C. R. Math. Rep. Acad. Sci. Canada Vol. **28**, (1), 2006 pp. 17–23

NONSTABILITY RESULTS IN THE THEORY OF CONVEX FUNCTIONS

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Presented by V. Dlab, FRSC

Abstract. We show that the inequality defining convex functions (convex in the sense of Wright) is not stable in infinitely-dimensional spaces. The inequality defining Jensen-convex functions is not stable either, even if its domain is a real interval.

Résumé. Nous montrons que l'inégalité définissant des fonctions convexes (convexes dans le sens de Wright) n'est pas stable dans les espaces à dimension infinie. L'inégalité définissant des fonctions convexes dans le sens de Jensen n'est pas stable non plus, même si son domaine est une intervalle réelle.

Introduction. Let $\mathbb{K}$ be a fixed number field contained in $\mathbb{R}$ (the set of all reals) and let X be a linear space over $\mathbb{K}$. A subset $D \subset X$ is said to be $\mathbb{K}$-convex iff for all $x, y \in D$ and each $\lambda \in \mathbb{K} \cap (0,1)$ we have $\lambda x + (1-\lambda)y \in D$. Assume that $\varepsilon \geq 0$ is a fixed real and D is a non-empty $\mathbb{K}$-convex subset of X. A function $f\colon D \to \mathbb{R}$ satisfying the inequality

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \varepsilon, \tag{1}$$

for all $x, y \in D$, $t \in (0,1)\cap\mathbb{K}$ is called ε-$\mathbb{K}$-convex. If (1) is fulfilled only for $t = \frac{1}{2}$ and all $x, y \in D$, then f is said to be an ε-J-convex function (ε-Jensen-convex function). We say that $f\colon D \to \mathbb{R}$ is an ε-$\mathbb{K}$-Wright-convex function iff it satisfies the inequality

$$f\big(tx + (1-t)y\big) + f\big((1-t)x + ty\big) \leq f(x) + f(y) + \varepsilon, \tag{2}$$

for all $x, y \in D$ and $t \in (0,1) \cap \mathbb{K}$. If $\mathbb{K} = \mathbb{R}$ we say shortly that f is ε-convex, ε-Wright-convex instead of ε-$\mathbb{R}$-convex, ε-$\mathbb{R}$-Wright-convex, respectively. Answering a problem which had been formulated by S. Ulam during a talk before a Mathematical Colloquium at the University of Winconsin, Madison, in 1940 (cf. also [10]), D. H. Hyers and S. Ulam [5] (also J. W. Green [4], P. W. Cholewa [2]) proved that if D is an open convex subset of $\mathbb{R}^n$ and $f\colon D \to \mathbb{R}$ is an ε-convex function, then there exist a convex function $g\colon D \to \mathbb{R}$ and a constant M (depending on the dimension n) such that

$$|f(x) - g(x)| \leq M, \quad x \in D. \tag{3}$$

Received by the editors on July, 2005; revised December 19, 2005.
AMS subject classification: Primary: 39B82; secondary: 39B62, 26A61.

It is no longer true if D is a subset of an infinite-dimensional real linear space. It was first observed by E. Casini and P. L. Papini [1] (see also [3], where versions of this statement were proved for specially constructed domains). Another counterexample (which works in every infinite-dimensional real linear space) was done by the first author of this paper [6]. In both cases the domain D of f was a rather thin subset of the space. In this note we show that every infinite-dimensional convex subset of a real linear (ly-topological) space admits such examples. In [9], the second author of this paper proved that if D is an open and convex subset of $\mathbb{R}^n$, then for every ε-Wright-convex function $f\colon D \to \mathbb{R}$ there exists a Wright-convex function $g\colon D \to \mathbb{R}$ (it means that g satisfies (2) for all $x, y \in D$ and every $t \in (0,1)$ with $\varepsilon = 0$) and a constant $M \geq 0$ fulfilling the estimation (3). We show that if D is an infinite-dimensional convex subset of a real linear space then this assertion does not hold. Moreover, we also show that there exists an ε-J-convex function $f\colon \mathbb{R} \supset (a,b) \to \mathbb{R}$ such that for every Jensen-convex function $g\colon (a,b) \to \mathbb{R}$, $-\infty \leq a < b \leq +\infty$ (*i.e.*, g satisfies the inequality (1) for $t = \frac{1}{2}$ and all $x, y \in (a,b)$ with $\varepsilon = 0$), we have

$$\sup\{|f(x) - g(x)|\,;\, x \in (a,b)\} = \infty.$$

Construction. We start with the well-known following lemma [8] (see also [6]).

