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JET SCHEMES, ARC SPACES AND THE NASH PROBLEM

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Abstract. This paper is an introduction to the jet schemes and the arc space of an algebraic variety. We also introduce the Nash problem on arc families.

Résumé. Ce papier constitue une introduction aux espaces de jets et à l'espace d'arcs d'une variété algébrique. Nous introduisons également le problème de Nash pour les familles d'arcs.

1. **Introduction.** The concepts of jet scheme and arc space over an algebraic variety or an analytic space were introduced by John Nash in a preprint in 1968, but only published in 1995 [36].

The study of these spaces was further developed by Kontsevich and by Denef and Loeser as the theory of motivic integration, see [28, 7, 8, 9, 10, 11]. These spaces are considered as a way to represent the nature of the singularities of the base space. In fact, papers [12, 13, 34, 35] by Mustaţă, Ein, and Yasuda show that geometric properties of the jet schemes determine certain properties of the singularities of the base space.

In this paper, we provide an introduction to the basic knowledge of these spaces and the Nash problem. A powerful tool for working on these spaces is the theory of motivic integration. We will not discuss this theory, as there are already very good introductory papers on motivic integration by A. Craw [5], W. Veys [48] and F. Loeser [32]. We delve into the basic study of the geometric structure of arc spaces and jet schemes. We also give an introduction to the Nash problem, which was posed in [36].

Throughout this paper the base field k is an algebraically closed field of arbitrary characteristic and a variety is an irreducible reduced scheme of finite type over k. A scheme of finite type over k is always separated over k.

We omit the proofs of statements whose references are thought to be easily accessible. We assume the reader has knowledge of Hartshorne's textbook [19].

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2. **Construction of jet schemes and arc spaces.**

DEFINITION 2.1. Let X be a scheme of finite type over k and $K \supset k$ a field extension. For $m \in \mathbb{N}$, a k-morphism $\operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ is called an m-jet of X and a k-morphism $\operatorname{Spec} K[[t]] \longrightarrow X$ is called an *arc* of X. We denote the unique point of $\operatorname{Spec} K[t]/(t^{m+1})$ by 0, the closed point of $\operatorname{Spec} K[[t]]$ by 0, and the generic point by η.

THEOREM 2.2. *Let X be a scheme of finite type over k. Let Sch be the category of k-schemes and Set the category of sets. Define a contravariant functor $F_m^X \colon \mathsf{Sch}/k \longrightarrow \mathsf{Set}$ by*

$$F_m^X(Z) = \operatorname{Hom}_k(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

Then F_m^X is representable by a scheme X_m of finite type over k, that is

$$\operatorname{Hom}_k(Z, X_m) \simeq \operatorname{Hom}_k\big(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X\big).$$

This X_m is called the space of m-jets *of X or the* m-jet scheme *of X.*

This proposition was proved in [4, p. 276]. In this paper, we prove it by a concrete construction for affine schemes X first and then patch those together for a general X. For our proof, we need some preparatory discussions.

Let X be a k-scheme. Assume that F_m^X is representable by X_m for every $m \in \mathbb{N}$. Then, for $m < m'$, the canonical surjection $k[t]/(t^{m'+1}) \longrightarrow k[t]/(t^{m+1})$ induces a morphism $\psi_{m',m} \colon X_{m'} \longrightarrow X_m$. Indeed, the canonical surjection $k[t]/(t^{m'+1}) \longrightarrow k[t]/(t^{m+1})$ induces a morphism

$$Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m'+1}) \leftarrow Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}),$$

for an arbitrary k-scheme Z. Therefore we have a map

$$\operatorname{Hom}_k\big(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m'+1}), X\big) \longrightarrow \operatorname{Hom}_k\big(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X\big)$$

which gives the map $\operatorname{Hom}_k(Z, X_{m'}) \longrightarrow \operatorname{Hom}_k(Z, X_m)$. Take, in particular, $X_{m'}$ as Z, $\operatorname{Hom}_k(X_{m'}, X_{m'}) \longrightarrow \operatorname{Hom}_k(X_{m'}, X_m)$. Then the image of $\mathrm{id}_{X_{m'}} \in \operatorname{Hom}(X_{m'}, X_{m'})$ by this map gives the required morphism.

This morphism $\psi_{m',m}$ is called a *truncation map*. In particular for $m = 0$, $\psi_{m',0} \colon X_{m'} \longrightarrow X$ is denoted by π_m. When we need to specify the scheme X, we denote it by π_{Xm}.

Actually $\psi_{m',m}$ "truncates" a power series in the following sense: A point α of $X_{m'}$ gives an m'-jet $\alpha \colon \operatorname{Spec} K[t]/(t^{m'+1}) \longrightarrow X$, which corresponds to a ring homomorphism $\alpha^* \colon A \longrightarrow K[t]/(t^{m'+1})$, where A is the affine coordinate ring of an affine neighborhood of the image of α. For every $f \in A$, let

$$\alpha^*(f) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m + \cdots + a_{m'} t^{m'}.$$

Then

$$\big(\psi_{m',m}(\alpha)\big)^*(f) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m.$$

This fact can be seen by letting $Z = \{\alpha\}$ in the above discussion.

As we did in the above argument, we denote the point of X_m corresponding to $\alpha\colon \operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ by the same symbol α. Then, we should note that $\pi_m(\alpha) = \alpha(0)$.

Proposition 2.3. *Let $f\colon X \longrightarrow Y$ be a morphism of k-schemes of finite type. Assume that the functors F_m^X and F_m^Y are representable by X_m and Y_m, respectively. Then a canonical morphism $f_m\colon X_m \longrightarrow Y_m$ is induced for every $m \in \mathbb{N}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ {\scriptstyle \pi_{Xm}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{Ym}} \\ X & \xrightarrow{f} & Y. \end{array}$$

Proof. Let $X_m \times \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow X$ be the "universal family" of m-jets of X, *i.e.*, it corresponds to the identity map in $\operatorname{Hom}_k(X_m, X_m)$. By composing this map with $f\colon X \longrightarrow Y$, we obtain a morphism

$$X_m \times \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow Y,$$

which gives a morphism $X_m \longrightarrow Y_m$. Pointwise, this morphism maps an m-jet $\alpha \in X_m$ of X to the composite $f \circ \alpha$ which is an m-jet of Y. To see this, just take a point $\alpha \in X_m$ and examine the image of $\{\alpha\} \times \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow Y$. The commutativity of the diagram follows from this description. ∎

Proposition 2.4. *For k-schemes X and Y, assume that the functors F_m^X and F_m^Y are representable by X_m and Y_m, respectively. If $f\colon X \longrightarrow Y$ is an étale morphism, then $X_m \simeq Y_m \times_Y X$, for every $m \in \mathbb{N}$.*

Proof. By the above proposition we have a commutative diagram:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

It is sufficient to prove that for every commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y_m \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y, \end{array}$$

there is a unique morphism $Z \longrightarrow X_m$ which is compatible with the projections to X and Y_m. Now we are given the following commutative diagram:

$$\begin{array}{ccc} Z & \longrightarrow & Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

As f is étale, there is a unique morphism $Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow X$ which makes the two triangles commutative. This gives the required morphism. ∎

As a corollary of this proposition, we obtain the following lemma.

