_i|| \le \delta_n \quad \text{for all integers } i \ge i_0.$$

When combined with (P2)(i), this implies that

$$||x_n - y_i|| \le 1/n \quad \text{for all integers } i \ge i_0. \tag{2.21}$$

In view of (2.21), for each pair of integers $i, j \ge i_0$,

$$||y_i - y_j|| \le ||y_i - x_n|| + ||x_n - y_j|| \le 2/n < \epsilon.$$

Since ϵ is an arbitrary positive number, we conclude that $\{y_i\}_{i=1}^{\infty}$ is a Cauchy sequence and therefore there exists

$$x_A = \lim_{i \to \infty} y_i. \tag{2.22}$$

Clearly, $Ax_A = x_A$. It is easy to see that x_A is the unique fixed point of A. Indeed, if it were not unique, then we would be able to construct a nonconvergent sequence $\{y_i\}_{i=0}^{\infty}$ satisfying (2.20).

By (2.21) and (2.22),

$$||x_A - x_n|| \le 1/n. \tag{2.23}$$

Now assume that

$$B \in \mathcal{U}_n, \quad z \in K, \quad \text{and} \quad ||z - Bz|| \le \delta_n. \tag{2.24}$$

By (P2)(i) and (2.24),

$$||z - x_n|| \le 1/n.$$

When combined with (2.23) and (2.19), this inequality implies that

$$||z - x_A|| \le ||z - x_n|| + ||x_n - x_A|| \le 2/n < \epsilon.$$

Finally, suppose that

$$B \in \mathcal{U}_n, \quad \{z_i\}_{i=0}^{q_n} \subset K, \quad \text{and} \quad Bz_i = z_{i+1}, i = 0, \dots, q_n - 1. \tag{2.25}$$

Then by (P2)(ii) and (2.25),

$$||z_{q_n} - x_n|| \le 1/n.$$

When combined with (2.23) and (2.19), this last inequality implies that

$$||z_{q_n} - x_A|| \le ||z_{q_n} - x_n|| + ||x_n - x_A|| \le 2/n < \epsilon.$$

This completes the proof of Theorem 1.1. ■

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