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STABLE RANK OF DEPTH TWO INCLUSIONS OF C^*-ALGEBRAS

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Abstract. Let $1 \in A \subset B$ be an inclusion of unital C^*-algebras of index-finite type and depth 2. Suppose that A is infinite dimensional, simple, with the property SP. We prove that if $\mathrm{tsr}(A) = 1$, then $\mathrm{tsr}(B) \leq 2$. An interesting special case is $B = A \rtimes_\alpha G$, where α is an action of a finite group G on $\mathrm{Aut}(A)$.

Résumé. Soit $1 \in A \subset B$ une inclusion de C^*-algèbres unitals du type indice-fini et de profondeur 2. On suppose que A est de dimension infinie, simple, et que A a la propriété SP. On démontre que, si $\mathrm{tsr}(A) = 1$, donc $\mathrm{tsr}(B) \leq 2$. Un cas intéressant est $B = A \rtimes_\alpha G$, oú α est une action d'un groupe fini G sur $\mathrm{Aut}(A)$.

1. **Introduction.** The notion of topological stable rank for a C^*-algebra A, denoted by $\mathrm{tsr}(A)$, was introduced by Rieffel, which generalizes the concept of dimension of a topological space [15]. He presented the basic properties and stability theorem related to K-Theory for C^*-algebras. In [15] he proved that $\mathrm{tsr}(A \rtimes_\alpha \mathbb{Z}) \leq \mathrm{tsr}(A) + 1$, and asked if an irrational rotation algebra A_θ has topological stable rank two. I. Putnam [14] gave a complete answer to this question, that is, $\mathrm{tsr}(A_\theta) = 1$. Moreover, Blackadar, Kumjian, and Rørdam [3] proved that every simple noncommutative torus has topological stable rank one. Naturally, we pose a question of how to compute topological stable rank of $A \rtimes_\alpha G$ for any discrete group G.

Blackadar proposed the question in [1] whether $\mathrm{tsr}(A \rtimes_\alpha G) = 1$ for any unital AF C^*-algebra A, finite group G, and action α of G on A. In [12] the authors presented a partial answer to an extended form of Blackadar's question using the C^*-index theory of Watatani [18]; that is, let $1 \in A \subset B$ be an inclusion of unital C^*-algebras and let $E\colon B \to A$ be a faithful conditional expectation of index-finite type. Suppose that the inclusion $1 \in A \subset B$ has depth 2 and A is tsr boundedly divisible with $\mathrm{tsr}(A) = 1$. Then $\mathrm{tsr}(B) \leq 2$. Here a C^*-algebra A is tsr *boundedly divisible* [16, Definition 4.1] if there is a constant K (> 0) such that for every positive integer m there is an integer $n \geq m$ such that A can

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be expressed as $M_n(B)$ for a C^*-algebra B with $\mathrm{tsr}(B) \leq K$. A typical such example is $B \otimes UHF$ for any unital C^*-algebra B. As a corollary, one has that if A is a tsr boundedly divisible, unital C^*-algebra with $\mathrm{tsr}(A) = 1$, G a finite group, and α an action of G on A, then $\mathrm{tsr}(A \rtimes_\alpha G) \leq 2$.

In this note we consider the generalized Blackadar question and get an optimal estimate in some sense: Let $1 \in A \subset B$ be an inclusion of unital C^*-algebras of index-finite type and depth 2. Suppose that A is infinite dimensional simple with $\mathrm{tsr}(A) = 1$ and the property SP. Then $\mathrm{tsr}(B) \leq 2$. In the case of crossed product algebras we conclude that $\mathrm{tsr}(A \rtimes_\alpha G) \leq 2$ for a simple unital C^*-algebra A with $\mathrm{tsr}(A) = 1$ and the property SP, and an action α from a finite group G on $\mathrm{Aut}(A)$.