Lemma 2.5. *Let $U \subset X$ be an open subset of a k-scheme X. Assume the functors F_m^X and F_m^U are representable by X_m and U_m, respectively. Then, $U_m = \pi_{Xm}^{-1}(U)$.*

Proof of Theorem 2.2. Since a k-scheme X is separated, the intersection of two affine open subsets is again affine. Therefore, by Lemma 2.5, it is sufficient to prove the representability of F_m^X for affine X. Let X be $\operatorname{Spec} R$, where $R = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. It is sufficient to prove the representability for an affine variety $Z = \operatorname{Spec} A$. Then we obtain that

$$(*)\ \operatorname{Hom}\bigl(Z \times \operatorname{Spec} k[t]/(t^{m+1}), X\bigr) \simeq \operatorname{Hom}\bigl(R, A[t]/(t^{m+1})\bigr)$$
$$\simeq \bigl\{\varphi \in \operatorname{Hom}\bigl(k[x_1, \dots, x_n], A[t]/(t^{m+1})\bigr) \bigm| \varphi(f_i) = 0 \text{ for } i = 1, \dots, r\bigr\}.$$
$$=: W.$$

If we write $\varphi(x_j) = a_j^{(0)} + a_j^{(1)} t + a_j^{(2)} t^2 + \cdots + a_j^{(m)} t^m$ for $a_j^{(l)} \in A$, it follows that

$$\varphi(f_i) = F_i^{(0)}(a_j^{(l)}) + F_i^{(1)}(a_j^{(l)}) t + \cdots + F_i^{(m)}(a_j^{(l)}) t^m$$

for polynomials $F_i^{(s)}$ in $a_j^{(l)}$'s. Then the above set $(*)$ is represented as follows:

$$W = \{\varphi \in \operatorname{Hom}(k[x_j, x_j^{(1)}, \dots, x_j^{(m)} \mid j = 1, \dots, n], A) \mid$$
$$\varphi(x_j^{(l)}) = a_j^{(l)}, F_i^{(s)}(a_j^{(l)}) = 0\}$$
$$= \operatorname{Hom}\Bigl(k[x_j, x_j^{(1)}, \dots, x_j^{(m)}] \Big/ \bigl(F_i^{(s)}(x_j^{(l)})\bigr), A\Bigr).$$

If we write $X_m = \operatorname{Spec} k[x_j, x_j^{(1)}, \ldots, x_j^{(m)}]/\big(F_i^{(s)}(x_j^{(l)})\big)$, the last set is bijective to $\operatorname{Hom}(Z, X_m)$. ■

Remark 2.6. The functor F_m^X is also representable even for k-schemes of non-finite type over k. The existence of jet schemes for a wider class of schemes is presented in [49].

Example 2.7. For $X = \mathbb{A}_k^n$, it follows that $X_m = \mathbb{A}_k^{n(m+1)}$. Indeed, this is the case that all $f_i = 0$, therefore all $F_i^{(s)} = 0$, in the proof of Theorem 2.2.

Example 2.8. Let X be a hypersurface in $\mathbb{A}_k^3$ defined by $f = xy + z^2 = 0$. Then X_2 is defined in $\mathbb{A}_k^9$ by

$$\begin{aligned} xy + z^2 &= x^{(1)}y + xy^{(1)} + 2zz^{(1)} \\ &= x^{(2)}y + x^{(1)}y^{(1)} + xy^{(2)} + z^{(1)}z^{(1)} + 2zz^{(2)} = 0. \end{aligned}$$

One can see that X_2 is irreducible and not normal. Indeed, as $X \setminus \{0\}$ is non-singular, $\pi_2^{-1}(X \setminus \{0\})$ is a 6-dimensional irreducible variety. On the other hand, $\pi_2^{-1}(0)$ is a hypersurface in $\mathbb{A}^6$, and therefore it is of dimension 5. Since X_2 is defined by 3 equations, every irreducible component of X_2 has dimension $\geq 9 - 3 = 6$. By this, $\pi_2^{-1}(0)$ does not produce an irreducible component of X_2, which yields the irreducibility of X_2. Looking at the Jacobian matrix, one can see that the singular locus of X_2 is $\pi_2^{-1}(0)$ which is of codimension one in X_2. Therefore, X_2 is not normal.

Let X_1 be the 1-jet scheme of X. Then for every closed point $x \in X$, the set of closed points of $\pi_1^{-1}(x)$ is the set of morphisms $\operatorname{Spec} k[t]/(t^2) \longrightarrow X$ with the image x. This set is the Zariski tangent space of X at x. Therefore, we can regard X_1 as the "tangent bundle" of X.

Example 2.9. Let X be a curve defined by $x^2 - y^2 - x^3 = 0$ in $\mathbb{A}_k^2$. Then $\pi_1^{-1}(X \setminus \{0\}) \longrightarrow X \setminus \{0\}$ is an $\mathbb{A}_k^1$-bundle, therefore $\pi_1^{-1}(X_{\mathrm{reg}})$ is 2-dimensional. On the other hand, $\pi_1^{-1}(0) \simeq \mathbb{A}_k^2$. Hence, X_1 has two irreducible components, $\overline{\pi_1^{-1}(X_{\mathrm{reg}})}$ and $\pi_1^{-1}(0)$.

Definition 2.10. The system $\{\psi_{m',m} \colon X_{m'} \longrightarrow X_m\}_{m<m'}$ is a projective system. Let $X_\infty = \varprojlim_m X_m$ and call it the *space of arcs* of X or *arc space* of X. Note that X_∞ is not of finite type over k if $\dim X > 0$.

