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CHARACTERIZATION OF COMPLEX SPACE FORMS IN TERMS OF CHARACTERISTIC VECTOR FIELDS ON GEODESIC SPHERES

Dedicated to Professor Tetsuo Furumochi on the occasion of his 60th birthday

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Presented by Edward Bierstone, FRSC

Abstract. Investigating geometric properties of characteristic vector fields on geodesic spheres in a complex space form, we characterize complex space forms in the class of Kähler manifolds.

Résumé. En étudiant des propriétés géométriques de champs de vecteurs caractéristiques sur des sphères géodésiques dans un espace complexe à courbure constante, on caractérise ces espaces dans la classe des variétés kähleriennes.

1. **Introduction.** Let (M, g, J) be a Kähler manifold of complex dimension n ($\geqq 2$) and $G_x(r)$ a geodesic sphere of radius r centered at $x \in M$. It is well known that every geodesic sphere of sufficiently small radius r has an almost contact metric structure (ϕ, ξ, η, g). Here ξ is the so-called characteristic vector field on $G_x(r)$ in M, which is defined by $\xi = -J\mathcal{N}$ with the outward unit normal vector field $\mathcal{N}$ on $G_x(r)$.

A complex n-dimensional complex space form $M_n(c)$ is a complete simply connected Kähler manifold of constant holomorphic sectional curvature c, which is congruent to either a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$, according as c is positive, zero or negative.

Geodesic spheres in $M_n(c)$ are nice objects in differential geometry. For example, it is well known that a geodesic sphere $G_x(r)$ $(0 < r < \pi/2)$ with $\tan^2 r > 2$ in $\mathbb{C}P^n(4)$ is a *Berger sphere*. We here explain this fact in detail. Sectional curvatures of $G_x(r)$ with $\tan^2 r > 2$ lie in the interval $[\delta K, K]$ with some $\delta \in (0, 1/9)$, but it has closed geodesics of length (say ℓ) shorter than $2\pi/\sqrt{K}$, where $K = 4 + \cot^2 r$ and $\ell = \pi \sin 2r$. These closed geodesics are integral curves of the characteristic vector field ξ of this manifold [12]. Moreover, every geodesic which is not congruent to such an integral curve has length which is longer than $2\pi/\sqrt{K}$ (see [6, Corollary 2.8]).

So in this paper, we pay attention to the characteristic vector field ξ of a geodesic sphere $G_x(r)$ in a complex space form $M_n(c)$. This vector field has

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other nice properties as follows.

(1) For each point $x \in M_n(c)$, any integral curve of ξ of every geodesic sphere $G_x(r)$ with sufficiently small radius r in $M_n(c)$ is a geodesic on $G_x(r)$. Moreover, every such integral curve is a circle of positive curvature on some totally geodesic complex curve $M_1(c)$ in $M_n(c)$.
(2) For each point $x \in M_n(c)$, the vector ξ of every geodesic sphere $G_x(r)$ with sufficiently small radius r in $M_n(c)$ is a Killing vector field, that is the Lie derivative $L_\xi g$ on $G_x(r)$ of the metric g with the direction ξ vanishes (*i.e.*, $L_\xi g = 0$).

Motivated by these facts, we shall provide a characterization of complex space forms in the class of Kähler manifolds. The purpose of this paper is to prove the following, which is an improvement of [4].

THEOREM. *For a Kähler manifold M of complex dimension n ($\geqq 2$), the following conditions are equivalent to each other.*

(i) *M has constant holomorphic sectional curvature.*
(ii) *For each point $x \in M$, any integral curve of ξ of every geodesic sphere $G_x(r)$ (with sufficiently small radius r in M) lies locally on some totally geodesic complex curve L in M.*
(iii) *For each point $x \in M$ and any point $p \in G_x(r)$, the geodesic $\gamma = \gamma(s)$ on every geodesic sphere $G_x(r)$ (with sufficiently small radius r in M) with initial condition $\gamma(0) = p$, $\dot\gamma(0) = \xi$ is a Frenet curve of proper order 2 in M.*
(iv) *For each point $x \in M$ the vector ξ of every geodesic sphere $G_x(r)$ (with sufficiently small radius r in M) is a Killing vector field.*

