

ON $L^{(r+1)}(\pi, 1/2)$

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ABSTRACT. Let r be the order of vanishing of the automorphic L -function $L(\pi, s)$ at $s = 1/2$. We study the non-vanishing of the derivative of order $r + 1$ of $L(\pi, s)$ at $s = 1/2$.

RÉSUMÉ. Soit r l'ordre d'annulation de la fonction L automorphe $L(\pi, s)$ à $s = 1/2$. Nous étudions la non-annulation de la dérivée d'ordre $r + 1$ de $L(\pi, s)$ à $s = 1/2$.

1. Introduction. Let F be a number field of degree d and $\pi = \bigotimes_{\nu} \pi_{\nu}$ be an irreducible cuspidal automorphic representation of GL_m over F with unitary central character and contragredient representation $\tilde{\pi}$. Let $L(\pi, s)$ and $L(\tilde{\pi}, s)$ be the associated L -functions to π and $\tilde{\pi}$. We have

$$L(\tilde{\pi}, s) = \overline{L(\pi, \bar{s})}.$$

It is known that $L(\pi, s)$ and $L(\tilde{\pi}, s)$ satisfy the functional equation

$$(1) \quad q^{s/2} L(\pi_{\infty}, s) L(\pi, s) = \omega q^{(1-s)/2} L(\tilde{\pi}_{\infty}, 1-s) L(\tilde{\pi}, 1-s),$$

where the positive integer q is the conductor of π , ω is the root number (a complex number of modulus 1) and

$$L(\pi_{\infty}, s) = \prod_{j=1}^{md} \pi^{-s/2} \Gamma\left(\frac{s + \mu_j}{2}\right), \quad L(\tilde{\pi}_{\infty}, s) = \prod_{j=1}^{md} \pi^{-s/2} \Gamma\left(\frac{s + \bar{\mu}_j}{2}\right).$$

Note that each side of equation (1) represents a meromorphic function in the whole complex plane with at most two simple poles. Moreover, by a theorem of Luo, Rudnick and Sarnak [LRS], we have

$$\Re \mu_j \geq \frac{1}{m^2 + 1} - \frac{1}{2}, \quad j = 1, \dots, md,$$

which implies that $L(\pi_{\infty}, s)$ and $L(\tilde{\pi}_{\infty}, s)$ are analytic and non-zero on the half plane $\Re s > \frac{1}{2} - \frac{1}{m^2 + 1}$. Let

$$r = \mathrm{ord}_{s=1/2} L(\pi, s).$$

So we have $L^{(i)}(\pi, 1/2) = 0$ for $0 \leq i \leq (r - 1)$ and $L^{(r)}(\pi, 1/2) \neq 0$. In this note we will exploit the functional equation (1) to investigate the possible values of $L^{(r+1)}(\pi, 1/2)$. In fact, we show that in several cases this value is non-zero.

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2. The main lemma. The following lemma is a generalization of Exercise 5.5.22 of [M].

LEMMA 2.1. *With the above notation we have*

$$\frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)} + \frac{L^{(r+1)}(\tilde{\pi}, 1/2)}{L^{(r)}(\tilde{\pi}, 1/2)} = -(r+1) \left(\log q + \frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)} \right).$$

PROOF. Let $A(s) = q^{s/2}L(\pi_\infty, s)$ and $B(s) = q^{s/2}L(\tilde{\pi}_\infty, s)$. So from (1) we have

$$A(1/2)L^{(r)}(\pi, 1/2) = \omega(-1)^r B(1/2)L^{(r)}(\tilde{\pi}, 1/2).$$

Similarly from (1) we have

$$\begin{aligned} A(1/2)L^{(r+1)}(\pi, 1/2) + (r+1)A'(1/2)L^{(r)}(\pi, 1/2) \\ = \omega(-1)^{r+1} (B(1/2)L^{(r+1)}(\tilde{\pi}, 1/2) + (r+1)B'(1/2)L^{(r)}(\tilde{\pi}, 1/2)). \end{aligned}$$

Dividing the latter equation to the former one yields

$$\frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)} + \frac{L^{(r+1)}(\tilde{\pi}, 1/2)}{L^{(r)}(\tilde{\pi}, 1/2)} = -(r+1) \left(\frac{A'(1/2)}{A(1/2)} + \frac{B'(1/2)}{B(1/2)} \right).$$

Now the result follows by calculating the logarithmic derivative of $A(s)B(s) = q^s L(\pi_\infty, s)L(\tilde{\pi}_\infty, s)$ at $s = 1/2$. \blacksquare

The following corollary is a direct consequence of the previous lemma.