Remark 2.11. One may be afraid that the projective limit scheme $\varprojlim_m X_m$ may not exist. But in our case we need not to worry, since for an affine scheme $X = \operatorname{Spec} A$, the m-jet scheme $X_m = \operatorname{Spec} A_m$ is affine for every m. Here, the morphisms $\psi_{m',m}^* \colon A_m \longrightarrow A_{m'}$ corresponding to $\psi_{m',m}$ are a direct system. It is well known that there is a direct limit $A_\infty = \varinjlim_m A_m$ in the category of k-algebras. The affine scheme $\operatorname{Spec} A_\infty$ is our projective limit of X_m. For a general k-scheme X, we have only to patch affine pieces $\operatorname{Spec} A_\infty$.

Using the representability of F_m^X, we obtain the following universal property of X_∞.

PROPOSITION 2.12. *Let X be a scheme of finite type over k. Then*

$$\mathrm{Hom}_k(Z, X_\infty) \simeq \mathrm{Hom}_k(Z \mathbin{\widehat{\times}}_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]], X)$$

for an arbitrary k-scheme Z, where $Z \mathbin{\widehat{\times}}_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]]$ means the formal completion of $Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]]$ along the subscheme $Z \times_{\mathrm{Spec}\, k} \{0\}$.

PROOF. By the representability of F_m^X we obtain an isomorphism of projective systems:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\mathrm{Hom}_k(Z, X_{m+1}) & \simeq & \mathrm{Hom}_k\big(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+2}), X\big) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_k(Z, X_m) & \simeq & \mathrm{Hom}_k\big(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X\big).
\end{array}$$

Then, we obtain an isomorphism of the projective limits:

$$\mathrm{Hom}_k(Z, \varprojlim_m X_m) \simeq \mathrm{Hom}_k\big(\varinjlim_m \big(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1})\big), X\big),$$

which gives the required isomorphism. ∎

REMARK 2.13. Consider the isomorphism of Proposition 2.12, in particular the case $Z = \mathrm{Spec}\, A$ for a k-algebra A, we obtain

$$\mathrm{Hom}_k(\mathrm{Spec}\, A, X_\infty) \simeq \mathrm{Hom}_k(\mathrm{Spec}\, A[[t]], X).$$

Here, we note that in general $A \otimes_k k[[t]] \not\simeq A[[t]] \simeq A \mathbin{\widehat{\otimes}}_k k[[t]]$, where $A \mathbin{\widehat{\otimes}}_k k[[t]]$ is the completion of $A \otimes_k k[t]$ by the ideal (t). Indeed, for example, for $A = k[x]$, the ring $A[[t]]$ contains $\sum_{i=0}^{\infty} f_i(x) t^i$ such that $\deg f_i$ are unbounded, while $A \otimes_k k[[t]]$ does not contain such an element.

Now consider the case $A = K$ for an extension field $K \supset k$; the bijection

$$\mathrm{Hom}_k(\mathrm{Spec}\, K, X_\infty) \simeq \mathrm{Hom}_k(\mathrm{Spec}\, K[[t]], X)$$

shows that a K-valued point of X_∞ is an arc $\mathrm{Spec}\, K[[t]] \longrightarrow X$.

DEFINITION 2.14. Denote the canonical projection $X_\infty \longrightarrow X_m$ induced from the surjection $k[[t]] \longrightarrow k[t]/(t^{m+1})$ by ψ_m and the composite $\pi_m \circ \psi_m$ by π. When we need to specify the base space X, we write it by π_X.

A point $x \in X_\infty$ gives an arc $\alpha_x\colon \operatorname{Spec} K[[t]] \longrightarrow X$ and $\pi(x) = \alpha_x(0)$, where K is the residue field at x. As the case of m-jets, we denote both $x \in X_\infty$ and α corresponding to x by the same symbol α.

For every $m \in \mathbb{N}$, $\psi_m(X_\infty)$ is a constructible set, since $\psi_m(X_\infty) = \psi_{m',m}(X_{m'})$ for sufficiently big m' [18].

Definition 2.15. Let σ_m denote the canonical morphism $X \longrightarrow X_m$ induced from the inclusion $k \hookrightarrow k[t]/(t^{m+1})$ ($m \in \mathbb{N} \cup \{\infty\}$). Here, we define $k[t]/(t^{m+1}) = k[[t]]$ for $m = \infty$. As $k \hookrightarrow k[t]/(t^{m+1})$ is a section of the projection $k[t]/(t^{m+1}) \longrightarrow k$, our morphism $\sigma_m\colon X \longrightarrow X_m$ is a section of $\pi_m\colon X_m \longrightarrow X$.

For a point $x \in X$, let K be the residue field at x; then

$$\sigma_m(x)\colon \operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$$

is an m-jet which factors through $\operatorname{Spec} K \longrightarrow X$ whose image is x. Therefore, $\sigma_m(x)$ is the constant m-jet at x, this is denoted by x_m.

Example 2.16. If $X = \mathbb{A}_k^n$, then

$$X_\infty = \operatorname{Spec} k[x_j, x_j^{(1)}, x_j^{(2)}, \dots \mid j = 1, \dots, n],$$

which is isomorphic to $\mathbb{A}_k^\infty = \operatorname{Spec} k[x_1, x_2, \dots, x_i, \dots]$. Here, we note that the set of closed points of $\mathbb{A}_k^\infty$ does not necessarily coincide with the set

$$k^\infty := \{(a_1, a_2, \dots) \mid a_i \in k\}$$

(see the following theorem).

Theorem 2.17. ([23, Propositions 2.10 and 2.11]) *Every closed point of $\mathbb{A}_k^\infty$ is a k-valued point if and only if the field k is not a countable.*

The concept "thin" in the following definition was first introduced in [12].

Definition 2.18. Let X be a variety over k. An arc $\alpha\colon \operatorname{Spec} K[[t]] \longrightarrow X$ is said to be *thin* if α factors through a proper closed subvariety of X. An arc which is not thin is called a *fat arc*.

An irreducible subset C in X_∞ is called a *thin set* if C is contained in Z_∞ for a proper closed subvariety $Z \subset X$. An irreducible subset in X_∞ which is not thin is called a *fat set*.

In case an irreducible subset C has the generic point $\gamma \in C$ (*i.e.*, the closure $\overline{\gamma}$ contains C), C is a fat set if and only if γ is a fat arc.

The following was proved in [24, Proposition 2.5].