2. Frenet curves of proper order 2.

A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M (with Riemannian connection ∇) parametrized by its arclength s is called a *Frenet curve of proper order* 2 if there exist a field of orthonormal frames $\{\dot\gamma(s), Y_s\}$ along γ and a positive smooth function $\kappa(s)$ satisfying the following system of ordinary differential equations:

$$\nabla_{\dot\gamma}\dot\gamma = \kappa(s)Y_s \quad \text{and} \quad \nabla_{\dot\gamma}Y_s = -\kappa(s)\dot\gamma. \tag{2.1}$$

The function κ is called the *curvature* of the Frenet curve γ of proper order 2. Here note that we do *not* allow the curvature $\kappa(s)$ to vanish at some point. Therefore curves with inflection points, such as $y = x^3$ on a Euclidean xy-plane, are not Frenet curves of proper order 2.

When the curvature κ is a constant function along γ, say k, the curve satisfying (2.1) is called a circle of curvature k on M. Needless to say a geodesic is regarded as a circle of null curvature.

3. Proof of the theorem. First of all we recall an expansion for the shape operator of a geodesic sphere due to Chen and Vanhecke (see [7, Theorem 3.1]). For a Riemannian manifold M of dimension greater than 2, we denote by $A_{x,r}$ the shape operator of $G_x(r)$ in M of sufficiently small radius r centered at $x \in M$ with respect to the outward unit normal vector field $\mathcal{N}$. We adopt the following signature of the Riemannian curvature tensor $\widetilde{R}$ of M: $\widetilde{R}(X,Y)Z = \widetilde{\nabla}_{[X,Y]}Z - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]Z$. The following is a key in our discussion.

Lemma. *For nonzero tangent vectors $v, w \in T_xM$ at a point $x \in M$, we choose a unit tangent vector $u \in T_xM$ orthogonal to both v and w. We denote by $v_r, w_r \in T_{\exp_x(ru)}M$ the parallel displacements of v, w along the geodesic segment $\exp_x(su)$, $0 \leqq s \leqq r$. Then for sufficiently small r we have*

$$g(A_{x,r}v_r, w_r) = \frac{1}{r}g(v,w) - \frac{r}{3}g\big(\widetilde{R}(u,v)w,u\big) + O(r^2). \tag{3.1}$$

In the following, we consider a geodesic sphere $G_x(r)$ in a Kähler manifold (M,g,J). The Riemannian connections $\widetilde{\nabla}$ of M and ∇ of $G_x(r)$ are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(A_{x,r}X, Y)\mathcal{N}, \tag{3.2}$$

$$\widetilde{\nabla}_X \mathcal{N} = -A_{x,r}X, \tag{3.3}$$

for vector fields X and Y on $G_x(r)$. The almost contact metric structure (ϕ, ξ, η, g) is given by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N}, \quad \eta(\xi) = 1, \tag{3.4}$$

for vector fields X and Y on $G_x(r)$, so that

$$\phi\xi = 0 \quad \text{and} \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the identity mapping of the tangent bundle $TG_x(r)$ of $G_x(r)$. Moreover, the differential of ξ is given as

$$\nabla_X \xi = \phi A_{x,r}X. \tag{3.5}$$

Indeed, it follows from (3.2), (3.3), (3.4) and the fact $\widetilde{\nabla}J = 0$ that

$$\begin{aligned}\nabla_X \xi &= \widetilde{\nabla}_X \xi - g(A_{x,r}X, \xi)\mathcal{N} = J\widetilde{\nabla}_X(-\mathcal{N}) + g(A_{x,r}X, J\mathcal{N})\mathcal{N} \\ &= JA_{x,r}X - g(JA_{x,r}X, \mathcal{N})\mathcal{N} = \phi A_{x,r}X.\end{aligned}$$

We say that a real hypersurface of a Kähler manifold is a *Hopf hypersurface* if its characteristic vector ξ is a principal curvature vector at each point of this real hypersurface.

