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HYPERBOLICITY OF QUADRATIC FIELDS, SEMIGROUP ALGEBRAS AND *RA*-LOOPS

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ABSTRACT. For the rational extension $K = \mathbb{Q}\sqrt{-d}$ with d a *square free* integer and R the ring of algebraic integers of K, we classify R and G such that $\mathcal{U}_1(RG)$ is a hyperbolic group. In particular, the unit group $\mathcal{U}_1(RK_8)$ is hyperbolic if and only if $d > 0$ and $d \equiv 7 \pmod 8$. In this case, the hyperbolic boundary $\partial(U_1(RK_8))$ is isomorphic to S^2, the two-dimensional sphere. Thus, $\mathcal{U}_1(RK_8)$ is a hyperbolic group of one end. Also, for a given division algebra of the quaternions, we construct two types of units of its $\mathbb{Z}$-orders: Pell's units and Gauss' units. Next, we classify the finite semigroups S such that for all $\mathbb{Z}$-orders Γ of the algebra $\mathbb{Q}S$, the unit group $\mathcal{U}(\Gamma)$ is hyperbolic. Finally, we classify the *RA*-loops L for which the unit loop of its integral loop ring does not contain a free abelian subgroup of rank two.

RÉSUMÉ. Nous classifions les anneaux d'entiers des extensions quadratiques rationelles, que nous noterons R, tel que le groupe d'unités $\mathcal{U}(RG)$ sur ces anneaux est hyperbolique pour un certain groupe G fixé. En particulier, le groupe $\mathcal{U}_1(RK_8)$ est hyperbolique si et seulement si $d > 0$ et $d \equiv 7 \pmod 8$. Dans ce cas, la frontière hyperbolique $\partial(U_1(RK_8))$ est isomorphe à la sphère S^2 de dimension 2. Nous considérons une algèbre de quaternions qui est aussi une algèbre de division. Pour un $\mathbb{Z}$-ordre de cette algèbre, nous présentons des constructions de deux types d'unités: les unités de Gauss et les unités de Pell. Par la suite, nous classifions les semi-groupes finis S dont l'algébre unitaire $\mathbb{Q}S$ verifie la propriété suivante: pour tout $\mathbb{Z}$-ordre Γ de $\mathbb{Q}S$ le groupe d'unités $\mathcal{U}(\Gamma)$ est hyperbolique. Dans le même contexte, nous classifions les *RA*-loops L dont le loop d'unités ne contient aucun sous-groupe abelien libre de rang 2.

1. **Introduction.** Hyperbolic groups were first defined and studied by Gromov [5]. If G is a finitely generated group with a symmetric system of generators S and $\mathcal{G}(G,S)$ is its Cayley graph with the length metric, then G is said to be hyperbolic if $\mathcal{G}(G,S)$ is a hyperbolic metric space.

Gromov showed (the flat plane theorem), that if Γ is a hyperbolic group, then it does not contain a free abelian group of rank 2, $\mathbb{Z}^2 \not\hookrightarrow \Gamma$ say. If G is a finite group such that $\mathcal{U}_1(\mathbb{Z}G)$ is hyperbolic, then $\mathbb{Q}G$ has at most one simple epimorphic image that is not a division ring, and it is isomorphic to $M_2(\mathbb{Q})$. This was first proved by Jespers [8], who also classified the finite groups G with non abelian free normal complement in $\mathcal{U}_1(\mathbb{Z}G)$, the group of normalized units of $\mathbb{Z}G$.

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Recently, Juriaans, Passi and Prasad [10] have classified the finite groups G for which the unit group $\mathcal{U}_1(\mathbb{Z}G)$ is hyperbolic. In Section 2, we extend this result, classifying, for the rational quadratic extensions $K = \mathbb{Q}(\sqrt{-d})$, the ring of algebraic integers R of K and the finite groups G such that $\mathcal{U}_1(RG)$ is hyperbolic.

