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THE AVALANCHE PRINCIPLE: FROM JOINT TO AVERAGED JOINT SPECTRAL RADIUS

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Abstract. The averaged joint spectral radius (AJSR) is defined. By using the *avalanche principle* we develop an effective algorithm to compute the averaged joint spectral radius for a pair of 2×2 matrices.

Résumé. Nous introduisons la notion de rayon spectral moyen d'un ensemble fini de matrices. En utilisant le *principe d'avalanche*, nous développons un algorithme efficace pour calculer le rayon spectral moyen d'une paire de matrices de tailles 2×2.

1. Introduction.

A *joint spectral radius* was defined by G-K. Rota and G. Strang in the 1960's [RS]. For a long time this interesting quantity was not investigated in mathematical researches. In the 1990's Daubechies and Lagarias and also Coleila and Heil, proved the importance of this object for Markov chains, random walks and in solving the central equation in wavelet theory: the *dilation equation* (alternatively: *the refinement equation* or *the scaling equation*). It was proved that the joint spectral radius can be used as a characteristic of continuity and Hölder continuity of the scaling vector (for more information about a multiresolution analysis see [CH1]). Moreover, it was shown that the joint spectral radius is an effective instrument in subdivision scheme analysis [CCS], [LL].

Recall that a multiresolutional analysis of $L^2(\mathbb{R})$ is an increasing sequence of closed subspaces

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

with $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ dense in $L^2(\mathbb{R})$, and such that

(i) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$;

(ii) there exists a function $\varphi(x) \in V_0$, called a *scaling function*, such that the family $\{\varphi(x-k), k \in \mathbb{Z}\}$ is an orthonormal basis in V_0.

Each multiresolution analysis in wavelet theory (see [D], [W]) determines the *dilation equation*:

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(2x-k).$$

A *scaling function* $\varphi(x)$ is a solution of the dilation equation. The coefficients $\{c_k\}$ are square-summable complex (in general) numbers. The function $\varphi(x)$ is

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a square integrable function that determines a *mother wavelet*,

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k c_{1-k}\varphi(2x-k),$$

such that the collection $\{\psi_{jk}(x) = \psi(2^j x - k)\}_{j,k}$ is an orthogonal basis in $L^2(\mathbb{R})$ (for more information about the scaling equation and mother wavelets, see [D], [W]). In order to ensure compact support for the wavelet ψ, we assume that the number of nonzero coefficients c_k is finite. Thus, the coefficients $\{c_k\}_{k=0}^N$ play a crucial role in the construction of multiresolutional analysis and wavelet orthonormal bases for the space $L^2(\mathbb{R})$. Then the question becomes: how can one choose the scalars $\{c_k\}_{k=0}^N$ in such a way that the resulting properties of the multiresolutional analysis will be good? The problem can be translated into matrix language. Two matrices T_0 and T_1 are associated to the dilation equation, $(T_0)_{ij} = c_{2j-i-1}$, $(T_1)_{ij} = c_{2j-i}$, and their interaction, via long products, determines many of the good (or bad) properties of the scaling function. One of the effective tools to investigate these long products is *the joint spectral radius* of the matrices T_0 and T_1 (restricted to a special subspace). The joint spectral radius can be considered as a generalization of spectral radius of a matrix to a set of matrices. For a finite set of matrices $M \subset M_d(\mathbb{C})$ we put

$$\Pi_n = \max_{A_1,\ldots,A_n\in M}\{\|A_1A_2\cdots A_n\|\}.$$

Definition 1.1. The joint spectral radius of the set M is defined by

$$\widehat{\rho}(M) = \limsup_{n\to\infty} \|\Pi_n\|^{\frac{1}{n}}.$$

Note that the quantity just defined is the generalization of the well-known spectral radius of a matrix A:

$$\rho(A) = \lim \|A\|^{\frac{1}{n}} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A \in M_n(\mathbb{C})\}.$$

Theorem 1.2. ([CH2, p. 177]) *If $\widehat{\rho}(T_0|_W, T_1|_W) < 1$, then there exists a continuous scaling function $\varphi(x)$ which is Hölder continuous with Hölder exponent α for every $0 \le \alpha < -\log_2 \widehat{\rho}(T_0|_W, T_1|_W)$, where W denotes a special subspace of C^N.*

However, it turns out that the joint spectral radius is hard to compute (see [BT1], [BT2]).

