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A RELATIVE DOUBLE COMMUTANT THEOREM FOR HEREDITARY SUB-C*-ALGEBRAS

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Abstract. We prove a double commutant theorem for hereditary subalgebras of a large class of C*-algebras, partially resolving a problem posed by Pedersen. Double commutant theorems originated with von Neumann, whose seminal result evolved into an entire field now called von Neumann algebra theory. Voiculescu proved a C*-algebraic double commutant theorem for separable subalgebras of the Calkin algebra. We prove a similar result for hereditary subalgebras which holds for more general corona C*-algebras. (It is not clear how generally Voiculescu's double commutant theorem holds.)

Résumé. Nous démontrons un théorème de commutant double (d'après Voiculescu et von Neumann) pour les sous-C*-algèbres héréditaires d'une C*-algèbre ⟪corona⟫, c'est-à-dire de l'algèbre $M(A)/A$ pour une C*-algèbre A. Les théorèmes de type commutant double ont commencé avec von Neumann, et son résultat séminal est maintenant la fondation de la théorie des algèbres de Neumann. Voiculescu a démontré un théorème de commutant double pour les sous-C*-algèbres séparables de l'algèbre $B(H)/K(H)$. Nous démontrons un résultat semblable pour les sous-C*-algèbres héréditaires des algèbres $M(A)/A$. Il n'est pas clair dans quel cadre le théorème de commutant double de Voiculescu est valable en général.

1. Introduction.

The most fundamental result in all of von Neumann algebra theory is perhaps von Neumann's double commutant theorem, published in 1929 (see [12]).

Theorem 1.1. *The double commutant of any sub-C*-algebra of $B(H)$ is equal to the weak operator closure of its unitization.*

Approximately half a century later, Voiculescu proved a remarkable and unexpected C*-algebraic version of the above theorem:

Theorem 1.2. ([11], [10]) *Consider the Calkin algebra $B(H)/K(H)$ associated with a separable infinite-dimensional Hilbert space H. The double commutant of a separable sub-C*-algebra is equal to its unitization.*

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The unitization referred to in these two statements is the concrete unitization obtained by adjoining the unit of the ambient algebra.

Recall that the multiplier algebra $M(B)$ of a given C*-algebra B is the idealizer of B in the double dual B^{**}. Since the multiplier algebra of the compact operators $K(H)$ is $B(H)$, we may reasonably regard the corona algebra $M(B)/B$ as a sweeping generalization of the Calkin algebra considered by Voiculescu. At a conference in 1988, Pedersen posed the problem of generalizing Voiculescu's theorem to the setting of corona algebras [8].

In this note, we consider an analogous question: we show that in many cases, the double commutant of a singly generated hereditary subalgebra H of a corona algebra is $H + Z$, where Z is the centre of the corona algebra. (This is new, we believe, even in the case of the Calkin algebra.)

2. Hereditary subalgebras with full annihilator

THEOREM 2.1. *Let H be a full hereditary subalgebra of a unital C*-algebra Q. Then an element x that commutes with H can be uniquely decomposed as $z + a$ where z is in the centre, $Z(Q)$, of Q and a is in the annihilator, $H^{\perp}$, of H.*

PROOF. Let us first prove the uniqueness of the decomposition. If $x = z_1+a_1 = z_2+a_2$ with z_i in the centre of Q and a_i in $H^{\perp}$, then $c = z_1-z_2 = a_2-a_1$ is in both $Z(Q)$ and $H^{\perp}$. We are to show that c is zero. If not, then by the Gelfand-Naimark theorem there is an irreducible representation π of Q such that $\pi(c)$ is non-zero. Then, as c is in the centre of Q, it follows that $\pi(c)$ is a scalar multiple of the unit of $\pi(Q)$. But since c is also in $H^{\perp}$, it follows that the subalgebra $\pi(H^{\perp})$ of $\pi(Q)$ contains the unit of $\pi(Q)$. Since $\pi(H)\pi(H^{\perp}) = 0$, it follows that $\pi(H) = 0$. This contradicts the assumption that H is full. Therefore, c must be zero.

Let us now prove existence. Given x that commutes with all of H, we notice that if $h \in H$, then $xh = x_1hx_2$ for any factoring $x = x_1x_2$ in H, from which it follows that xh is in H. The case of action on the right is similar, and so x can be regarded as an element of $M(H)$. Clearly x is central as an element of $M(H)$.

By one of Pedersen's early results, as H is full, the natural map $t \mapsto t \cap H$ from $\operatorname{Prim} Q$ to $\operatorname{Prim} H$ is a homeomorphism [7]. Since this map is compatible with the map from $Z(Q)$ to $Z\big(M(H)\big)$ constructed in the previous paragraph, it follows by the extension of the Dauns-Hofmann Theorem given in [3] that this map is an isomorphism of C*-algebras, and in particular is surjective.

Let us denote the (unique) pre-image of $x \in Z(M(H))$ under this isomorphism by $c \in Z(Q)$, and set $x - c = a$.