Proposition 2.19. *Let X be a variety over k and $\alpha\colon \operatorname{Spec} K[[t]] \longrightarrow X$ an arc. Then the following hold:*

(i) *α is a fat arc if and only if the ring homomorphism $\alpha^*\colon \mathcal{O}_{X,\alpha(0)} \longrightarrow K[[t]]$ induced from α is injective;*
(ii) *Assume that α is fat. For an arbitrary proper birational morphism $\varphi\colon Y \longrightarrow X$, α is lifted to Y.*

REMARK 2.20. A fat set in X_∞ for a variety X introduces a discrete valuation on the rational function field $K(X)$ of X. We do not give the construction of the valuation here; the reader may refer to [24]. A Nash component (see the next section) is a fat set, and the Nash map (see the next section) is just the correspondence that associates a fat set to the valuation induced from the fat set [24].

EXAMPLE 2.21. A typical example of a fat set is an irreducible *cylinder* (*i.e.*, the pull back $\psi_m^{-1}(S)$ of a constructible set $S \subset X_m$) for a non-singular X. Actually, take an m-jet $\alpha_m\colon \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow X$ in C; then at a neighborhood of $x = \alpha_m(0) = \pi_m(\alpha_m)$, X is étale over $\mathbb{A}_k^n$. Therefore, we may assume that $X = \mathbb{A}_k^n$ and $x = 0$. Assume that $\psi_m^{-1}(\alpha_m)$ is thin. Then it is contained in Z_∞ for some proper closed subset $Z \subset X$. Let the m-jet α_m correspond to a ring homomorphism

$$\alpha_m^*\colon k[x_1, \dots, x_n] \longrightarrow k[t]/(t^{m+1}), \quad \alpha_m^*(x_i) = \sum_{j=1}^{m} a_i^{(j)} t^j.$$

Let $x_i^{(j)}$ be an indeterminate for every $i = 1, \dots, n$ and $j \geq m+1$. Let

$$\alpha^*\colon k[x_1, \dots, x_n] \longrightarrow k\big(x_i^{(j)} \mid i = 1, \dots, n, j \geq m+1\big)[[t]]$$

be an arc defined by

$$\alpha^*(x_i) = \sum_{j=1}^{m} a_i^{(j)} t^j + \sum_{j=m+1}^{\infty} x_i^{(j)} t^j.$$

Let $\alpha^*(f) = F_0(a_i^{(j)}, x_i^{(j)}) + F_1(a_i^{(j)}, x_i^{(j)})t + \cdots + F_\ell(a_i^{(j)}, x_i^{(j)})t^\ell + \cdots$ for $f \in I_Z$. Then as the $x_i^{(j)}$ are indeterminates, there is an ℓ such that $F_\ell \neq 0$. Hence, we obtain $\alpha \in \psi_m^{-1}(C)$ such that $\alpha \notin Z_\infty$.

EXAMPLE 2.22. ([6]) For a singular variety X, an irreducible cylinder is not necessarily fat. Indeed, let X be the Whitney Umbrella, which is a hypersurface defined by $xy^2 - z^2 = 0$ in $\mathbb{A}_k^3$. For $m \geq 1$, let

$$\alpha_m^*\colon k[x, y, z]/(xy^2 - z^2) \longrightarrow k[t]/(t^{m+1})$$

be the m-jet defined by $\alpha_m(x) = t$, $\alpha_m(y) = 0$, $\alpha_m(z) = 0$. Then the cylinder $\psi_m^{-1}(\alpha_m)$ is contained in $\operatorname{Sing}(X)_\infty$, where $\operatorname{Sing}(X) = (y = z = 0)$. This is proved as follows: Let an arbitrary $\alpha \in \psi_m^{-1}(\alpha_m)$ be induced from

$$\alpha^*\colon k[x, y, z] \longrightarrow k[[t]]$$

with

$$\alpha^*(x) = \sum_{j=1}^{\infty} a_j t^j, \alpha^*(y) = \sum_{j=1}^{\infty} b_j t^j, \alpha^*(z) = \sum_{j=1}^{\infty} c_j t^j,$$

where we note that $a_1 = 1$. Then the condition $\alpha^*(xy^2 - z^2) = 0$ implies that the initial term of $\alpha^*(xy^2)$ and that of $\alpha^*(z^2)$ cancel each other. If $\alpha^*(y) \neq 0$, then the order of $\alpha^*(xy^2)$ is odd, while if $\alpha^*(z) \neq 0$, the order of $\alpha^*(z^2)$ is even. Hence if $\alpha^*(y) \neq 0$ or $\alpha^*(z) \neq 0$, then the initial term of $\alpha^*(xy^2)$ and that of $\alpha^*(z^2)$ do not cancel each other. Therefore, $\alpha^*(y) = \alpha^*(z) = 0$, which shows that $\psi_m^{-1}(\alpha_m) \subset \operatorname{Sing}(X)_\infty$.

3. Properties of jet schemes and arc spaces.

Consider

$$\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} k[s, s^{-1}]$$

as a multiplicative group scheme. For $m \in \mathbb{N} \cup \{\infty\}$, the morphism

$$k[t]/(t^{m+1}) \longrightarrow k[s, s^{-1}, t]/(t^{m+1})$$

defined by $t \mapsto s \cdot t$ gives an action

$$\mu_m \colon \mathbb{G}_m \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow \operatorname{Spec} k[t]/(t^{m+1})$$

of $\mathbb{G}_m$ on $\operatorname{Spec} k[t]/(t^{m+1})$. Therefore, it gives an action

$$\mu_{Xm} \colon \mathbb{G}_m \times_{\operatorname{Spec} k} X_m \longrightarrow X_m$$

of $\mathbb{G}_m$ on X_m. As μ_m is extended to a morphism:

$$\overline{\mu}_m \colon \mathbb{A}^1 \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow \operatorname{Spec} k[t]/(t^{m+1}),$$

we obtain the extension

$$\overline{\mu}_{Xm} \colon \mathbb{A}^1 \times_{\operatorname{Spec} k} X_m \longrightarrow X_m \tag{3.1}$$

of μ_{Xm}.

Note that $\overline{\mu}_{Xm}(\{0\} \times \alpha) = x_m$, where x_m is the trivial m-jet on $x = \alpha(0) \in X$. Therefore, every orbit $\mu_{Xm}(\mathbb{G}_m \times \{\alpha\})$ contains the trivial m-jet on $\alpha(0)$ in its closure.