Theorem 1.3. ([BT2]) *Unless $P = NP$, the joint spectral radius $\widehat{\rho}$ of two matrices is not polynomial-time approximable.*

This work is a first step in developing some alternative approach that we believe will be useful for characterizing the properties of the scaling function and, on the other hand, will deal with more typical objects in the probabilistic sense. The basic tool of our approach is the *avalanche principle* that was established by M. Goldstein and W. Schlag [GS]. This principle proved its effectiveness in studying the *Lyapunov exponent* which is intimately connected to the joint spectral radius (see [BT2]).

THEOREM 1.4. (Avalanche Principle [GS, Prop. 2.2]) *Let $A_1, A_2, \ldots, A_n$ be a sequence of unimodular 2×2 matrices. Suppose that*

(i) $\min_{1\le j\le n} \|A_j\| \ge \mu > n$,
(ii) $\max_{1\le j\le n-1}[\log\|A_{j+1}\| + \log\|A_j\| - \log\|A_{j+1}A_j\|] < \frac{1}{2}\log\mu$.

Then

$$\left|\log\|A_n\cdots A_1\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\right| < c\frac{n}{\mu}.$$

To define the *averaged joint spectral* radius we need several auxiliary notions and results.

DEFINITION 1.5. Let Ω_n denote the set of all the words of length n composed from the letters A and B, where $A, B \in M_d(\mathbb{C})$, and let Π_n be an element of Ω_n. We put $\gamma_n(A,B) = 2^{-n}\cdot\sum_{\Pi_n\in\Omega_n} n^{-1}\log\|\Pi_n\|$.

In the next lemma we adduce some well-known facts about subadditive sequences.

LEMMA 1.6.

(i) $(n+m)\gamma_{n+m} \le n\gamma_n + m\gamma_m$, i.e., $\{n\gamma_n\}_{n=1}^{\infty}$ *is a subadditive sequence.*
(ii) $\gamma_n \le \log\max\{\|A\|, \|B\|\}$.
(iii) *There exists* $\gamma(A,B) = \lim_{n\to\infty}\gamma_n = \inf_n \gamma_n$.

PROOF. The first and the second assertions are trivial. The third assertion is a well-known fact for subadditive sequences, and we omit the proof. ■

Now we are ready to give the key definition of this paper.

DEFINITION 1.7. Let $A, B \in M_n(\mathbb{C})$. An *averaged joint spectral radius* of the matrices A and B is $\widetilde{\rho}(A,B) = e^{\gamma(A,B)}$.

The next theorem is the main result of the paper, and is proved in Section 3.

THEOREM 1.8. *Let $A, B \in M_2(C)$. Suppose that*

(i) $\min\{\|A\|, \|B\|\} \ge \mu_0 > n$,
(ii) $\max[\log\|C_1\| + \log\|C_2\| - \log\|C_2C_1\|] < \frac{1}{2}\log\mu_0$ *for any* $C_1, C_2 \in \{A, B\}$.

For any natural number n, denote by $\mu^*(n)$ the greatest multiple of n which is smaller than $\mu(n) = \mu_0^{\frac{n}{2}}$. Then

$$\gamma_{\mu^*(n)} - (2\gamma_{2n} - \gamma_n) < \mathbf{C}\frac{n}{\mu^*(n)},$$

where $\mathbf{C}$ *denotes an absolute constant.*

In the next section, we slightly improve Theorem 1.4 in order to study the hereditary character of the avalanche condition when the length of a product increases. In Section 3 we will prove Theorem 1.8.

2. **Around the avalanche principle: some deterministic developments.** In this section we develop the deterministic contents of the avalanche principle and explain its self-reproducing character. In our first lemma we show that the norm growth of long matrix products, satisfying an avalanche condition, is exponentially fast. It is not a surprising fact, but it is important for our further calculations.

Lemma 2.1. *Let $C_1, \dots, C_n \in M_2(\mathbb{C})$, $n > 1$. Suppose that*

(i) *all of $C_1, \dots, C_n$ are unimodular;*

(ii) $\min_{i=\overline{1,n}} \|C_i\| \ge \mu_0 > n$;

(iii) *the avalanche condition,* $\log\|C_i\| + \log\|C_j\| - \log\|C_iC_j\| < \frac{1}{2}\log\mu_0$, *holds for all* $1 \le i, j \le n$.