(ii) $\Rightarrow$ (i) Let $\gamma = \gamma(s)$ be an integral curve (parametrized by its arclength s) of the vector ξ satisfying the condition (ii). We note that the complex curve L has the Riemannian connection $\widetilde{\nabla}$, since L is totally geodesic in the ambient Kähler manifold M (with Riemannian connection $\widetilde{\nabla}$). Then the curve γ satisfies the following equation:

$$\widetilde{\nabla}_{\xi_\gamma}\xi_\gamma = \kappa_\gamma(s)\mathcal{N}_\gamma \quad (= \kappa_\gamma(s)J\xi_\gamma), \tag{3.6}$$

where $\xi_\gamma = \dot{\gamma}(s)$ and $\kappa_\gamma(s)$ is a smooth function on the curve γ. On the other hand, from (3.2) and (3.5) we see that

$$\widetilde{\nabla}_{\xi_\gamma}\xi_\gamma = \phi A_{x,r}\xi_\gamma + g(A_{x,r}\xi_\gamma, \xi_\gamma)\mathcal{N}_\gamma. \tag{3.7}$$

Comparing the tangential components of (3.6) and (3.7), we obtain $\phi A_{x,r}\xi_\gamma = 0$, so that ξ_γ is principal. Since γ is an arbitrary integral curve of ξ, we know that our geodesic sphere $G_x(r)$ is a Hopf hypersurface of the Kähler manifold M. Hence due to the results of [3], [8] we can see that M has constant holomorphic sectional curvature. However, in the following we prove this fact in detail for readers.

Given a unit tangent $v \in T_xM$ we take a unit tangent vector $w \in T_xM$ which is orthogonal to v and Jv, and use the Lemma by putting $u = Jv$. Since u_r is a normal vector of $G_x(r)$ in M at $y = \exp_x(ru)$, the vector $v_r = -Ju_r$ is the characteristic vector of $G_x(r)$ at y, so that v_r is a principal curvature vector of $G_x(r)$. This, together with equation (3.1), shows that the curvature tensor $\widetilde{R}$ of M satisfies $g\big(\widetilde{R}(u, Ju)w, u\big) = 0$. This means that $\widetilde{R}(u, Ju)u$ is proportional to Ju for every unit vector u at each point x of M, so that our Kähler manifold M has constant holomorphic sectional curvature (see [11]).

(iii) $\Rightarrow$ (i) It follows from (3.2) and (3.3) that

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = g(A_{x,r}\dot{\gamma}, \dot{\gamma})\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_{\dot{\gamma}}\mathcal{N} = -A_{x,r}\dot{\gamma}. \tag{3.8}$$

On the other hand, by hypothesis the curve γ, considered as a curve in the ambient manifold M, satisfies

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \kappa(s)Y_s \quad \text{and} \quad \widetilde{\nabla}_{\dot{\gamma}}Y_s = -\kappa(s)\dot{\gamma}, \tag{3.9}$$

where $\kappa(s)$ is a positive smooth function on γ and $\{\dot{\gamma}(s), Y_s\}$ is the Frenet frame along γ. Then, from (3.8) and (3.9) we find that $A_{x,r}\dot{\gamma}(s)$ is proportional to $\dot{\gamma}(s)$ for each s, that is the vector $\dot{\gamma}(s)$ is a principal curvature vector for any s, so that in particular at the point $p = \gamma(0)$ the vector $\dot{\gamma}(0) = \xi$ is principal. Thus we can see that our geodesic sphere $G_x(r)$ is a Hopf hypersurface in the Kähler manifold M. Therefore by virtue of the above discussion we obtain the desirable conclusion (i).

(iv) $\Rightarrow$ (i) For all vectors X, Y on $G_x(r)$, from (3.5) we have

$$\begin{aligned} 0 = (L_\xi g)(X,Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= g(\phi A_{x,r} X, Y) + g(X, \phi A_{x,r} Y) \\ &= g\big((\phi A_{x,r} - A_{x,r}\phi)X, Y\big). \end{aligned}$$

Thus the condition (iv) is equivalent to the equality $\phi A_{x,r} - A_{x,r}\phi = 0$, so that in particular $\phi A_{x,r}\xi = 0$ at every point of $G_x(r)$, which shows that our geodesic sphere $G_x(r)$ is a Hopf hypersurface. Therefore the manifold M has constant holomorphic sectional curvature.

(i) $\Rightarrow$ (ii), (iii), (iv) Since our discussion is local, without loss of generality we may suppose that our manifold M is either $\mathbb{C}^n$, $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$. When $c = 0$, our geodesic sphere $G_x(r)$ is nothing but a standard hypersphere (of radius r) in $\mathbb{C}^n$ embedded as a totally umbilic hypersurface, so that the equality $\phi A_{x,r} = A_{x,r}\phi$ holds on $G_x(r)$. When $c \neq 0$, the geodesic sphere $G_x(r)$ is *not* totally umbilic in $M_n(c)$, but $\phi A_{x,r} = A_{x,r}\phi$ holds (see [10]). Thus we obtain the condition (iv).