COROLLARY 2.2.

$$q \neq \exp \left(- \left(\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)} \right) \right) \Rightarrow L^{(r+1)}(\pi, 1/2) \neq 0.$$

From now on, let $\psi(s) = \Gamma'(s)/\Gamma(s)$ and $\mu_j = \sigma_j + it_j$. An automorphic representation π is called *tempered* if $\sigma_j = 0$ for $j = 1, \dots, md$. We have the following.

COROLLARY 2.3. *If π is tempered we have*

$$\Re \left(\frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)} \right) = 0 \iff q = (2\pi)^{md} \exp \left(\frac{\pi}{2} \sum_{j=1}^{md} \operatorname{sech}(\pi t_j) \right) \exp \left(- \sum_{j=1}^{md} \Re \left(\psi \left(\frac{1}{2} + it_j \right) \right) \right).$$

PROOF. From Lemma 2.1, we have

$$\begin{aligned}
(2) \quad \Re\left(\frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)}\right) = 0 &\iff q = \exp\left(-\left(\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)}\right)\right) \\
&\iff q = \pi^{md} \exp\left(-\frac{1}{2} \sum_{j=1}^{md} \left(\psi\left(\frac{1}{4} + i\frac{t_j}{2}\right) + \psi\left(\frac{1}{4} - i\frac{t_j}{2}\right)\right)\right) \\
&\iff q = (2\pi)^{md} \exp\left(\frac{\pi}{2} \sum_{j=1}^{md} \tan\left(\frac{\pi}{4} + i\frac{\pi t_j}{2}\right)\right) \exp\left(-\sum_{j=1}^{md} \psi\left(\frac{1}{2} + it_j\right)\right).
\end{aligned}$$

The last equivalence is a consequence of calculating the logarithmic derivative of the identity

$$(3) \quad \left(\cos \frac{\pi s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \Gamma(s) = \sqrt{\pi} 2^{s-1} \Gamma\left(\frac{s}{2}\right)$$

[D, p. 73] at $s = 1/2 + it_j$. Next note that

$$(4) \quad \Re\left(\tan\left(\frac{\pi}{4} + i\frac{\pi t_j}{2}\right)\right) = \operatorname{sech}(\pi t_j).$$

Now the results follows from (2), (4) and the fact that

$$\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)}$$

is real. ■

In the rest of this paper, we apply the above corollaries in some special cases and as a result we prove that for Dirichlet L -functions and modular L -functions $L^{(r+1)}(\pi, 1/2) \neq 0$. We also investigate the situation for the L -functions associated to Maass forms.

3. GL(1).

PROPOSITION 3.1. *If χ is a primitive Dirichlet character mod q , then*

$$L^{(r+1)}(\chi, 1/2) \neq 0.$$

PROOF. We have

$$L(\pi_\infty, s) = L(\tilde{\pi}_\infty, s) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) & \text{if } \chi(-1) = 1 \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) & \text{if } \chi(-1) = -1. \end{cases}$$

Then

$$\begin{aligned} \exp\left(-\left(\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)}\right)\right) &= \begin{cases} \pi e^{-\psi(1/4)} & \text{if } \chi(-1) = 1 \\ \pi e^{-\psi(3/4)} & \text{if } \chi(-1) = -1 \end{cases} \\ &= \begin{cases} 8\pi e^{\gamma + \frac{\pi}{2}} & \text{if } \chi(-1) = 1 \\ 8\pi e^{\gamma - \frac{\pi}{2}} & \text{if } \chi(-1) = -1. \end{cases} \end{aligned}$$

Here γ is the Euler constant and we used the identities $\psi(\frac{1}{4}) = -\gamma - \frac{\pi}{2} - 3 \log 2$ and $\psi(\frac{3}{4}) = -\gamma + \frac{\pi}{2} - 3 \log 2$ (see [W]). Now Corollary 2.2 implies the result. \blacksquare

4. GL(2), the holomorphic case.

PROPOSITION 4.1. *If f is a holomorphic cuspidal newform of weight k and level q and nebentypus χ , then $L^{(r+1)}(f, 1/2) \neq 0$.*

PROOF. By employing the Legendre duplication formula, we have

$$\begin{aligned} L(\pi_\infty, s) &= L(\tilde{\pi}_\infty, s) = \pi^{-s} \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right) \\ &= \frac{\sqrt{\pi}}{2^{\frac{k-3}{2}}} (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right). \end{aligned}$$