In Section 3, for a division ring $\mathbf{H}(K)$, we construct some units of the $\mathbb{Z}$-order $\mathbf{H}(R) \subset \mathbf{H}(K)$ as follows: for a given Pell equation, each of its solutions over $\mathbb{Z}(\sqrt{d})$ generates units $u \in \mathcal{U}(\mathbf{H}(R))$ of norm 1, which we define as *Pell units*. Furthermore, we construct units $u \in \mathcal{U}(\mathbf{H}(R))$ of norm -1 which give rise to what we call *Gauss units*. Using these units we show that $\mathcal{U}(RK_8)$ has a free subgroup of rank 2 when $d \equiv 7 \pmod 8$.

In Section 4, we give the structure of the finite dimensional algebras $\mathcal{A}$ over $\mathbb{Q}$ such that for every $\mathbb{Z}$-order $\Gamma \subset \mathcal{A}$, the group $\mathcal{U}(\Gamma)$ is hyperbolic, and we say that such algebras have the *hyperbolic property*. We also classify the finite semigroups S for which the unitary algebra $\mathbb{Q}S$ has the hyperbolic property; an example of such an algebra is when $\mathcal{U}(\Gamma)$ is a torsion group. Jespers and Wang [9] classified the finite semigroups Σ for which $\mathcal{U}(\mathbb{Z}\Sigma)$ is finite. Therefore, such semigroup algebras $\mathbb{Q}\Sigma$ are examples of algebras that have the hyperbolic property. We extend this class of semigroups Σ and describe the finite semigroups S whose rational semigroup algebra has the hyperbolic property. Finally, we give some properties of the idempotents of the maximal subgroups of S. In the last section we classify the RA-loops L, such that, $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(\mathbb{Z}L)$.

2. The rings R with $\mathcal{U}_1(RG)$ hyperbolic. We define the set of *square-free integers* to be $\mathcal{D} = \{d \in \mathbb{Z} \setminus \{-1, 0\} : c^2 \nmid d$ for all integers c with $c^2 \neq 1\}$. We let K be the quadratic extension $\mathbb{Q}(\sqrt{-d})$ and $R := I_K$ its ring of algebraic integers. The cyclic group of order n is denoted by C_n and the quaternion group of order 8 is denoted $K_8 := \{\pm 1, \pm i, \pm j, \pm k\}$.

If G is a finite abelian group, then the unit group $\mathcal{U}_1(RG)$ is a hyperbolic group if and only if its free rank is at most 1. In [13], it is shown that it is sufficient to consider the cyclic groups of order $2, 3, 4, 5, 6$ or 8. Thus, for the abelian groups G of [10, Theorem 3] which classify the finite groups G such that $\mathcal{U}_1(\mathbb{Z}G)$ is a hyperbolic group, the free rank of $\mathcal{U}_1(RG)$ is calculated. When G is one of the non-abelian groups of [10, Theorem 3], we show that in case $\mathcal{U}_1(RG)$ is hyperbolic, K is an imaginary quadratic extension and $G = K_8$. To prove the converse, we use a geometric approach.

Definition 2.1. Let K be an algebraic number field and R its ring of algebraic integers. For $a, b \in K$, we denote by $\mathbf{H}(K) = \left(\frac{a,b}{K}\right)$ the *generalized quaternion algebra, i.e.,* $\mathbf{H}(K)$ is the K-algebra

$$\mathbf{H}(K) = K\big[i, j : i^2 = a, j^2 = b, -ji = ij =: k\big].$$

The set $\{1, i, j, k\}$ is a K-*basis* of $\mathbf{H}(K)$. If $a, b \in R$, then

$$\mathbf{H}(R) = R\big[i, j : i^2 = a, j^2 = b, -ji = ij =: k\big].$$

The *norm* of $x = x_1 + x_i i + x_j j + x_k k \in \mathbf{H}(K)$ is $\eta(x) = x_1^2 - a x_i^2 - b x_j^2 + ab x_k^2$. In what follows, we consider $\mathbf{H}(K) = K[i, j : i^2 = -1, j^2 = -1, -ji = ij =: k]$.

