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HILBERT MODULES OVER A C*-ALGEBRA OF STABLE RANK ONE

Dedicated to the memory of Israel Halperin

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Abstract. It is shown that for countably generated Hilbert C*-modules over a C*-algebra of stable rank one (*i.e.*, a C*-algebra in which the invertible elements are dense) the relation of compact inclusion up to isomorphism is cancellative, in a certain weak but natural sense. This generalizes the well-known fact that cancellation is valid in the abelian semigroup of isomorphism classes of finitely generated projective modules over such a C*-algebra.

Résumé. Il est démontré que la relation d'inclusion compacte entre modules de Hilbert dénombrablement engendrés sur une C*-algèbre de rang stable égal à un est cancellative, dans un sens faible mais naturel. Ceci généralise un résultat bien connu pour le cas des modules projectifs finiment engendrés.

1. Hilbert C*-modules—say countably generated ones to avoid cardinality questions—over a given C*-algebra form a very natural class of objects. (See *e.g.* [8]; see also [6], [5].) Just as for finitely generated projective modules over an arbitrary ring, the isomorphism classes of countably generated Hilbert A-modules for a given C*-algebra A form in a natural way an abelian semigroup with zero—which one may consider in addition to be ordered by inclusion. It was shown in [3] that this semigroup, in the case that A is stable and of stable rank one, coincides with an invariant introduced by Cuntz in [4], now often referred to as the Cuntz semigroup.

The Cuntz semigroup was shown recently in [10] to distinguish certain C*-algebras (of stable rank one) which otherwise had similar properties. It was also shown recently in [2] to provide sufficient information to determine isomorphism for certain other C*-algebras of stable rank one, most notably, AI algebras (*i.e.*, the C*-algebra inductive limits of sequences of direct sums of matrix algebras over the interval). In this result, a certain weak cancellative property (of the Cuntz semigroup of a C*-algebra of stable rank one) was used, which will now be established.

Recall that, in [3], the Cuntz semigroup was shown to have a considerably more subtle structure than just that of ordered semigroup. Namely, it was

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shown that suprema of countable increasing sequences always exist and the corresponding operation is compatible with addition, and it was shown furthermore that every element is the supremum of an increasing sequence of elements each compactly contained in the next, where x is compactly contained in y, written $x \ll y$, if $x \leq y$ and, whenever $y \leq \sup z_n$ for an increasing sequence (z_n), eventually $x \leq z_n$. (This additional structure is especially interesting as, with this structure—*i.e.*, considered as a functor from C*-algebras to the category based on this structure—as shown in [3] the Cuntz semigroup preserves inductive limits of sequences.)

Theorem. *Let A be a C*-algebra of stable rank one, and let a and b be elements of the Cuntz semigroup of A. Suppose that there exist elements c and d such that $a + c \leq d \ll d \leq b + c$. It follows that $a \ll b$.*

Proof. By Theorem 1 of [3] there exist rapidly increasing sequences (b_n) and (c_n) in $\mathcal{C}u\,A$ with $\sup b_n = b$ and $\sup c_n = c$. (Recall that by rapidly increasing is meant $b_n \ll b_{n+1}$ and $c_n \ll c_{n+1}$.) Then $\sup(b_n + c_n) = b + c$, and so by hypothesis, for some n we have $d \leq b_n + c_n$.

From now on, let n be fixed.

Let E, F, and G be countably generated Hilbert A-modules belonging to the classes a, b, and c in $\mathcal{C}u\,A$. By Theorems 1 and 3 of [3] there exist subobjects F_n and G_n of F and G belonging to the classes b_n and c_n which are compactly contained in F and G in the concrete sense, *i.e.*, are such that there exist self-adjoint compact endomorphisms f and g of F and G equal to the identity on F_n and G_n respectively. Also by Theorems 1 and 3 of [3], since $d \leq b + c$, there exists a subobject H of $F_n \oplus G_n$ belonging to the class d, and since $d \ll d$ (one may say, d is compact), necessarily H is compactly contained in itself in the concrete sense (one may perhaps again say, H is compact). In other words, the identity operator on H is a compact endomorphism; denote this by h. Again by Theorems 1 and 3 of [3], since $a + b \leq d$ there is a subobject of H isomorphic to $E \oplus G$; let us denote this by K.

Considering h as a compact endomorphism of $F \oplus G$ by the canonical extension ensured by Theorem 6 of [2], we have

$$(f + g)h = h.$$

(Indeed, recall that h as an endomorphism of H is a norm limit of finite rank endomorphisms of H, and this approximation is preserved by the canonical extension process; in the case that h is of finite rank—determined by finitely many elements of H—its extension clearly still has range in H, and this is therefore also true in general. The equation $(f + g)h = h$ then holds, since $f + g$ is the identity on $F_n \oplus G_n \supseteq H$.)