Next, we shall check the condition (ii). When $c = 0$, every integral curve ξ_γ on a standard hypersphere $G_x(r)$ is a great circle of $G_x(r)$ in the sense of Euclidean geometry. Hence such a curve is a circle of curvature $1/r$ on some complex line $\mathbb{C}^1$. When $c > 0$ (resp. $c < 0$), our integral curve ξ_γ is a circle of curvature $\sqrt{c}\cot(\sqrt{c}r)$ (resp. $\sqrt{|c|}\coth(\sqrt{|c|}r)$) on some complex line $\mathbb{C}P^1(c)$ (resp. $\mathbb{C}H^1(c)$) of the ambient space $\mathbb{C}P^n(c)$ (resp. $\mathbb{C}H^n(c)$), where $0 < r < \pi/\sqrt{c}$ (resp. $0 < r < \infty$). This fact is due to [2], [5]. Therefore we have verified condition (ii).

Our geodesic sphere $G_x(r)$ is a Hopf hypersurface in $M_n(c)$, since the geodesic sphere $G_x(r)$ satisfies $\phi A_{x,r} = A_{x,r}\phi$. This, together with (3.5), yields that every integral curve of ξ is a geodesic on the geodesic sphere $G_x(r)$. Hence, from the uniqueness theorem for geodesics and our discussion we can verify the condition (iii). ■

4. **Remarks.** Geodesic spheres $G_x(r)$ of a nonflat complex space form $M_n(c)$ ($= \mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$) are typical examples of naturally reductive Riemannian homogeneous manifolds which are not Riemmanian symmetric spaces. Hence every geodesic γ of such a geodesic sphere $G_x(r)$ is a *homogeneous curve* on $G_x(r)$, namely γ is an orbit of some one-parameter subgroup of the isometry group $\mathrm{I}\big(G_x(r)\big)$ of $G_x(r)$. So, in particular every integral curve of the characteristic vector field ξ of $G_x(r)$ is a homogeneous curve.

When $c > 0$, each integral curve of ξ satisfies the following equation in the ambient space $\big(\mathbb{C}P^n(c), J\big)$:

$$\widetilde{\nabla}_\xi \xi = \sqrt{c}\cot(\sqrt{c}r)\cdot J\xi. \tag{4.1}$$

When $c < 0$, each integral curve of ξ satisfies the following equation in the ambient space $(\mathbb{C}H^n(c), J)$:

$$\widetilde{\nabla}_\xi \xi = \sqrt{|c|}\coth(\sqrt{|c|}r) \cdot J\xi. \tag{4.2}$$

On the other hand, due to results of [9] we find that there exist many homogeneous curves γ's on $G_x(r)$ in $M_n(c)$, $c \neq 0$ satisfying the following three conditions.

(i) The curve γ is not a geodesic on $G_x(r)$.
(ii) The curve γ, considered as a curve in $M_n(c)$, $c \neq 0$, satisfies the following ordinary differential equation to that of (4.1) or (4.2). That is, when $c > 0$, the curve γ satisfies

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \sqrt{c}\cot(\sqrt{c}r) \cdot J\dot\gamma \quad \text{or} \quad \widetilde{\nabla}_{\dot\gamma}\dot\gamma = -\sqrt{c}\cot(\sqrt{c}r) \cdot J\dot\gamma,$$

and, when $c < 0$, the curve γ satisfies

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \sqrt{|c|}\coth(\sqrt{|c|}r) \cdot J\dot\gamma \quad \text{or} \quad \widetilde{\nabla}_{\dot\gamma}\dot\gamma = -\sqrt{|c|}\coth(\sqrt{|c|}r) \cdot J\dot\gamma.$$

(iii) The curve γ is not congruent to each integral curve of ξ with respect to $\mathrm{I}(G_x(r))$.

It is interesting to investigate homogeneous curves which are not geodesics on $G_x(r)$ in a nonflat complex space form. Adachi [1] studied curve theory of such geodesic spheres from this point of view.

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