Definition 2.2. ([11]) The least natural number s for which the equation

$$-1 = a_1^2 + a_2^2 + \cdots + a_s^2, a_j \in K, \quad 1 \le j \le s$$

is soluble is called the *stufe* of K, denoted $s(K)$. When this equation admits no solution, we set $s := \infty$ and K is called *formally real*.

Rajwade [11] proved that if the quadratic extension $\mathbb{Q}(\sqrt{-d}), d \in \mathcal{D} \cap \mathbb{Z}^+$ has $s(K) = 4$, then $d \equiv 7 \pmod 8$. Using this we prove in [13] that the quaternion algebra $\mathbf{H}(K)$ over K is a division ring if and only if $d \equiv 7 \pmod 8$, and as a corollary we obtain that if $d \not\equiv 7 \pmod 8$, then $\mathcal{U}(RK_8)$ is not hyperbolic. Defining a proper action of the group $SL_1(\mathbf{H}(R)) := \{x \in \mathbf{H} : \eta(x) = 1\}$ over the three-dimensional hyperbolic space $\mathbb{H}$, and using a result of Gromov about the fundamental group of a closed n-dimension riemannian manifold of constant negative sectional curvature, we prove that if $d \equiv 7 \pmod 8$, then the group $\mathcal{U}(RK_8)$ is hyperbolic.

Theorem 2.3. ([13, Theorem 1.7.5]) *Let R be the ring of algebraic integers of a rational quadratic extension $K = \mathbb{Q}(\sqrt{-d}), d \in \mathcal{D}$. The unit group $\mathcal{U}_1(RG)$ is hyperbolic if and only if G is one of the groups listed below and R (or K) is determined by the respective value of d.*

(i) *$G \in \{C_2, C_3\}$ and any d.*
(ii) *G is an abelian group of exponent dividing n, for $n = 2$ and $d > 0$, or $n = 6$ and $d = 3$, or $n = 4$ and $d = 1$.*
(iii) *$G = C_4$ and $d > 0$.*
(iv) *$G = C_8$ and $d = 1$.*
(v) *$G = K_8$ and $s(K) = 4$, that is, $d > 0$ and $d \equiv 7 \pmod 8$.*

For a metric space X, let the maps $r_1, r_2 \colon [0, \infty[\longrightarrow X$ be proper, that is, $r_i^{-1}(C)$ is compact for each compact $C \subseteq X$. Two rays are *equivalent* if for each compact set $C \subset X$ there exists $N \in \mathbb{N}$, such that $r_i([N, \infty[), i = 1, 2$, are in the same path connected component of $X \setminus C$. The equivalence class of r is denoted by $\mathrm{end}(r)$; $\mathrm{End}(X)$ denotes the set of equivalence classes and $|\mathrm{End}(X)|$ is the number of ends of X. For a finitely generated group Γ and $\mathcal{G}$ its Cayley graph, we define $\mathrm{Ends}(\Gamma) := \mathrm{Ends}(\mathcal{G})$ [1, 5].

Corollary 2.4. *The group $\mathcal{U}(RK_8)$ is hyperbolic if and only if $d > 0$ and $d \equiv 7 \pmod 8$. Furthermore, the hyperbolic boundary $\partial(\mathcal{U}(RK_8)) \cong S^2$, the two dimensional euclidean sphere, and $\mathcal{U}(RK_8)$ has one end.*

Corollary 2.5. *Let $d \equiv 7 \pmod 8$; if $u_1 \cdots u_n \in \mathcal{U}(RK_8)$, then there exists $m \in \mathbb{N}$ such that $\langle u_1^m, \ldots, u_n^m \rangle$ is a free group of rank less than or equal to n.*

3. The Pell and Gauss units. Corrales *et al.* [2] determined generators of a subgroup of finite index of $\mathcal{U}(\mathbf{H}(\mathbb{Z}(\frac{1+\sqrt{-7}}{2})))$, whose units all have norm 1. In our classification we obtain the groups $\mathcal{U}(RK_8)$ for which R is the ring of algebraic integers of $\mathbb{Q}(\sqrt{-d})$ such that $d > 0$, $d \equiv 7 \pmod 8$, and these groups are commensurable to the groups $\mathcal{U}(\mathbf{H}(R))$, that is, they have isomorphic subgroups of finite index. Thus, it is natural to consider the unit groups $\mathcal{U}(\mathbf{H}(\mathbb{Z}(\frac{1+\sqrt{-d}}{2})))$, $d \in \mathcal{D} \cap \mathbb{Z}^+$ when $d \equiv 7 \pmod 8$.