LEMMA. *Let D be a subset of a real normed space X and let $f\colon D \to \mathbb{R}$ be a function bounded from below satisfying Lipschitz condition with a constant $L \in [0,\infty)$. Then the function $F\colon X \to \mathbb{R}$ defined by the formula*

$$F(x) := \inf\{L\|x - y\| + f(y)\,;\, y \in D\}, \quad x \in X,$$

is an extension of f to the whole space X preserving the Lipschitz condition.

Note that if f is an ε-$\mathbb{K}$-convex (ε-J-convex, ε-$\mathbb{K}$-Wright-convex) then F is ε-$\mathbb{K}$-convex (ε-J-convex, ε-$\mathbb{K}$-Wright-convex), too. In fact, assume (for example) that f is an ε-$\mathbb{K}$-convex function. Let $x, y \in X$ be fixed and take arbitrary $\eta > 0$. According to definition of F one can find $u, v \in D$ such that

$$F(x) + \eta > L\|x - u\| + f(u) \quad \text{and} \quad F(y) + \eta > L\|y - v\| + f(v).$$

By $\mathbb{K}$-convexity of D, $\lambda u + (1-\lambda)v \in D$, for every $\lambda \in \mathbb{K} \cap (0,1)$. According to ε-$\mathbb{K}$-convexity of f we obtain

$$\begin{aligned}
&\lambda F(x) + (1-\lambda)F(y) + \varepsilon \\
&\quad > L\lambda\|x - u\| + \lambda f(u) - \lambda\eta + L(1-\lambda)\|y - v\| + (1-\lambda)f(v) - (1-\lambda)\eta + \varepsilon \\
&\quad \geq L\big\|\big(\lambda x + (1-\lambda)y\big) - \big(\lambda u + (1-\lambda)v\big)\big\| + f\big(\lambda u + (1-\lambda)v\big) - \eta \\
&\quad \geq \inf\{L\|\lambda x + (1-\lambda)y - z\| + f(z)\,;\, z \in D\} - \eta,
\end{aligned}$$

from which ε-$\mathbb{K}$-convexity of F easily follows.

Let X be an infinite-dimensional linear space over the field $\mathbb{K}$ and let D be an infinite-dimensional $\mathbb{K}$-convex subset of X, *i.e.*, the dimension of the space generated by D over $\mathbb{K}$ is infinite. Without loss of generality we may assume that

$$0 \in D.$$

For an arbitrary positive integer n we denote by $\{e_1^n, \dots, e_n^n\}$ the standard basis of $\mathbb{K}^n$. By $\|x\|$ we will denote the maximum norm in $\mathbb{R}^n$, *i.e.*,

$$\|x\| := \max\{|x_1|, \dots, |x_n|\}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

We define a function $\varphi_n \colon [0,1]^n \to \mathbb{R}$ by the formula

$$\varphi_n(x) := -\log_2\Big(\frac{1}{n} + \alpha_n \max\{x_1, \dots, x_n\}\Big),$$

where $x = \Sigma_{i=1}^n x_i e_i^n$, $\alpha_n := 1 - \frac{1}{n}$. We have

$$\varphi_n(x) \geq 0, \quad \text{and} \quad \varphi_n(0) = \log_2 n.$$

For arbitrary $x, y \in [0,1]^n$, $x = \sum_{i=1}^n x_i e_i^n$, $y = \sum_{i=1}^n y_i e_i^n$ and $\lambda \in \mathbb{K} \cap (0,1)$ we obtain

$$\begin{aligned}
&\varphi_n\big(\lambda x + (1-\lambda)y\big) \\
&\quad = -\log_2\Big(\frac{1}{n} + \alpha_n \max\{\lambda x_1 + (1-\lambda)y_1, \dots, \lambda x_n + (1-\lambda)y_n\}\Big) \\
&\quad \leq -\log_2\Big(\lambda\frac{1}{n} + \alpha_n \max\{\lambda x_1, \dots, \lambda x_n\}\Big) \\
&\quad = -\log_2\Big(\frac{1}{n} + \alpha_n \max\{x_1, \dots, x_n\}\Big) - \log_2 \lambda \\
&\quad = \varphi_n(x) - \log_2 \lambda.
\end{aligned}$$