2. Preliminaries.

Let A be a unital C^*-algebra and $Lg_n(A)$ be the set of elements (b_i) of A^n such that

$$Ab_1 + Ab_2 + \cdots + Ab_n = A.$$

Then the *topological stable rank* of A, $\mathrm{tsr}(A)$, is defined to be the least integer n such that the set $Lg_n(A)$ is dense in A^n. Topological stable rank of a non-unital C^*-algebra is defined by the topological stable rank of its unitization algebra $\tilde{A}$. Note that $\mathrm{tsr}(A) = 1$ is equivalent to having the dense set of invertible elements in $\tilde{A}$.

Let A be a C^*-algebra. A is said to have the *property* SP if any non-zero hereditary C^*-subalgebra of A has non-zero projection. It is well known that if A has real rank zero, that is, any self-adjoint element can be approximated by self-adjoint elements with finite spectra, then A has the property SP. (See [4].)

The inclusion $1 \in A \subset B$ of unital C^*-algebras of index-finite type is said to have *finite depth* k if the derived tower obtained by iterating the basic construction

$$A' \cap A \subset A' \cap B \subset A' \cap B_2 \subset A' \cap B_3 \subset \cdots$$

satisfies $(A' \cap B_k)e_k(A' \cap B_k) = A' \cap B_{k+1}$, where $\{e_k\}_{k \geq 1}$ are projections obtained by iterating the basic construction, so that $B_1 = B, e_1 = e_A$, and $B_{k+1} = C^*(B_k, e_k)$ for $k \geq 1$. Let $E_{k+1} \colon B_{k+1} \to B_k$ be a faithful conditional expectation correspondent to e_{k+1} for $k \geq 1$. (See [5].)

When G is a finite group and α an action of G on A, it is well known that an inclusion $1 \in A \subset A \rtimes_\alpha G$ is of depth 2. (See [12, Lemma 3.1].)

3. Main result.

The following result is contained in [12, Theorem 5.1].

PROPOSITION 3.1. (*cf.* [12, Theorem 5.1]) *Let* $1 \in A \subset B$ *be an inclusion of unital* C^**-algebras of index-finite type and depth* 2. *Suppose that* $\mathrm{tsr}(A) = 1$. *Then we have*

$$\sup_{p \in P(A)} \mathrm{tsr}(pBp) < \infty,$$

where $P(A)$ *denotes the set of all projections in* A.

The following main theorem is an extended version of [12, Theorem 5.1].

THEOREM 3.2. *Let* $1 \in A \subset B$ *be an inclusion of unital* C^**-algebras of index-finite type and depth* 2*. Suppose that* A *is infinite dimensional simple with* $\mathrm{tsr}(A) = 1$ *and the property* SP*. Then* $\mathrm{tsr}(B) \leq 2$.

PROOF. Since $1 \in A \subset B$ is an inclusion of simple C^*-algebras of index-finite type and depth 2, from Proposition 3.1 we have

$$\sup_{p \in P(A)} \mathrm{tsr}(pBp) < \infty.$$

Set $K = \sup_{p \in P(A)} \mathrm{tsr}(pBp)$.

Since A is simple with the property SP, there is a sequence of mutually orthogonal equivalent projections $\{p_i\}_{i=1}^N$ in A such that $N > K$. (For example, see [9, Lemma 3.5.7].)

Set $p = \sum_{i=1}^N p_i$. Then pBp has a matrix unit such that

$$pBp \cong M_N(p_1 B p_1).$$

Then, using [15, Theorem 6.1]

$$\begin{aligned}
\mathrm{tsr}(pBp) &= \mathrm{tsr}\big(M_N(p_1 B p_1)\big) \\
&= \left\{ \frac{\mathrm{tsr}(p_1 B p_1) - 1}{N} \right\} + 1 \\
&\leq \left\{ \frac{K}{N} \right\} + 1 = 2,
\end{aligned}$$

where $\{a\}$ denotes least integer greater than a. Since A is simple, p is a full projection in A, and moreover, in B. Hence from [2, Theorem 4.5] we have

$$\mathrm{tsr}(B) \leq \mathrm{tsr}(pBp) \leq 2. \qquad \blacksquare$$

COROLLARY 3.3. *Let* A *be a simple* C^**-algebra with* $\mathrm{tsr}(A) = 1$ *and having the property* SP*, and* α *an action of a finite group* G *on* $\mathrm{Aut}(A)$*. Then*

$$\mathrm{tsr}(A \rtimes_\alpha G) \leq 2.$$

PROOF. If A is finite dimensional, then so is $A \rtimes_\alpha G$, and $\mathrm{tsr}(A \rtimes_\alpha G) = 1$. Hence we may assume that A is infinite dimensional.