Let R be a ring and G be a group. For a unit $u \in \mathcal{U}(RG)$, writing $u = \sum_{g\in G} u_g g$, the set $\text{supp}(u) := \{g \in G : u_g \neq 0\}$ is called the *support* of the unit u. For $u = u_1 + u_i i + u_j j + u_k k \in \mathcal{U}(\mathbf{H}(R))$, the set $\text{supp}(u)$ is defined similarly according to the K-basis $\{1, i, j, k\}$.

Proposition 3.1. *Let* $u = u_1 + u_i i + u_j j + u_k k \in \mathcal{U}(\mathbf{H}(R))$ *with norm* $\eta(u)$. *The following conditions hold:*

(i) $u^2 = 2u_1 u - \eta(u)$.
(ii) *If* $d \equiv 7 \pmod 8$ *and* $\eta(u) = 1$, *then* u *is torsion if and only if* $u_1 \in \{-1, 0, 1\}$. *Thus, the order* $o(u)$ *is either* $o(u) = 4, 2$ *or* 1.
(iii) *If* $d \equiv 7 \pmod 8$ *and* $\eta(u) = -1$, *then* $o(u) = \infty$.

Let $\mathbb{L} := \mathbb{Q}(\sqrt{d})$ and $\xi \neq \psi \in \{1, i, j, k\}$. For $\epsilon = x + y\sqrt{d} \in \mathcal{U}(I_{\mathbb{L}})$, we denote $u_{(\epsilon)} := y\sqrt{-d}\xi + x\psi \in \mathbf{H}(K)$.

Proposition 3.2. *Let* $d \equiv i \pmod 4), i \in \{2, 3\}$ *and* $\xi \neq \psi \in \{1, i, j, k\}$. *The following conditions hold:*

(i) $u_{(\epsilon)} \in \mathcal{U}(\mathbf{H}(R))$ *if and only if* $\epsilon = p + m\sqrt{d} \in \mathcal{U}(I_{\mathbb{L}})$.
(ii) *If* $1 \notin \text{supp}(u)$, *then* $u_{(\epsilon)}$ *is torsion.*
(iii) *If* $\mu, \nu \in \mathcal{U}(I_{\mathbb{L}})$ *and* $1 \in \text{supp}(u_{(\mu)}) = \text{supp}(u_{(\nu)})$, *then* $u_{(\mu)}u_{(\nu)} = u_{(\mu\nu)}$.
(iv) *If* $1 \in \text{supp}(u_{(\epsilon)})$, *then* $\langle u_{(\epsilon)} \rangle = \{u_{(\epsilon^n)}, n \in \mathbb{Z}\}$.
(v) *For* $d \equiv 3 \pmod 4$ *and* $\mathbb{F} := \mathbb{Q}(\sqrt{2d})$,

$$u := m\sqrt{-d}\xi + p\psi + (1-p)\phi \in \mathcal{U}(\mathbf{H}(R)) \Leftrightarrow \epsilon = (2p-1) + m\sqrt{2d} \in \mathcal{U}(I_{\mathbb{F}}).$$

Theorem 3.3. *Let* $\mathbf{H}(K)$ *be a division ring. If* $x + y\sqrt{d} \in \mathcal{U}(I_{\mathbb{L}})$, *then*

$$u = \begin{cases} \frac{y}{2}\sqrt{-d} + (\frac{y}{2}\sqrt{-d})i + (\frac{1\pm x}{2})j + (\frac{1\mp x}{2})k & \text{if } y \equiv 0 \pmod 2, \\ xy\sqrt{-d} + (xy\sqrt{-d})i + (\frac{1\pm(x^2+y^2d)}{2})j + (\frac{1\mp(x^2+y^2d)}{2}k) & \text{if } y \equiv 1 \pmod 2. \end{cases}$$

is a unit in $\mathbf{H}(R)$.