So

$$\exp\left(-\left(\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)}\right)\right) = (2\pi)^2 \exp(-2\psi(k/2)).$$

We know that

$$(5) \quad \psi(z) = \log z + O(1/|z|)$$

for $|z| \rightarrow \infty$ in the sector $-\pi + \delta < \arg z < \pi - \delta$ for any fixed $\delta > 0$ (see [M, Exercise 6.3.17]). So

$$\lim_{k \rightarrow \infty} (2\pi)^2 \exp(-2\psi(k/2)) = 0.$$

More precisely, by evaluating $f(k) = (2\pi)^2 \exp(-2\psi(k/2))$ for integer k and using Maple, we can see that for integer $1 \leq k \leq 13$, $f(k)$ is not an integer, and for integer $k > 13$, we have $0 < f(k) < 1$. So $f(k)$ never is an integer and therefore $q \neq f(k)$. Thus Corollary 2.2 implies the result. \blacksquare

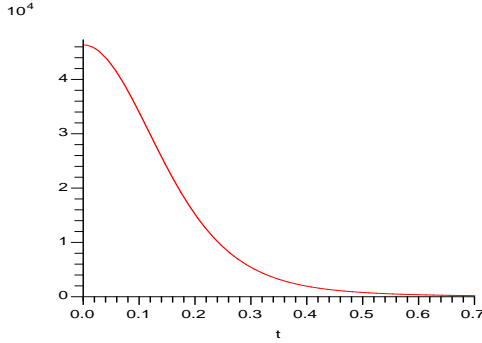
5. **GL(2), the real analytic case.** For $t \geq 0$, let

$$g(t) = (2\pi)^2 \exp(\pi \operatorname{sech}(\pi t)) \exp\left(-2\Re\left(\psi\left(\frac{1}{2} + it\right)\right)\right).$$

From (5) and $\lim_{t \rightarrow \infty} \operatorname{sech}(t) = 0$ we have

$$\lim_{t \rightarrow \infty} g(t) = 0.$$

By employing Maple one can show that $g(0) = 46,368.09\dots$, and $0 < g(t) < 1$ for $t \geq 6.29$. The following is the graph of $g(t)$ for $0 \leq t \leq 0.7$.



From here it is clear that for any integer $1 \leq q \leq 46,368$ there exists a unique $0 < t_q < 6.29$ such that $g(t_q) = q$.

PROPOSITION 5.1. *If f is an even Maass cuspidal newform of weight zero and level q , nebentypus χ and eigenvalue λ , then, under the assumption of the Selberg Eigenvalue Conjecture, we have the following.*

(i) *If $q \leq 46,368$, then*

$$\Re\left(\frac{L^{(r+1)}(f, 1/2)}{L^{(r)}(f, 1/2)}\right) = 0 \iff \lambda = \frac{1}{4} + t_q^2.$$

(ii) *If $q \geq 46,369$ or $\lambda \geq 6.54$, then*

$$L^{(r+1)}(f, 1/2) \neq 0.$$

PROOF. Let λ be the eigenvalue corresponding to f , then $\lambda = \frac{1}{4} + t^2$. In this case for the ∞ factors in the functional equation (1) we have $\mu_1 = it$ and $\mu_2 = -it$. More precisely, we have

$$L(\pi_\infty, s) = L(\tilde{\pi}_\infty, s) = \pi^{-s} \Gamma\left(\frac{s+it}{2}\right) \Gamma\left(\frac{s-it}{2}\right).$$

From the Selberg Eigenvalue Conjecture we know that t is real, so by Corollary 2.3 and the definition of the function $g(t)$ we have

$$\begin{aligned} \Re\left(\frac{L^{(r+1)}(f, 1/2)}{L^{(r)}(f, 1/2)}\right) &= 0 \\ \iff q &= (2\pi)^2 \exp(\pi \operatorname{sech}(\pi t)) \exp\left(-2\Re\left(\psi\left(\frac{1}{2} + it\right)\right)\right) \\ \iff q &= g(t). \end{aligned}$$

But $q = g(t)$ if and only if $t = t_q$. ■

6. Conclusion. Our observations here indicate that $L^{(r+1)}(\pi, 1/2) \neq 0$ in several cases. We end this note by raising the following question. Is there an automorphic representation π such that $L^{(r+1)}(\pi, 1/2) = 0$?

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