PROPOSITION 3.1. *For $m \in \mathbb{N} \cup \{\infty\}$, let $Z \subset X_m$ be a $\mathbb{G}_m$-invariant closed subset. Then the image $\pi_m(Z)$ is closed in X. In particular, the image $\pi_m(Z)$ of an irreducible component Z of X_m is closed in X.*

PROOF. Let $Z \subset X_m$ be an $\mathbb{G}_m$-invariant closed subset. Then we obtain

$$\overline{\mu}_{Xm}(\mathbb{A}^1 \times Z) = Z.$$

On the other hand, $\overline{\mu}_{Xm}(\{0\} \times Z) = \sigma_m \circ \pi_m(Z)$ by the note after (3.1). Therefore, as Z is closed, it follows that

$$Z \supset \overline{\sigma_m \circ \pi_m(Z)} \supset \sigma_m\big(\overline{\pi_m(Z)}\big),$$

which yields $\pi_m(Z) \supset \overline{\pi_m(Z)}$. ■

PROPOSITION 3.2. *Let $f\colon X \longrightarrow Y$ be a morphism of k-schemes of finite type. Then a canonical morphism $f_\infty\colon X_\infty \longrightarrow Y_\infty$ is induced such that the following diagram is commutative:*

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ {\scriptstyle \pi_{Xm}}\downarrow & & \downarrow{\scriptstyle \pi_{Ym}} \\ X & \xrightarrow{f} & Y. \end{array}$$

PROOF. The morphism f_∞ is induced as the projective limit of f_m $(m \in \mathbb{N})$. ■

PROPOSITION 3.3. *Let $f\colon X \longrightarrow Y$ be a proper birational morphism of k-schemes of finite type such that $f|_{X \setminus W} : X \setminus W \simeq Y \setminus V$, where $W \subset X$ and $V \subset Y$ are closed. Then f_∞ gives a bijection $X_\infty \setminus W_\infty \longrightarrow Y_\infty \setminus V_\infty$.*

PROOF. Let $\alpha \in Y_\infty \setminus V_\infty$. Then $\alpha(\eta) \in X \setminus V$. As $X \setminus W \simeq Y \setminus V$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec} K((t)) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} K[[t]] & \xrightarrow{\alpha} & X. \end{array}$$

Then, as f is a proper morphism, by the valuative criteria of properness, there is a unique morphism $\tilde{\alpha}\colon \operatorname{Spec} K[[t]] \longrightarrow Y$ such that $f \circ \tilde{\alpha} = \alpha$. This shows the bijectivity as required. ■

The following is the version for $m = \infty$ of Proposition 2.4.

Proposition 3.4. *If $f\colon X \longrightarrow Y$ is an étale morphism, then*

$$X_\infty \simeq Y_\infty \times_Y X.$$

Proof. As $\varprojlim_m (Y_m \times_Y X) = (\varprojlim_m Y_m) \times_Y X$, the case $m = \infty$ is reduced to the case $m < \infty$, which is proved in Proposition 2.4. ∎

Proposition 3.5. *There is a canonical isomorphism $(X \times Y)_m \simeq X_m \times Y_m$, for every $m \in \mathbb{N} \cup \{\infty\}$.*

Proof. For an arbitrary k-scheme Z and $m < \infty$,

$$\mathrm{Hom}_k(Z, X_m \times Y_m) \simeq \mathrm{Hom}_k(Z, X_m) \times \mathrm{Hom}_k(Z, Y_m),$$

and the right-hand side is isomorphic to

$$\begin{aligned} &\mathrm{Hom}_k\big(Z \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,k[t]/(t^{m+1}), X\big) \times \mathrm{Hom}_k\big(Z \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,k[t]/(t^{m+1}), Y\big) \\ &\qquad \simeq \mathrm{Hom}_k\big(Z \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,k[t]/(t^{m+1}), X \times Y\big) \\ &\qquad \simeq \mathrm{Hom}_k\big(Z, (X \times Y)_m\big). \end{aligned}$$

The case $m = \infty$ follows from this. ∎

Proposition 3.6. *Let $f\colon X \longrightarrow Y$ be an open immersion (resp. closed immersion) of k-schemes of finite type. Then the induced morphism $f_m\colon X_m \longrightarrow Y_m$ is also an open immersion (resp. closed immersion) for every $m \in \mathbb{N} \cup \{\infty\}$.*

Proof. The open case follows from Lemma 2.5 and Proposition 3.4. For the closed case, we may assume that Y is affine. If Y is defined by f_i $(i = 1, \ldots, r)$ in an affine space, then X is defined by f_i $(i = 1, \ldots, r, \ldots, u)$ with $r \le u$ in the same affine space. Then Y_m is defined by $F_i^{(s)}$ $(i = 1, \ldots, r,\ s \le m)$ and X_m is defined by $F_i^{(s)}$ $(i = 1, \ldots, r, \ldots, u,\ s \le m)$ in the corresponding affine space. This shows that X_m is a closed subscheme of Y_m. ∎

Remark 3.7. In the above proposition we see that the properties of open or closed immersion of the base spaces is inherited by the morphism of the space of jets and arcs. But some properties are not inherited. For example, surjectivity and closedness are not inherited.

Example 3.8. There is an example that $f\colon X \longrightarrow Y$ is surjective and closed but $f_\infty\colon X_\infty \longrightarrow Y_\infty$ is neither surjective nor closed. Let $X = \mathbb{A}^2_{\mathbb{C}}$ and $G = \left\langle \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon^{n-1} \end{smallmatrix}\right) \right\rangle$ be a finite cyclic subgroup in $\mathrm{GL}(2, \mathbb{C})$ acting on X, where $n \ge 2$ and ϵ is a primitive n-th root of unity. Let $Y = X/G$ be the quotient of X by the action of G. Then it is well known that the singularity appearing in Y is an A_{n-1}-singularity. Then the canonical projection $f\colon X \longrightarrow Y$ is closed and surjective. We will see that these two properties are not inherited by $f_\infty\colon X_\infty \longrightarrow Y_\infty$. Let p be the image $f(0) \in Y$. Then by the commutativity of

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y, \end{array}$$

we obtain $\pi_X^{-1}(0) = f_\infty^{-1} \circ \pi_Y^{-1}(p)$. Here, $\pi_X^{-1}(0)$ is irreducible, since X is non-singular. On the other hand $\pi_Y^{-1}(p)$ has $(n-1)$-irreducible components by [36], [21]. Therefore the morphism f_∞ is not surjective for $n \geq 3$. As $X \setminus \{0\} \longrightarrow Y \setminus \{p\}$ is étale, the morphism $(X \setminus \{0\})_\infty \longrightarrow (Y \setminus \{p\})_\infty$ is also étale by Proposition 3.4. Since Y_∞ is irreducible, f_∞ is dominant. Therefore, f_∞ is not closed.

Next we think of the irreducibility of the arc space or jet schemes. The following theorem was proved in [27]. In [22] we gave another proof by using [21, Lemma 2.12] and a resolution of the singularities. Here we present a proof without using a resolution.