We observe that $\epsilon := x + y\sqrt{d} \in \mathcal{U}(I_{\mathbb{L}})$ if and only if $\epsilon \cdot \bar{\epsilon} = 1 = x^2 - y^2 d$, $x, y \in \mathbb{Z}$, since $d \equiv 3 \pmod 4$ and d is a positive integer. This equation is called the Pell equation, and it is well known that it has a solution, (see [3, VI.§19]). According to the definition of the unit u, as in the last theorem, the solution of the Pell equation determines the coefficients of u. Also, as in Proposition

3.2(v), due to the unit u definition, the associated *invertible* ϵ has $\{1, \sqrt{2d}\}$ as its integral basis. The units constructed by the proper Pell equation are defined as follows.

Definition 3.4. The units given above are called *Pell units.* For $l \in \{2,3\}$, a *Pell l-unit* is a unit whose support has cardinality l, and the unique non-integer coefficient is of the form $m\sqrt{-d}$.

We note that the set $\{u \in \mathcal{U}(\mathbf{H}(R)) : u \text{ is a Pell unit}\}$ is not a group. For instance, when $d \equiv 2 \pmod 4$ or $d \equiv 3 \pmod 4$ and $\mathbf{H}(K)$ is the quaternion algebra over K, if u is a Pell unit with $|\operatorname{supp}(u)| = 4$, then u^2 is not a Pell unit. We remark, however, that item (iv) of Proposition 3.2 shows that if u is a Pell 2-unit, then all powers of u are Pell 2-units.

Next we construct units $u := m\sqrt{-d} + pi + qj + rk \in \mathbf{H}(R) \subset \mathbf{H}(K)$ of norm $\eta(u) = -1$. Thus $p^2 + q^2 + r^2 = m^2 d - 1 = n$ is a sum of three integer squares. Writing $n = 4^a n'$ so that $4 \nmid n'$ and $a \geq 0$, by [12, Theorem 1] (quoted as Gauss' Theorem), n is a sum of three integer squares if and only if $n' \not\equiv 7 \pmod 8$. This leads to the following theorem.

Theorem 3.5. *Let $\mathbf{H}(K)$ be a division ring. If $m \equiv 2 \pmod 4$, then there exist integers p, q, r, such that, $u = m\sqrt{-d} + pi + qj + rk \in \mathcal{U}(\mathbf{H}(R))$.*

Definition 3.6. Let u be unit of $\mathbf{H}(R)$ whose support has cardinality $l > 1$. If the unique non-integer coefficient of u is of the form $m\sqrt{-d}$ where $m^2 \pm 1$ is a sum of three square integers, then we call u a *Gauss unit*, or a *Gauss l-unit.*

Proposition 3.7. *Let u be a unit of norm $\eta(u) = 1$, $l \in \{2,3\}$, and $\mathbf{H}(K)$ a division ring. Then u is a Pell l-unit if and only if u is a Gauss l-unit.*

Theorem 3.8. *Let $d \equiv 7 \pmod 8$. If $u, v \in \mathcal{U}(\mathbf{H}(R))$ are Gauss 2-units, and $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \{1\}$, then there exists $m \in \mathbb{N}$ such that $\langle u^m, v^m \rangle$ is a free group of rank 2.*

4. **Semigroup algebras.** We will consider $\mathcal{A}$ a unitary finitely generated $\mathbb{Q}$-algebra and denote by $\mathcal{S}(\mathcal{A})$, respectively $J(\mathcal{A})$, the semisimple subalgebra, respectively the Jacobson radical, of $\mathcal{A}$ and by $E(\mathcal{A}) := \{ E_1, \ldots, E_N\}, N \in \mathbb{Z}^+$, the set of the central primitive idempotents of the semisimple algebra $\mathcal{S}(\mathcal{A})$. A classical result of Wedderburn–Malĉev states that $\mathcal{A} \cong \mathcal{S}(\mathcal{A}) \oplus J(\mathcal{A})$, as a vector space. As a result, we have that $\mathcal{A}$ is an artinian algebra and thus its radical is a nilpotent ideal. We denote $T_2(\mathbb{Q}) := \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right)$ the algebra of 2×2 upper triangular matrices over $\mathbb{Q}$, with the usual matrix multiplication.