Writing the preceding equation as $hf + hg = h$, and recalling that the image of h is contained in H, note that the closure of the image of hg is a subobject of H isomorphic to a subobject of $0 \oplus G \subseteq F \oplus G$ (by means of the partially isometric part of hg). This subobject of H—let us denote it by H_0—is therefore isomorphic to a subobject of $0 \oplus G \subseteq E \oplus G = K \subseteq H$ (where we are choosing an identification of K as $E \oplus G$)—let us denote this by G_0—say by means of an isomorphism $v\colon H_0 \to G_0$. By Theorem 5 of [2] (or by Theorem 2 below), there exists an automorphism u of H which extends v approximately on any given finite subset of the domain H_0 of v. Since hg is compact, as a homomorphism from $F \oplus G$ to H_0, it can be approximated in norm by a finite-rank homomorphism between these modules, the image of which is then the submodule of H_0 generated by a finite subset. Choosing u such that its image on each of these finitely many elements of H_0 is close to an element of $G_0 \subseteq 0 \oplus G \subseteq H$ (namely, the image of this element under v), we may suppose that the adjoint of uhg, the compact endomorphism ghu^* of H, is small in norm on $E \oplus 0 \subseteq H$.

Since (see Theorem 5 of [2] and Theorem 2 below) u may be chosen to be an inner automorphism of H, *i.e.*, to belong to the C*-algebra of compact endomorphisms of H with unit adjoined, and in particular (by Theorem 6 of [2]) may be chosen to extend to $F \oplus G$, as an inner automorphism and in particular as an adjointable endomorphism, say denoted still by u, we have $fhu^* + ghu^* = hu^*$ on $F \oplus G$. Since hu^* as an endomorphism of $F \oplus G$ is isometric on $H \subseteq F \oplus G$ (because both h and u are, and take H into itself) and ghu^* as an endomorphism of $F \oplus G$ is small in norm on $E \oplus 0 \subseteq H$, it follows that the endomorphism fhu^* of $F \oplus G$ is bounded below on $E \oplus 0 \subseteq H$.

The restriction, say k, of fhu^* to $E \oplus 0 \subseteq H$ is therefore a homomorphism from E into F which is bounded below. It follows, even though k is presumably not adjointable (and so its partially isometric part is not an isomorphism), that E is isomorphic to the image of k, a subobject of F, as desired. Indeed, k is an A-linear topological vector space isomorphism of the countably generated Hilbert C*-module E onto the Hilbert C*-module kE, and so by Theorem 4.1 of [7], E and kE are isomorphic as Hilbert C*-modules (*i.e.*, as A-valued inner product spaces). (Alternatively, by the somewhat more elementary Corollary 8 of [2], E is isomorphic to a subobject of kE, which is all that is needed.) (By the same token, kE is isomorphic to a subobject of E, and since A has stable rank one it in fact then follows by Theorem 3 of [3] that kE and E are isomorphic!)

2. Theorem ([3], [2]). *Let A be a C*-algebra of stable rank one. Let F be a (right) Hilbert A-module, and let E_1 and E_2 be closed submodules of F, isomorphic as Hilbert A-modules. For any isomorphism u from E_1 to E_2 there exists an automorphism v of F such that the restriction $v|E_1$ of v to E_1 agrees arbitrarily closely with u in the topology of pointwise convergence in norm (for maps from E_1 to F). The automorphism v of F may be chosen to be inner, i.e., to belong to the C*-algebra of compact endomorphisms of F with unit adjoined.*

Proof. This result appears in the proof of Theorem 3 of [3] and is stated explicitly as Theorem 5 of [2]. (It is well known—see Proposition 6.5.1 of [1]—in the case that F and E_1 and E_2 are algebraically finitely generated and projective, *i.e.*, arise from projections in the stabilization, $A \otimes \mathcal{K}$, of A, and may be stated in that case as the cancellative property for the abelian semigroup of (algebraic) isomorphism classes of such modules—equivalently, as cancellation for Murray–von Neumann equivalence classes of projections in A—which implies this also for projections in the stabilization of A, since (by 3.3 of [9]) this C*-algebra still has stable rank one.)

(Strictly speaking, Theorem 5 of [2] has three parts, of which the present result constitutes only one—but as shown in [2] the other two parts follow immediately from this one. The main result of the present note uses only this form of the statement.)

Added August 25, 2007: Recently, Rørdam and Winter have (independently) obtained a considerably stronger result than Theorem 1 above: namely, in the setting of Theorem 1, the conclusion $a \ll b$ still holds if it is just assumed that $a + c \ll b + c$. (This fact, also conjectured by the author, is not obviously identical with the result of Rørdam and Winter—Proposition 4.3 of *The Jiang–Su algebra revisited* (in preparation)—but in view of the results of [3] is easily seen to be equivalent to it.)

This implication of course may fail if the C*-algebra is not assumed to have stable rank one. It would appear to be an interesting question whether the validity of this weak form of cancellation in the Cuntz semigroup of a given C*-algebra implies that the C*-algebra must have stable rank one.

This result, when used in place of the present Theorem 1, permits a slight strengthening of the main result of [2] (Theorems 4 and 11 of [2]), as is explained in an added note at the end of that article.

Leonel Robert has shown that this cancellation result can be deduced—by adjoining a unit to the C*-algebra—from the weaker version in Theorem 1.

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