THEOREM 3.9. *If the characteristic of k is zero, then the space of arcs of a variety X is irreducible.*

PROOF. By [21, Lemma 2.12] we obtain the following:

(1) Given any arc $\phi\colon \operatorname{Spec} k'[[s]] \longrightarrow X$, we construct an arc Φ such that $\phi \in \overline{\{\Phi\}}$ and $\Phi(0) = \Phi(\eta) = \phi(\eta)$.

(2) We construct an arc Ψ such that $\Phi \in \overline{\{\Psi\}}$ and $\Psi(\eta) \in X \setminus \operatorname{Sing} X$.

Now for this Ψ we apply the procedure (1) again, obtaining a new arc Ψ' such that $\Psi \in \overline{\{\Psi'\}}$ and $\Psi'(0) = \Psi'(\eta) = \Psi(\eta) \in X \setminus \operatorname{Sing} X$. If we let $\pi(\Psi') = \Psi'(0) = \lambda$, then $\Psi' \in \pi^{-1}(\lambda)$. As $\lambda \in X \setminus \operatorname{Sing} X$, it follows that

$$\Psi' \in \pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\rho)},$$

where ρ is the generic point of X. This yields $\phi \in \overline{\pi^{-1}(\rho)}$, which is an irreducible closed subset. ∎

EXAMPLE 3.10. ([21, Example 2.13]) If the characteristic of k is $p > 0$, X_∞ is not necessarily irreducible. For example, the hypersurface X defined by $x^p - y^p z = 0$ has an irreducible component in $(\operatorname{Sing} X)_\infty$ which is not in the closure of $X_\infty \setminus (\operatorname{Sing} X)_\infty$.

Example 3.11. ([23]) Let X be a toric variety over an algebraically closed field of arbitrary characteristic. Then X_∞ is irreducible.

Next let us think of m-jet schemes. A space of m-jets is not necessarily irreducible even if the characteristic of k is zero (see Example 2.9).

Theorem 3.12. ([34]) *If X is a variety of locally complete intersection over an algebraically closed field of characteristic zero, then X_m is irreducible for all $m \geq 1$ if and only if X has rational singularities.*

Another situation in which a geometric property of space of jets determines the singularities on the base space is as follows.

Theorem 3.13. ([14]) *Let X be a reduced divisor on a nonsingular variety over $\mathbb{C}$. Then X has terminal singularities if and only if X_m is normal for every $m \in \mathbb{N}$.*

4. **Introduction to the Nash problem.** In this section, we assume the existence of resolutions of singularities. It is sufficient to assume that the characteristic of k is zero. One of the most mysterious and fascinating problems in arc spaces is the Nash problem, which was posed by Nash in a preprint in 1968. It is a question about the Nash components and the essential divisors. First we introduce the concept of essential divisors.

Definition 4.1. Let X be a variety, $g\colon X_1 \longrightarrow X$ a proper birational morphism from a normal variety X_1, and $E \subset X_1$ an irreducible exceptional divisor of g. Let $f\colon X_2 \longrightarrow X$ be another proper birational morphism from a normal variety X_2. The birational map $f^{-1} \circ g\colon X_1 \dashrightarrow X_2$ is defined on a (nonempty) open subset E^0 of E. Because of Zariski's main theorem, the "undefined locus" of a birational map between normal varieties is of codimension ≥ 2. The closure of $(f^{-1} \circ g)(E^0)$ is called the *center* of E on X_2.

We say that E *appears in* f (or in X_2), if the center of E on X_2 is also a divisor. In this case the birational map $f^{-1} \circ g\colon X_1 \dashrightarrow X_2$ is a local isomorphism at the generic point of E and we denote the birational transform of E on X_2 again by E. For our purposes $E \subset X_1$ is identified with $E \subset X_2$. Such an equivalence class is called an *exceptional divisor over* X.

Definition 4.2. Let X be a variety over k and let $\operatorname{Sing} X$ be the singular locus of X. In this paper, by a *resolution* of the singularities of X we mean a proper, birational morphism $f\colon Y \longrightarrow X$ with Y non-singular such that the restriction $Y \setminus f^{-1}(\operatorname{Sing} X) \longrightarrow X \setminus \operatorname{Sing} X$ of f is an isomorphism.

Definition 4.3. An exceptional divisor E over X is called an *essential divisor* over X if for every resolution $f\colon Y \longrightarrow X$, the center of E on Y is an irreducible component of $f^{-1}(\operatorname{Sing} X)$.

For a given resolution $f\colon Y \longrightarrow X$, the center of an essential divisor is called an *essential component* of Y.

PROPOSITION 4.4. *Let $f\colon Y \longrightarrow X$ be a resolution of the singularities of a variety X. The set*

$$\mathcal{E} = \mathcal{E}_{Y/X} = \Big\{ \textit{irreducible components of } f^{-1}(\operatorname{Sing} X) \\ \textit{which are centers of essential divisors over } X \Big\}$$

corresponds bijectively to the set of all essential divisors over X.

In particular, the set of essential divisors over X is a finite set.

PROOF. The map

$$\{\text{essential divisors over } X\} \longrightarrow \mathcal{E}_{Y/X}, \quad E \mapsto \text{center of } E \text{ on } Y$$

is surjective by the definition of essential components. To prove the injectivity, take an essential component C and then blow up $Y' \longrightarrow Y$ with center C. Then there is a unique divisor $E \subset Y'$ dominating C. Let $Y'' \longrightarrow Y'$ be a resolution of the singularities of Y'. Then E is the unique exceptional divisor on Y'' that dominates C. Therefore, every exceptional divisor over X with center $C \subset Y$ has center contained in E on a resolution Y'' of the singularities of X. Therefore, by the definition of essential divisor, this E is the unique essential divisor whose center on Y is C. ■

C. Bourvier and G. Gonzalez-Sprinberg also introduced "essential divisors" and "essential components" in [2] and [3], but we should note that their definitions are different from ours. In order to distinguish them we give different names to their "essential divisors" and "essential components".

DEFINITION 4.5. ([2], [3]) An exceptional divisor E over X is called a *BGS-essential divisor* over X if E appears in every resolution. An exceptional divisor E over X is called a *BGS-essential component* over X if the center of E on every resolution f of the singularity of X is an irreducible component of $f^{-1}(E')$, where E' is the center of E on X.

We will see how different they are from our essential divisors and essential components. First we see that they coincide for the 2-dimensional case. To show this we need to introduce the concept of minimal resolution.