Definition 4.1. Let $\mathcal{A}$ be a finite dimensional algebra over $\mathbb{Q}$, and let Γ be a $\mathbb{Z}$-order of $\mathcal{A}$. If $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(\Gamma)$, we say $\mathcal{A}$ has the *hyperbolic property.*

If the algebra $\mathcal{A}$ has the hyperbolic property and it is non-semisimple with radical J, then we prove that $J^2 = 0$. In this condition we can consider J as vector subspace of $\mathcal{A}$ such that there exists $j_0 \in \mathcal{A}$ and $J = \langle j_0, j_0^2 = 0\rangle_{\mathbb{Q}}$, the space generated by j_0 over $\mathbb{Q}$. Furthermore, $1+J \cong \mathbb{Q}$ as a multiplicative group. For an idempotent $E \in E(\mathcal{A})$, since $J = \langle j_0\rangle$ is an ideal of $\mathcal{A}$, we have $j_0 \cdot E \in J$. Thus there exists $\lambda \in \mathbb{Q}$ such that $j_0 \cdot E = \lambda j_0$. Likewise $E \cdot j_0 = \mu j_0, \mu \in \mathbb{Q}$. The following proposition shows that the left or right action of $\{j_0\}$ over $E(\mathcal{A})$ is non-trivial for a unique idempotent $E_i \in E(\mathcal{A})$.

Proposition 4.2. *Let $\mathcal{A}$ be a finite dimensional non-semisimple $\mathbb{Q}$-algebra with $J(\mathcal{A}) = \langle j_0\rangle_{\mathbb{Q}}$ and $N = |E(\mathcal{A})|$. The following conditions hold.*

(i) *For all $x \in \mathcal{A}$, there exist $\lambda_x, \mu_x \in \mathbb{Q}$ such that $xj_0 = \lambda_x j_0$ and $j_0 x = \mu_x j_0$.*
(ii) *If x is an idempotent, then $\lambda_x, \mu_x \in \{0, 1\}$.*
(iii) *There exist unique $E, F \in E(\mathcal{A})$ such that $Ej_0 \neq 0$ and $j_0 F \neq 0$.*
(iv) *If $E = F$, then J is central.*
(v) *If J is non-central, then, up to an index reordering, we can suppose that $E = E_1$, $F = E_N$, and $E_1 j_0 = j_0 E_N = j_0$.*

This enables us to give a structure for the algebras $\mathcal{A}$ with the hyperbolic property.

Theorem 4.3. *Let $\mathcal{A}$ be a finite dimensional $\mathbb{Q}$-algebra. If $\mathcal{A}_i$ is a simple epimorphic image of $\mathcal{A}$, denote by F_i a maximal subfield of $\mathcal{A}_i$ and $\Gamma_i \subset \mathcal{A}_i$ a $\mathbb{Z}$-order. The following conditions hold.*

(i) *The algebra $\mathcal{A}$ has the hyperbolic property and is semisimple with no nonzero nilpotent elements if and only if $\mathcal{A} = \bigoplus \mathcal{A}_i$, where $\mathcal{A}_i$ is a division ring and there exists at most one index i_0 such that $\mathcal{U}(\Gamma_{i_0})$ is hyperbolic and infinite.*
(ii) *The algebra $\mathcal{A}$ has the hyperbolic property and is semisimple with nonzero nilpotent elements if and only if $\mathcal{A} = (\bigoplus \mathcal{A}_i) \oplus M_2(\mathbb{Q})$.*
(iii) *The algebra $\mathcal{A}$ has the hyperbolic property and is non-semisimple with central radical J if and only if $\mathcal{A} = (\bigoplus \mathcal{A}_i) \oplus J$.*
(iv) *The algebra $\mathcal{A}$ has the hyperbolic property and is non-semisimple with non-central radical if and only if $\mathcal{A} = (\bigoplus \mathcal{A}_i) \oplus T_2(\mathbb{Q})$.*

For each item above, F_i is an imaginary quadratic field and $\mathcal{A}_i$ is either an imaginary quadratic field or a totally definite quaternion algebra. Furthermore, every simple epimorphic image of $\mathcal{A}$ in the direct sum is an ideal of $\mathcal{A}$.