In a similar way we get

$$\varphi_n\big(\lambda x + (1-\lambda)y\big) \leq \varphi_n(y) - \log_2(1-\lambda).$$

This implies that

$$\begin{aligned}
\varphi_n\big(\lambda x + (1-\lambda)y\big) &\leq \lambda\varphi_n(x) + (1-\lambda)\varphi_n(y) - \log_2 \lambda^\lambda (1-\lambda)^{1-\lambda} \\
&\leq \lambda\varphi_n(x) + (1-\lambda)\varphi_n(y) + 1,
\end{aligned}$$

and therefore φ_n is an 1-convex function. It is easily seen that φ_n fulfils the Lipschitz condition with a constant L_n in $[0,1]^n$. On account of the Lemma the function $\psi_n \colon \mathbb{R}^n \to \mathbb{R}$ given by the formula

$$\psi_n(x) := \inf\{L_n\|x - y\| + \varphi_n(y)\ ;\ y \in [0,1]^n\}, \quad x \in \mathbb{R}^n,$$

is an extension of φ_n on the whole space $\mathbb{R}^n$ and, moreover, ψ_n is a 1-convex function. Let us define

$$\sigma_n = (1, \ldots, 1) = \sum_{i=1}^{n} e_i^n,$$
$$\psi_n^*(x) := \psi_n(\sigma_n - x), \quad x \in \mathbb{R}^n,$$

and let $Z := \operatorname{lin}_{\mathbb{K}}(D)$ denotes the linear space (over $\mathbb{K}$) generated by D. Observe that $\psi_n^*(0) = 0$. Since the dimension of the space $(Z, \mathbb{K})$ is infinite, one can choose an infinite sequence $(w_i)_{i\in\mathbb{N}}$ of linearly independent elements of D. Put

$$W := \{w_i \; ; \; i \in \mathbb{N}\} \quad \text{and} \quad W' := \{w' := 2^{-i}w_i \; ; \; i \in \mathbb{N}\}.$$

Note that every finite sum of elements of the set W' belongs to the set D, because it is a convex combination of elements of the set $W \subset D$ and $0 \in D$. The linear space $Y := \operatorname{lin}_{\mathbb{K}}(W')$ is a linear subspace of X and W' is a basis of Y. Let H_0 be a subset of X such that $H_0 \cap W' = \varnothing$ and $H := W' \cup H_0 = \{h_t, t \in T\}$ is a basis of the space X. Put

$$T_n := \Big\{\frac{n(n-1)}{2} + 1, \ldots, \frac{n(n+1)}{2}\Big\},$$
$$H_n := \{w_i' \in W' \; ; \; i \in T_n\},$$
$$Y_n := \operatorname{lin}_{\mathbb{K}}(H_n), \quad n \in \mathbb{N},$$
$$Y_0 := \operatorname{lin}_{\mathbb{K}}(H_0).$$

For an arbitrary positive integer n the dimension of the space Y_n is equal to n, and the set H_n is a basis of Y_n. Moreover, functions $I_n \colon Y_n \to \mathbb{K}^n \subset \mathbb{R}^n$ given by the formula

$$I_n(x) := \sum_{i=1}^{n} \lambda_{m_i^n} e_i^n,$$

where $x = \sum_{j\in T_n} \lambda_j w_j' \in Y_n$, $\lambda_{m_i^n} \in \mathbb{K}$, $w'_{m_i^n} \in H_n$, $m_i^n = \frac{n(n-1)}{2} + i$, $i=1, \ldots, n$, form isomorphisms of the spaces $(Y_n, \mathbb{K})$ onto $(\mathbb{K}^n, \mathbb{K})$.