DEFINITION 4.6. A resolution $f\colon Y \longrightarrow X$ of the singularities of X is called the *minimal resolution* if for any resolution $g\colon Y' \longrightarrow X$, there is a unique morphism $Y' \longrightarrow Y$ over X.

It is known that for a surface X the minimal resolution $f\colon Y \longrightarrow X$ exists. It is characterized by the fact that Y has no exceptional curve of the first kind over X.

For a higher dimensional variety X, the minimal resolution does not necessarily exist. For example, $X = \{xy - zw = 0\} \subset \mathbb{A}^4$ has two resolutions, neither of which dominates the other. These two resolutions are obtained as follows: first take a blow-up $f\colon \tilde{Y} \longrightarrow X$ at the origin of X which has a unique singular point at the origin. Then f is a resolution of the singularity of X and the exceptional divisor E of f is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Here we have two contractions $g_1\colon Y_1 \longrightarrow X$, $g_2\colon Y_2 \longrightarrow X$ whose restrictions are the first projection $p_1\colon E = \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ and the second projection $p_2\colon E = \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, respectively. Both Y_i's are non-singular, therefore the f_i's are resolutions of the singularity of X. It is clear that there is no morphism between Y_1 and Y_2 over X.

Proposition 4.7. *If X is a surface, then each set of "essential divisors", "BGS-essential divisors" and "BGS-essential components" are bijective with the set of the components of the fiber $f^{-1}(\operatorname{Sing} X)$, where $f\colon Y \longrightarrow X$ is the minimal resolution. These are also essential components on the minimal resolution.*

Remark 4.8. The four concepts "essential divisor", "essential component", "BGS-essential divisor" and "BGS-essential component" are mutually different in general.

First, our essential component is different from the others, because it is a closed subset on a specific resolution and the others are all equivalence classes of divisors.

Next, a BGS-essential divisor is different from a BGS-essential component or an essential divisor. Indeed, for $X = (xy - zw = 0) \subset \mathbb{A}^4_k$, the exceptional divisor obtained by a blow-up at the origin is the unique essential divisor and also the unique BGS-essential component, while there is no BGS-essential divisor, since X has a resolution whose exceptional set does not contain a divisor.

Finally a BGS-essential component and an essential divisor are different. Indeed, consider a cone generated by $(0,0,1)$, $(2,0,1)$, $(1,1,1)$, $(0,1,1)$ in $\mathbb{R}^3$. It is well known that a cone generated by integer points in a real Euclidean space defines an affine toric variety (see [15, 38] for a basic notion of toric variety). Let X be the affine toric variety defined by this cone. Then the canonical subdivision adding a 1-dimensional cone $\mathbb{R}_{\geq 0}(1,0,1)$ is a resolution of X. As the singular locus of X is of dimension one, there is no small resolution. Therefore, the divisor $D_{(1,0,1)}$ is the unique essential divisor, while $D_{(1,1,2)}$ and $D_{(2,1,2)}$ are BGS-essential components by the criterion [2, Theorem 2.3].

Definition 4.9. Let X be a variety and $\pi\colon X_\infty \longrightarrow X$ the canonical projection. An irreducible component C of $\pi^{-1}(\operatorname{Sing} X)$ is called a *Nash component* if it contains an arc α such that $\alpha(\eta) \notin \operatorname{Sing} X$. This is equivalent to saying $C \not\subset (\operatorname{Sing} X)_\infty$.

The following lemma was quoted earlier for the irreducibility of the space of arcs (Theorem 3.9).

LEMMA 4.10. ([21]) *If the characteristic of the base field k is zero, then every irreducible component of* $\pi^{-1}(\operatorname{Sing} X)$ *is a Nash component.*

We note that this lemma does not hold for the positive characteristic case. Indeed, Example 3.10 is an example where $\pi^{-1}(\operatorname{Sing} X)$ has an irreducible component which is not a Nash component.

Let $f\colon Y \longrightarrow X$ be a resolution of the singularities of X and E_l $(l = 1, \dots, r)$ the irreducible components of $f^{-1}(\operatorname{Sing} X)$. Now we are going to introduce a map $\mathcal{N}$ which is called the Nash map

$$\left\{\begin{matrix}\text{Nash components}\\ \text{of the space of arcs}\\ \text{of } X\end{matrix}\right\} \xrightarrow{\mathcal{N}} \left\{\begin{matrix}\text{essential}\\ \text{components}\\ \text{on } Y\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{essential}\\ \text{divisors}\\ \text{over } X\end{matrix}\right\}.$$

4.1. Construction of the Nash map. The resolution $f\colon Y \longrightarrow X$ induces a morphism $f_\infty\colon Y_\infty \longrightarrow X_\infty$ of schemes. Let $\pi_Y\colon Y_\infty \longrightarrow Y$ be the canonical projection. As Y is non-singular, $(\pi_Y)^{-1}(E_l)$ is irreducible for every l. Denote by $(\pi_Y)^{-1}(E_l)^o$ the open subset of $(\pi_Y)^{-1}(E_l)$ consisting of the points corresponding to arcs $\beta\colon \operatorname{Spec} K[[t]] \longrightarrow Y$ such that $\beta(\eta) \notin \bigcup_l E_l$. Let C_i $(i \in I)$ be the Nash components of X. Denote by C_i^o the open subset of C_i consisting of the points corresponding to arcs $\alpha\colon \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha(\eta) \notin \operatorname{Sing} X$. As C_i is a Nash component, we have $C_i^o \neq \varnothing$. The restriction of f_∞ gives

$$f'_\infty\colon \bigcup_{l=1}^{r} (\pi_Y)^{-1}(E_l)^o \longrightarrow \bigcup_{i \in I} C_i^o.$$

By Proposition 3.3, f'_∞ is surjective. Hence, for each $i \in I$ there is a unique l_i such that $1 \le l_i \le r$ and the generic point β_{l_i} of $(\pi_Y)^{-1}(E_{l_i})^o$ is mapped to the generic point α_i of C_i^o. By this correspondence $C_i \mapsto E_{l_i}$ we obtain a map

$$\mathcal{N}\colon \left\{\begin{matrix}\text{Nash components}\\ \text{of the space of arcs}\\ \text{through } \operatorname{Sing} X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{irreducible}\\ \text{components}\\ \text{of } f^{-1}(\operatorname{Sing} X)\end{matrix}\right\}.$$

LEMMA 4.11. *The map* $\mathcal{N}$ *is an injective map to the subset* {*essential components on* Y}.