This element $a \in Q$ certainly multiplies H into itself, as both x and (for the same reason) c do, and as an element of $M(H)$ it is zero by construction. Thus, a is in the annihilator of H, and the decomposition $x = a + c$ has the desired properties. ∎

Now recall Pedersen's result:

THEOREM 2.2. ([8]) *If H is a singly generated hereditary subalgebra of a corona algebra (of a σ-unital C*-algebra), then $H^{\perp\perp} = H$.*

Our first result on double commutatants follows:

THEOREM 2.3. *Let H be a singly generated hereditary subalgebra of the corona C*-algebra of some σ-unital C*-algebra, and suppose that the annihilator of H is full. Then the double commutant H'' of H inside the corona is equal to $H + Z$, where Z is the centre of the corona.*

PROOF. Let x be an element of H''. Note that x commutes with the annihilator $H^{\perp}$, since after all, the elements of the annihilator commute with the elements of H. We may thus apply our Theorem 2.1 to decompose x as $a + z$ with a annihilating $H^{\perp}$ and z in the centre of the corona. But a is then in $H^{\perp\perp}$, and by Theorem 2.2 this algebra is equal to H. We conclude that H'' is contained in $H + Z$. On the other hand, it is routine to verify that both Z and H are contained in H''. ■

3. **The case of extremally disconnected primitive ideal space.** We shall now remove the fullness condition on the annihilator of the given hereditary subalgebra, replacing it by a condition on the primitive ideal space of the corona algebra. Namely, we shall impose the well-known condition that the space be extremally disconnected, *i.e.,* that the closure of every open set be open. An extremally disconnected first countable Hausdorff space must be discrete, but of course primitive ideal spaces are not usually either Hausdorff or first countable. The most important special case of relevance to C*-algebraic problems is probably the case of a prime C*-algebra. (To see that a prime C*-algebra has extremally disconnected primitive ideal space, recall that open sets in the primitive ideal space of a prime C*-algebra are either empty or dense; in either case, the closure of an open set is both open and closed.)

Let us consider the question of characterizing C*-algebras for which the corona C*-algebra is prime.

THEOREM 3.1. ([1]) *The corona of a primitive σ-unital C*-algebra is prime.*

One would perhaps expect the converse of this result to hold, but the situation is complicated by the fact that if I is a strictly closed ideal of a C*-algebra A, then A and A/I have isomorphic corona algebras. If there are no non-zero strictly closed ideals, then, indeed, primeness of the corona implies that the given algebra is prime (and hence, if separable, it is primitive). We remark that an ideal is strictly closed if and only if the associated open projection is majorized by an algebra element [1, Proposition 1.1.3].

The natural conjecture that the corona of a separable C*-algebra with extremally disconnected primitive ideal space also has extremally disconnected primitive ideal space is false: consider the case $A = C_0(\mathbb{N})$. It can be shown [5] that the Stone–Čech corona of the natural numbers $\mathbb{N}$ is (surprisingly) not extremally disconnected. In this case, it is even true that the algebra and the multiplier algebra (and also the corona algebra) have real rank zero.

Theorem 3.1 gives a large supply of C*-algebras whose corona has extremally disconnected primitive ideal space, and it seems of interest that our Theorem 2.3 generalizes to coronas with this property.

The hypothesis on the primitive ideal space is applied by means of the following basically topological lemma, which can be obtained from [2, Propositions 3.2.4 and 3.2.3]. (For the convenience of the reader, we include a summary of the proof given in [2].)

LEMMA 3.2. ([2]) *The following conditions are equivalent, for a C*-algebra A with primitive ideal space V:*

(i) Any element of the centre of the multiplier algebra of an ideal comes from an element of the centre of the multiplier algebra $M(A)$ of A.

(ii) The primitive ideal space V is extremally disconnected.

The extension of a central multiplier from an ideal is unique if and only if the ideal is essential.

PROOF. Let J be a given ideal and let z_0 be a given element of the centre of $M(J)$. By Dixmier's extension of the Dauns-Hofmann Theorem [3, 7], the element z_0 is a continuous bounded function on the primitive ideal space of J. Recall that the primitive ideal space of J is an open subset of V. Note that we may as well assume that J is essential, replacing J by $J + J^{\perp}$ and defining z_0 to be zero on $J^{\perp}$.

Now we apply the theorem that a space is extremally disconnected if and only if any continuous bounded function on a dense open set can be extended to a continuous bounded function on the whole space (see paragraphs 1.4 and 1.H.6 of [5]). Conversely, if the property (i) holds for all essential ideals, we deduce again by Dixmier's extension of the Dauns-Hofmann Theorem that the primitive ideal space V has the requisite extension property.

The uniqueness stated in the last part of the lemma is straightforward. ■

THEOREM 3.3. *Let Q be a unital C*-algebra with extremally disconnected primitive ideal space. If x commutes with a hereditary sub-C*-algebra H, then $x = z + a$ for some a in $H^{\perp}$ and some central element $z \in Q$. The decomposition is unique if and only if the ideal generated by H in Q is essential.*

PROOF. To show existence, we notice as before that x multiplies H into H. Denote by m the element of $Z\big(M(H)\big)$ thus obtained (from x). By Dixmier's extension of the Dauns-Hofmann Theorem [3, 7], an element of $Z\big(M(H)\big)$ is equivalently a continuous function on the open subset of $\mathrm{Prim}Q$ that corresponds to the ideal I generated by H in Q, and this element is still a central multiplier. Applying Lemma 3.2, we obtain an element z of the centre of Q. Then $x = (x - z) + z$ is our desired decomposition. ■

Specializing to the case of corona algebras and repeating the proof of Theorem 2.3, we have our main result:

THEOREM 3.4. *Let Q be the corona algebra of some σ-unital C^*-algebra. Suppose that Q has extremally disconnected primitive ideal space. Then the double commutant of a singly generated hereditary subalgebra H is $H + Z$, where Z is the centre of Q.*

One particularly simple special case of the above theorem is as follows:

COROLLARY 3.5. *Let A be a σ-unital primitive C^*-algebra. If H is a singly generated hereditary subalgebra of $M(A)/A$, then H'' is equal to the unitization $\mathbb{C}1_{M(A)/A} + H$ of H.*

This is deduced by noting that by Theorem 3.1 the corona algebra $M(A)/A$ is prime, thus having trivial centre and extremally disconnected primitive ideal space.

Note that if A is simple it is certainly primitive, so in this case we recover the earlier result [6].

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