Then

$$\left|\log\|C_n \cdots C_1\| + \sum_{j=2}^{n-1}\log\|C_j\| - \sum_{j=1}^{n-1}\log\|C_{j+1}C_j\|\right| < c\frac{n}{\mu_0},$$

$$\|C_n \cdots C_1\| \ge \mu(n) = \mu_0^{\frac{n}{2}}.$$

Proof. The first inequality follows immediately from [GS, Prop. 2.2]. To prove the second inequality, note that

$$\log\|C_n \cdots C_1\| > \sum_{j=1}^{n-1}\log\|C_{j+1}C_j\| - \sum_{j=2}^{n-1}\log\|C_j\| - c\frac{n}{\mu_0}.$$

Set

$$S = \sum_{j=1}^{n-1}\log\|C_{j+1}C_j\| - \sum_{j=2}^{n-1}\log\|C_j\|.$$

For even n we have

$$\begin{aligned} S &= \sum_{j=1}^{n/2}\log\|C_{2j}C_{2j-1}\| - \sum_{j=1}^{n/2-1}\left(\log\|C_{2j+1}\| + \log\|C_{2j}\| - \log\|C_{2j+1}C_{2j}\|\right) \\ &> \frac{3}{4}n\log\mu_0 - \frac{n-2}{4}\log\mu_0 = \frac{n+1}{2}\log\mu_0. \end{aligned}$$

If n is odd, then

$$S = \sum_{j=1}^{(n-1)/2} \log \|C_{2j+1}C_{2j}\| - \sum_{j=1}^{(n-1)/2} \left(\log \|C_{2j}\| + \log \|C_{2j-1}\| - \log \|C_{2j}C_{2j-1}\|\right) + \log \|C_1\|$$
$$> \frac{3(n-1)}{4} \log \mu_0 - \frac{n-1}{4} \log \mu_0 + \log \mu_0 = \frac{n+1}{2} \log \mu_0.$$

Thus $S > \frac{n+1}{2} \log \mu_0$, which implies

$$\log \|C_n \cdots C_1\| > \frac{n+1}{2} \log \mu_0 - c\frac{n}{\mu_0}$$

and completes the proof. ■

Remark. Throughout this paper, $\mu(n)$ denotes the quantity $\mu_0^{\frac{n}{2}}$. Lemma 2.2 indicates the self-reproducing character of the avalanche principle.

Lemma 2.2. *Let $C_1, \dots, C_n \in M_2(\mathbb{C})$ as in Lemma 2.1, and let $2 < k < n-2$. Then* $\log \|C_n \cdots C_1\| - \log \|C_n \cdots C_{k+1}\| - \log \|C_k \cdots C_1\| < c\frac{2n}{\mu_0} + \frac{1}{2} \log \mu_0$.

Proof. Let

$$S_1 = \sum_{j=1}^{k-1} \log \|C_{j+1}C_j\| - \sum_{j=2}^{k-1} \log \|C_j\|,$$
$$S_2 = \sum_{j=k+1}^{n-1} \log \|C_{j+1}C_j\| - \sum_{j=k+2}^{n-1} \log \|C_j\|.$$

By Lemma 2.1

$$\left|\log \|C_k \cdots C_1\| - S_1\right| < c\frac{k}{\mu_0}, \qquad \left|\log \|C_n \cdots C_{k+1}\| - S_2\right| < c\frac{n-k}{\mu_0}.$$

Obviously,

$$\sum_{j=2}^{n-1} \log \|C_j\| - \sum_{j=1}^{n-1} \log \|C_{j+1}C_j\| = -S_1 - S_2 + \log \|C_k\| + \log \|C_{k+1}\| - \log \|C_{k+1}C_k\|.$$

Hence

$$\left|\log \|C_n \cdots C_1\| - S_1 - S_2 + \log \|C_k\| + \log \|C_{k+1}\| - \log \|C_{k+1}C_k\|\right| < c\frac{n}{\mu_0}.$$

Since

$$\log\|C_k\| + \log\|C_{k+1}\| - \log\|C_{k+1}C_k\| < \frac{1}{2}\log\mu_0,$$

it follows that

$$\left|\log\|C_n\cdots C_1\| - S_1 - S_2\right| < c\frac{n}{\mu_0} + \frac{1}{2}\log\mu_0.$$

Finally,

$$\log\|C_n\cdots C_1\| - \log\|C_k\cdots C_1\| - \log\|C_n\cdots C_{k+1}\| < c\frac{2n}{\mu_0} + \frac{1}{2}\log\mu_0,$$

as claimed. ■

Remark. Thus, matrices satisfying the avalanche principle generate words having the same property:

$$\log\|C_n\cdots C_1\| - \log\|C_n\cdots C_{k+1}\| - \log\|C_k\cdots C_1\| < \frac{1}{2}\log\mu(\min\{k, n-k\}),$$

for sufficiently large μ_0.