In what follows, S denotes a finite semigroup, $\mathbb{Q}S$ denotes a unitary semigroup algebra over $\mathbb{Q}$, $\mathcal{M}^0(G; n, n; P)$ denotes the Rees semigroup with structural group G, and P denotes an $n \times n$ sandwich matrix. The groups S_3 and D_4 are the dihedral groups of order 6 and 8, respectively, and $\mathbb{Q}_{12} \cong C_3 \rtimes C_4$.

Theorem 4.4. *The algebra $\mathbb{Q}S$ has no nonzero nilpotent element and has the hyperbolic property if and only if S is an inverse semigroup admitting a* *principal series whose principal factors are isomorphic to groups G and K, listed below, with at most one occurrence of K:*

(i) *G is an abelian group of exponent dividing 4 or 6;*
(ii) *G is a Hamiltonian 2-group;*
(iii) $K \in \{C_5, C_8, C_{12}\}$.

Theorem 4.5. *Let $\mathbb{Q}S$ be an algebra with nonzero nilpotent elements. The algebra $\mathbb{Q}S$ is semisimple with the hyperbolic property if and only if S admits a principal series whose principal factors are isomorphic to groups G and a semigroup K as listed below, with exactly one occurrence of K.*

(i) *G is an abelian group of exponent dividing 4 or 6;*
(ii) *G is a Hamiltonian 2-group;*
(iii) *K is a group of the set $\{S_3, D_4, Q_{12}, C_4 \rtimes C_4\}$;*
(iv) *K is one of the Rees semigroups:*

$$\mathcal{M}^0(\{1\}; 2, 2; I_d) = M \quad or \quad \mathcal{M}^0(\{1\}; 2, 2; \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = M_{12},$$

which is an ideal of S.

In particular, S is the disjoint union of the groups G and the semigroup K.

We proved that a finite dimensional $\mathbb{Q}$-algebra with the hyperbolic property has a nice Wedderburn–Malĉev decomposition. We recall the idempotent decomposition of Proposition 4.2, for a non-semisimple algebra $\mathbb{Q}S$ with the hyperbolic condition: $1 = \sum_{1<i<N} E_i + E$, since $E = E_1 + E_N$. Now let $e \in \mathbb{Q}S$ be any idempotent; then $e = \sum_{1<i<N} eE_i + eE$, where $(eE_i)^2 = eE_i \in \mathcal{A}_i$, a division ring, for all $1 \le i \le N-1$. Therefore, $eE_i \in \{E_i, 0\}$, and hence $e = \sum E_{i_l} + eE$. Giving a more explicit description of certain idempotents, we shall describe some subsemigroups appearing naturally in a finite semigroup whose rational semigroup algebra has the hyperbolic property.

Proposition 4.6. *Let $S = \bigcup G_i \cup \{\theta, j_0\}$ be a semigroup. If $e_i \in G_i$ is the group identity element of the group G_i, then e_i has one of the following expressions:*

$$\sum E_{i_l} + E_1 + \lambda j_0, \quad \sum E_{i_l} + E_N + \mu j_0, \quad \sum E_{i_l} + E_1 + E_N, \quad \sum E_{i_l}$$

with $1 < i_l < N$. Moreover, the last two expressions are central idempotents.