Since, from [12, Lemma 3.1], $A \subset A \rtimes_\alpha G$ is an inclusion of finite index and depth 2, the corollary follows from Theorem 3.2. $\blacksquare$

REMARK 3.4. If A has tracial topological rank zero, then tsr(A)=1 and A has the property SP. (For example see [8] and [9, Lemma 3.6.6 and Theorem 3.6.10].) Hence by Corollary 3.3, if A is a simple C^*-algebra of tracial topological rank zero and α is an action of a finite group G on Aut(A), then $\mathrm{tsr}(A \rtimes_\alpha G) \leq 2$.

REMARK 3.5. If a given C^*-algebra A has only the condition of $\mathrm{tsr}(A) = 1$, the estimate in Corollary 3.3 is best possible. Indeed, in [1, Example 8.2.1], Blackadar constructed an symmetry action α on CAR such that

$$(C[0,1] \otimes CAR) \rtimes_{id\otimes\alpha} Z_2 \cong C[0,1] \otimes B,$$

where B is the the tensor product of the Bunce–Deddens algebra of type 2^∞ and the CAR algebra. Then since $K_1(B)$ is non-trivial, we know that

$$\mathrm{tsr}(C[0,1] \otimes B) = 2.$$

(See also [10, Proposition 5.2].)

REFERENCES

1. B. Blackadar, *Symmetries of the CAR algebra.* Ann. of Math. (2) **131** (1990), 589–623.
2. ———, *The stable rank of full corners in C^*-algebras.* Proc. Amer. Math. Soc. **132** (2004), 2945–2950.
3. B. Blackadar, A. Kumjian, and M. Rørdam, *Approximately central matrix units and the structure of noncommutative tori.* K-Theory **6** (1992), 267–284.
4. L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero.* J. Funct. Anal. **99** (1991), 131–149.
5. F. M. Goodman, P. de la Harpe and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras.* Mathematical Sciences Research Institute Publications **14**, Springer–Verlag, New York, 1989.
6. M. Izumi, *Inclusions of simple C^*-algebras.* J. Reine Angew. Math. **547** (2002), 97–138.
7. J. A. Jeong, H. Osaka, T. Teruya and N. C. Phillips, *Cancellation of crossed product algebras.* In preparation.
8. A. Kumjian, *An involutive automorphism of the Bunce–Deddens algebra.* C. R. Math. Rep. Acad. Sci. Canada **10** (1988), 217–218.
9. H. Lin, *Introduction to the classification of amenable C^*-algebras.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
10. M. Nagisa, H. Osaka and N. C. Phillips, *Ranks of algebras of continuous C^*-algebra valued functions.* Canad. J. Math. **53** (2001), 979–1030.
11. H. Osaka and N. C. Phillips, *Crossed products of simple C^*-algebras with tracial rank one by actions with tracial Roklin property.* In preparation.
12. H. Osaka and T. Teruya, *Topological stable rank of inclusions of unital C^*-algebras*, Internat. J. Math. **17** (2006), 19–34.
13. N. C. Phillips, *Crossed products by finite cyclic group action with the tracial Rokhlin property.* Unpublished preprint, arXiv:math.OA/0306410.
14. I. Putnam, *The invertible elements are dense in the irrational rotation algebras.* J. Reine Angew. Math. **410** (1990), 160–166.
15. M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras.* Proc. London Math. Soc. **46** (1983), 301–333.

16. ______, *The homotopy groups of the unitary groups of non-commutative tori.* J. Operator Theory **17** (1987), 237–254.

17. A. S. Toms, *Cancellation does not imply stable rank one.* arXiv:math.OA/0509107.

18. Y. Watatani, *Index for C^*-algebras.* Mem. Amer. Math. Soc. **83**, no. 424, Amer. Math. Soc., Providence, RI, 1990.

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