PROOF. Let $\mathcal{N}(C_i) = E_{l_i}$. Denote the generic point of C_i by α_i and the generic point of $(\pi_Y)^{-1}(E_l)$ by β_l. If $E_{l_i} = E_{l_j}$ for $i \neq j$, then $\alpha_i = f'_\infty(\beta_{l_i}) = f'_\infty(\beta_{l_j}) = \alpha_j$, a contradiction.

To prove that the $\{E_{l_i} : i \in I\}$ are essential components on Y, let $Y' \longrightarrow X$ be another resolution and $\tilde{Y} \longrightarrow X$ a divisorial resolution which factors through both Y and Y'. Let $E'_{l_i} \subset Y'$ and $\tilde{E}_{l_i} \subset \tilde{Y}$ be the irreducible components of the exceptional sets corresponding to C_i. Then we can see that E_{l_i} and E'_{l_i} are the image of $\tilde{E}_{l_i}$. This shows that $\tilde{E}_{l_i}$ is an essential divisor over X and therefore E_{l_i} is an essential component on Y. ■

PROBLEM 4.12. *Is the Nash map*

$$\left\{\begin{matrix}\text{Nash components}\\ \text{of the space of arcs}\\ \text{through } \operatorname{Sing} X\end{matrix}\right\} \xrightarrow{\mathcal{N}} \left\{\begin{matrix}\text{essential}\\ \text{components}\\ \text{on } Y\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{essential}\\ \text{divisors}\\ \text{over } X\end{matrix}\right\}.$$

bijective?

After the preprint in which Nash posed this problem was circulated in 1968, Bouvier, Gonzalez-Sprinberg, Hickel, Lejeune-Jalabert, Nobile, Reguera-Lopez and others (see [2, 17, 20, 29, 30, 31, 37, 42]) worked on the arc space of a singular variety related to this problem.

Recently, the Nash problem has been answered affirmatively for a toric variety of arbitrary dimension, but has been negatively answered in general by Ishii and Kollár [21].

Here, we show known results for this problem.

THEOREM 4.13. ([36]) *The Nash problem is answered affirmatively for an A_n-singularity ($n \in \mathbb{N}$), where an A_n-singularity is the hypersurface singularity defined by $xy - z^{n+1} = 0$ in $\mathbb{A}^3_k$.*

THEOREM 4.14. ([42]) *The Nash problem is answered affirmatively for a minimal surface singularity. Here, a minimal surface singularity means a rational surface singularity with the reduced fundamental cycle. The fundamental cycle is induced by M. Artin (see* [1] *for the definition).*

THEOREM 4.15. ([31], [43]) *The Nash problem is answered affirmatively for a sandwiched surface singularity. Here, a sandwiched surface singularity means the formal neighborhood of a singular point on a surface obtained by blowing up a complete ideal in the local ring of a closed point on a non-singular algebraic surface. A complete ideal is defined by Zariski and Samuel (see* [50, Vol. II, Appendix 4]*), but the idea of a sandwiched singularity is that it is a singularity which is birationally sandwiched by non-singular surfaces.*

These are results on rational surface singularities. The following gives an affirmative answer for some non-rational surface singularities:

THEOREM 4.16. ([40]) *The Nash problem is answered affirmatively for a normal surface singularity with the reduced fiber E of the singular point on the minimal resolution such that $E \cdot E_i < 0$ for every irreducible component E_i of E.*

This result is generalized in [33] to a wider class of surface singularities. We omit the statement, since it is not simple.

The following results are for arbitrary dimension.

THEOREM 4.17. ([21]) *The Nash problem is answered affirmatively for a toric singularity of arbitrary dimension.*

Theorem 4.18. ([24]) *The Nash problem is answered affirmatively for a non-normal toric variety of arbitrary dimension.*

We have a notion of the local Nash problem which is a slight modification of the Nash problem [25].

Theorem 4.19. ([25]) *The local Nash problem holds true for a quasi-ordinary singularity. Here, a quasi-ordinary singularity is a hypersurface singularity which is a finite cover over a non-singular variety with the normal crossing branch locus. We note that a quasi-ordinary singularity is not necessarily normal.*

The paper [41] gives an affirmative answer to the Nash problem for a certain class of higher dimensional non-toric singularities.

So far we have seen the affirmative answers. But there are negative examples given in [21].

Example 4.20. Let X be a hypersurface defined by $x_1^3+x_2^3+x_3^3+x_4^3+x_5^6=0$ in $\mathbb{A}^5_{\mathbb{C}}$. Then the number of the Nash components is one, while the number of the essential divisors is two. Therefore the Nash map is not bijective.

By the above example we can construct counterexamples to the Nash problem for any dimension greater than 3. At this moment the Nash problem is still open for two- and three-dimensional varieties. Now we can formulate a new version of the Nash problem.

Problem 4.21. *What is the image of the Nash map? For two- and three-dimensional cases, does the image of the Nash map coincide with the set of essential divisors?*

Related to this problem, we have one characterization of the image of the Nash map given by Reguera [44]. To formulate her result, we introduce the concept of "wedge".

Definition 4.22. Let $K \supset k$ be a field extension. A K-*wedge* of X is a k-morphism $\gamma\colon \operatorname{Spec} K[[\lambda, t]] \longrightarrow X$. A K-wedge γ can be identified to a $K[[\lambda]]$-point on X_∞. We call the *special arc* of γ the image in X_∞ of the closed point 0 of $\operatorname{Spec} K[[\lambda]]$. We call the *generic arc* of γ the image in X_∞ of the generic point η of $\operatorname{Spec} K[[\lambda]]$.

Theorem 4.23. ([44]) *Let E be an essential divisor over X and $f\colon Y \longrightarrow X$ a resolution of the singularities of X on which E appears. Let $\alpha \in X_\infty$ be the generic point of $f_\infty\big(\pi_Y^{-1}(E)\big)$ and k_E the residue field of α. Then the following conditions are equivalent.*

(i) *E belongs to the image of the Nash map.*
(ii) *For any resolution of the singularities $g\colon Y' \longrightarrow X$ and for any field extension K of k_E, any K-wedge γ on X whose special arc is α and whose generic arc belongs to $\pi_X^{-1}(\operatorname{Sing} X)$, lifts to Y'.*
(iii) *There exists a resolution of the singularities $g\colon Y' \longrightarrow X$ satisfying condition* (ii)*.*

As a corollary of this theorem, we also obtain Theorem 4.15.

There are some generalizations of the Nash problem for a pair (X, Z) consisting of a variety X and a closed subset Z (see [39, 16]). These promise to be a new area of active research.

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