Now we define functions $F_n \colon Y_n \to [0, \infty)$ by the formula

$$F_n(x) := \psi_n^*\big(I_n(x)\big), \quad x \in Y_n.$$

Functions F_n, $n \in \mathbb{N}$, are 1-$\mathbb{K}$-convex in Y_n, and $F_n(0) = 0$, $n \in \mathbb{N}$. Since

$$X = Y_n \oplus \operatorname{lin}_{\mathbb{K}}(H \setminus H_n) \quad \text{(direct sum)},$$

each element $x \in X$ has a unique representation of the form $x = x_n^1 + x_n^0$, where $x_n^1 \in Y_n$ and $x_n^0 \in \operatorname{lin}_{\mathbb{K}}(H \setminus H_n)$. In the next step we define the functions $G_n \colon X \to [0, \infty)$ by the formula

$$G_n(x) := F_n(x_n^1), \quad x = x_n^1 + x_n^0,\ x_n^1 \in Y_n,\ x_n^0 \in \operatorname{lin}_{\mathbb{K}}(H \setminus H_n).$$

Observe that G_n are 1-$\mathbb{K}$-convex, and $G_n(0) = 0$, $n \in \mathbb{N}$.

Every element $x \in X$ treated as an element of the linear space over $\mathbb{K}$ has a unique representation of the form

$$x = \sum_{n\in\mathbb{N}} x_n + x_0, \quad x_n \in Y_n,\ x_0 \in Y_0,\ n \in \mathbb{N}, \tag{4}$$

where the sum on the right hand side has a finite number of summands different from zero (in particular every summand may be equal to zero). Let us put

$$G(x) := \max\{G_n(x)\ ;\, n \in \mathbb{N}\}, \tag{5}$$

if x is of the form (4). Function $G\colon X \to [0, \infty)$ is a 1-$\mathbb{K}$-convex and $G(0) = 0$.

Consequences.

THEOREM. *Let $(X, \mathbb{K})$ be a linear space (over $\mathbb{K}$) and let $D \subset X$ be an infinite dimensional $\mathbb{K}$-convex subset of X. For arbitrary $\varepsilon > 0$ there exists an ε-$\mathbb{K}$-convex function $f\colon D \to \mathbb{R}$ such that for every $\mathbb{K}$-convex function $g\colon D \to \mathbb{R}$ we have*

$$\sup\{|f(x) - g(x)|\ ;\, x \in D\} = \infty.$$

PROOF. Without loss of generality we may assume that $0 \in D$ and $\varepsilon = 1$. Put

$$f := G_{|D},$$

where G is a function obtained in the previous section. For indirect proof assume that there exists a $\mathbb{K}$-convex function $g\colon D \to \mathbb{R}$ (*i.e.*, g fulfils condition (1) with $\varepsilon = 0$, $x, y \in D$, $t \in (0, 1) \cap \mathbb{K}$) and a constant $M > 0$ such that condition (3) is fulfilled. Fix a positive integer n such that

$$\log_2 n > 2M + 1. \tag{6}$$

Putting

$$u_n := \Big(1 - \frac{1}{n}\Big) \sum_{j\in T_n} w'_j, \quad v_n^i := \sum_{j\in T_n} w'_j - w'_{m_i^n}, \quad i = 1, \ldots, n,$$

we observe that

$$\begin{aligned} u_n &= \frac{n-1}{n} \sum_{j\in T_n} w'_j = \frac{1}{n}\Big(n \sum_{j\in T_n} w'_j - \sum_{i\in T_n} w'_i\Big) \\ &= \frac{1}{n}\Big(\sum_{i=1}^{n}\Big(\sum_{j\in T_n} w'_j - w'_{m_i^n}\Big)\Big) = \frac{1}{n}\sum_{i=1}^{n} v_n^i. \end{aligned}$$

Since $u_n, v_n^i \in D$, $i = 1, \ldots, n$, by virtue of (3) we have

$$\begin{aligned} f(u_n) = G(u_n) \le g(u_n) + M &= g\Big(\frac{1}{n}\sum_{i=1}^{n} v_n^i\Big) + M \\ &\le \frac{1}{n}\sum_{i=1}^{n} g(v_n^i) + M \le \frac{1}{n}\Big[\sum_{i=1}^{n}\big(G(v_n^i) + M\big)\Big] + M \\ &= \frac{1}{n}\sum_{i=1}^{n} G(v_n^i) + 2M. \end{aligned}$$

Moreover, since $v_n^i \in Y$, we have

$$\begin{aligned} G(v_n^i) = G_n(v_n^i) = \psi_n^*\big(I_n(v_n^i)\big) &= \psi_n^*\Big(I_n\Big(\sum_{j\in T_n} w'_j - w'_{m_n^i}\Big)\Big) \\ = \psi_n^*\Big(\sum_{j=1, j\ne i}^{n} e_j^n\Big) &= \psi_n(e_i^n) = \varphi_n(e_i^n) = -\log_2\Big(\frac{1}{n} + \Big(1 - \frac{1}{n}\Big)\Big) = 0. \end{aligned}$$