Theorem 2.3. *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of matrices from $M_2(\mathbb{C})$. Suppose that*

(i) *A_i is unimodular for every $i \in \mathbb{N}$,*
(ii) *each pair A_i, A_j satisfies the avalanche principle:*

$$\log\|C_i\| + \log\|C_j\| - \log\|C_iC_j| < \frac{1}{2}\log\mu_0,$$

where $\mu_0 = \min_n\|C_n\|$ is a sufficiently large positive number.

Then for every $n, m \in \mathbb{N}$ with $2 < k < n-2$ and $l = \min\{k, n-k\}$, the following inequalities hold:

$$\|A_{m+k}\cdots A_{m+1}\| > \mu(k),$$

$$\log\|A_{m+n}A_{m+k+1}\| + \log\|A_{m+k}\cdots A_{m+1}\| - \log\|A_{m+n}\cdots A_{m+1}\| < \frac{1}{2}\log\mu(l).$$

Proof. Apply Lemma 2.2. ■

3. The averaged joint spectral radius. Proof of Theorem 1.8. In this section we prove Theorem 1.8. The theorem provides an effective algorithm for computing AJSR in the 2×2 case.

For $\Pi_{\mu^*(n)} = \Pi_1 \Pi_2 \cdots \Pi_m$, where $m = \frac{\mu^*(n)}{n}$, we define f_i, a map from $\Omega_{\mu^*(n)}$ to Ω_n, by $f_i(\Pi_{\mu^*(n)}) = \Pi_i$, $i = 1, \dots, m$. Then by combining Theorem 2.3 and the avalanche principle, we have

$$
\begin{aligned}
\gamma_{\mu^*(n)} &= 2^{-\mu^*(n)} \sum_{\Omega_{\mu^*(n)}} \mu^*(n)^{-1} \log \|\Pi_{\mu^*(n)}\| \\
&= \frac{2^{-\mu^*(n)}}{\mu^*(n)} \sum_{\Omega_{\mu^*(n)}} \Big[\sum_{i=1}^{m-1} \log \|f_{i+1}(\Pi_{\mu^*(n)}) f_{i+1}(\Pi_{\mu^*(n)})\| \\
&\qquad - \sum_{i=1}^{m-1} \log \|f_i(\Pi_{\mu^*(n)})\| + \frac{cm}{\mu^*(n)} \Big] \\
&= \frac{2^{-\mu^*(n)}}{\mu^*(n)} \sum_{i=1}^{m-1} \Big[\sum_{\Omega_{\mu^*(n)}} \log \|f_{i+1}(\Pi_{\mu^*(n)}) f_{i+1}(\Pi_{\mu^*(n)})\| \\
&\qquad - \sum_{\Omega_{\mu^*(n)}} \log \|f_i(\Pi_{\mu^*(n)})\| + \frac{cm}{\mu^*(n)} \Big] \\
&= \frac{2^{-\mu^*(n)}}{\mu^*(n)} \sum_{i=1}^{m-1} \big[2^{\mu^*(n)} 2n\gamma_{2n} - 2^{\mu^*(n)} n\gamma_n \big] + \frac{cm}{(\mu^*(n))^2} \\
&= \frac{1}{\mu^*(n)} \sum_{i=1}^{m-1} [2n\gamma_{2n} - n\gamma_n] + \frac{cm}{(\mu^*(n))^2} \\
&= \frac{2n(m-1)\gamma_{2n}}{\mu^*(n)} - \frac{n(m-1)\gamma_n}{\mu^*(n)} + \frac{cm}{(\mu^*(n))^2} \\
&= \frac{2(m-1)\gamma_{2n}}{m} - \frac{(m-1)\gamma_n}{m} + \frac{cm}{(\mu^*(n))^2} \\
&= 2\gamma_{2n} - \gamma_n + \mathbf{C} \frac{n}{\mu^*(n)}.
\end{aligned}
$$

This completes the proof of Theorem 1.8. ■

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