Theorem 4.7. *The algebra $\mathbb{Q}S$ is non-semisimple with the hyperbolic property if and only if there exists a unique nonzero nilpotent element $j_0 \in S$ such that the subsemigroup $\mathfrak{I} = \{\theta, j_0\}$ is an ideal of S and $S \setminus \mathfrak{I}$ admits a principal series whose principal factors are isomorphic to abelian groups of exponent dividing 4 or 6, or a Hamiltonian 2-group. In particular, $S/\mathfrak{I}$ is the disjoint union* *of its maximal subgroups such that if $e_1 \in G_1$ and $e_N \in G_N$ are the respective group identity elements, then $e_1 j_0 = j_0 e_N = j_0$. Writing*

$$e_1 = \sum_{1<1_l<N} E_{1_l} + E_1 + \lambda j_0, \lambda \in \mathbb{Q},$$

$$e_N = \sum_{1<N_l<N} E_{N_l} + E_N + \mu j_0, \mu \in \mathbb{Q},$$

then exactly one of the following holds:

(i) $e_1 e_N = 0 \Leftrightarrow e_N e_1 = 0$ *and* $\lambda + \mu = 0$*;* $T_2 \cong \{e_1, e_N, j_0, \theta\}$ *is such that* $\mathbb{Q}T_2 \cong T_2(\mathbb{Q})$.

(ii) *If* $e_N e_1 \neq 0$*, then* $e_1 e_N = e_N e_1 =: e_3$ *and* $\lambda + \mu = 0$*;* $T_2' = \{e_1, e_N, e_3, j_0, \theta\}$ *is a subsemigroup of* S *and* $\mathbb{Q}T_2' \cong \mathbb{Q} \oplus \mathbb{Q} \oplus T_2(\mathbb{Q})$.

(iii) $e_N e_1 = 0 \Leftrightarrow e_1 e_N = j_0 \Leftrightarrow \lambda + \mu = 1$*;* $\hat{T}_2 = \{e_1, e_N, j_0, \theta\}$*, and* $\mathbb{Q}\hat{T}_2 \cong T_2(\mathbb{Q})$.

The semigroups T_2, T_2' *and* $\hat{T}_2$ *are non isomorphic.*

5. **The quasi-hyperbolic unit loop $\mathcal{U}(\mathbb{Z}L)$ of an RA-loop L.** A loop L is a nonempty set, with a closed binary operation $\cdot$, such that the equation $a \cdot b = c$ has a unique solution $b \in L$ when $a, c \in L$ are known, and a unique solution $a \in L$ when $b, c \in L$ are known, and with a two-sided identity element 1. We say that a loop L is quasi-hyperbolic if $\mathbb{Z}^2 \not\hookrightarrow L$. Defining $[x, y, z] := (xy)z - x(yz)$, recall that a ring A is *alternative* if $[x, x, y] = [y, x, x] = 0$ for every $x, y \in A$. An RA-loop is a loop whose loop ring RL over some commutative, associative and unitary ring R of characteristic not equal to 2 is alternative, but not associative. The basic reference is [4].

In this section we classify the RA-loops L such that the loop of units of $\mathbb{Z}L$, say $\mathcal{U}(\mathbb{Z}L)$, is quasi-hyperbolic.

LEMMA 5.1. *Let L be a finite RA-loop. The loop $\mathcal{U}(\mathbb{Z}L)$ is quasi-hyperbolic if and only if $\mathcal{U}(\mathbb{Z}L)$ is trivial.*

For a theoretical group property $\mathcal{P}$, a group G is virtually $\mathcal{P}$ if it has a subgroup of finite index, say H, with property $\mathcal{P}$.

THEOREM 5.2. *Let L be an RA-loop. Then $\mathcal{U}(\mathbb{Z}L)$ is quasi-hyperbolic if and only if*

(i) *L is a finite loop or a loop whose center is virtually cyclic,*

(ii) *the torsion subloop $T(L)$ of L is such that if $T(L)$ is a group, then it is an abelian group of exponent dividing 4 or 6 or a Hamiltonian 2-group whose subgroups are all normal in L, and*

(iii) *if $T(L)$ is a loop, then it is a Hamiltonian Moufang 2-loop whose subloops are all normal in L.*

Under these conditions we also have that $\mathcal{U}_1(\mathbb{Z}L) = L$.

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