This means that

$$f(u_n) \le 2M. \tag{7}$$

On the other hand we obtain the following inequality

$$\begin{aligned} f(u_n) = G(u_n) &= \psi_n^*\Big(\Big(1 - \frac{1}{n}\Big)\sum_{j=1}^{n} e_j^n\Big) = \psi_n\Big(\frac{1}{n}\sum_{j=1}^{n} e_j^n\Big) \\ &= \varphi_n\Big(\frac{1}{n}\sum_{j=1}^{n} e_j^n\Big) = -\log_2\Big(\frac{2}{n} - \frac{1}{n^2}\Big) \ge -\log_2\frac{2}{n} = \log_2 n - 1, \end{aligned}$$

which jointly with (7) implies that

$$\log_2 n - 1 \le f(u_n) \le 2M.$$

This contradicts (6) and ends the proof of our theorem. ■

As an immediate consequence we obtain:

Corollary 1. *Let X be a real linear space, and let $D \subset X$ be an infinitely-dimensional subset of X. The inequality defining convex functions $f\colon D \to \mathbb{R}$ is not stable in the sense of Hyers and Ulam.*

Since every open and convex subset D of the space $\mathbb{R}^n$ treated as a linear space over $\mathbb{Q}$ (the set of all rationals) has infinite dimension, our theorem implies also the following corollary.

Corollary 2. *Let D be an infinitely-dimensional over $\mathbb{Q}$ and convex subset of $\mathbb{R}^n$ (for example convex and open). There exists an ε-J-convex function $f\colon D \to \mathbb{R}$ such that for every J-convex function $g\colon D \to \mathbb{R}$ we have*

$$\sup\{|f(x) - g(x)| \,;\, x \in D\} = \infty.$$

It follows from the definitions that every ε-$\mathbb{K}$-convex function is also a 2ε-$\mathbb{K}$-Wright-convex function. Note also that every $\mathbb{K}$-Wright-convex function is J-convex. Thus we can repeat the argument used in our theorem to obtain the following:

Corollary 3. *Let X be a linear space over $\mathbb{K}$ and let D be an infinite-dimensional subset of X. Then there exists an ε-$\mathbb{K}$-Wright-convex function such that for every $\mathbb{K}$-Wright-convex function $g\colon D \to \mathbb{R}$ we have*

$$\sup\{|f(x) - g(x)| \,;\, x \in D\} = \infty.$$

Particularly, Corollaries 1 and 3 say that in every infinite-dimensional case the inequalities defining the notions of the convexity and Wright-convexity, as well, are not stable in the sense of Hyers and Ulam. Similarly, Corollary 2 says that the inequality defining the notion of Jensen-convexity on an open and convex domain is not stable in the Hyers–Ulam sense.

References

1. E. Casini and P. L. Papini, *A counterexample to the infinity version of the Hyers and Ulam stability theorem.* Proc. Amer. Math. Soc. **118** (1993), 885–890.
2. P. W. Cholewa, *Remarks on the stability of functional equations.* Aequationes Math. **27** (1984), 76–86.
3. R. Ger, *Almost Approximately Convex Functions.* Math. Slovaca **38** (1988), 61–78.
4. J. W. Green, *Approximately convex functions.* Duke Math. J. **19** (1952), 499–504.
5. D. H. Hyers and S. M. Ulam, *Approximately convex functions.* Proc. Amer. Math. Soc. **3** (1952), 821–828.
6. Z. Kominek, *Report and abstracts of the meeting of functionals analysis and applications.* Extracta Math. **14** (1999), 69–98.
7. M. Laczkovich, *The local stability of convexity, affinity and of the Jensen equation.* Aequationes Math. **58** (1999), 135–142.
8. E. J. McShane, *Extension of range of functions.* Bull. Amer. Math. Soc. **40** (1934), 837–842.
9. J. Mrowiec, *On the stability of Wright-convex functions.* Aequationes Math. **65** (2003), 158–164.
10. S. Ulam, *A collection of mathematical problems.* Interscience, New York, 1960.

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