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HYPERCENTRAL UNIT GROUPS AND THE HYPERBOLICITY OF A MODULAR GROUP ALGEBRA

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Abstract. We classify groups G such that the unit group $\mathcal{U}_1(\mathbb{Z}G)$ is hypercentral. In the second part, we classify groups G whose modular group algebras KG have hyperbolic unit group $\mathcal{U}_1(KG)$.

Résumé. Nous classifions les groupes G tels que le groupe unité $\mathcal{U}_1(\mathbb{Z}G)$ est hypercentral. Dans la deuxième partie, nous classifions les groupes G dont l'algèbre du groupe modulaire KG a un groupe unité $\mathcal{U}_1(KG)$ hyperbolique.

1. **Introduction.** We denote by $\Gamma = \mathcal{U}_1(\mathbb{Z}G)$ the group of units of augmentation one of the integral group ring $\mathbb{Z}G$ of G. Let

$$1 = \mathcal{Z}_0(\Gamma) \leq \mathcal{Z}_1(\Gamma) \leq \cdots \leq \mathcal{Z}_k(\Gamma) \leq \cdots$$

denote the upper central series of Γ. $\mathcal{Z}_n(\Gamma)$ will denote the n-th centre of Γ and we define $\mathcal{Z}_\infty(\Gamma) = \bigcup_{n\in\mathbb{N}} \mathcal{Z}_n(\Gamma)$. An element in $\mathcal{Z}_\infty(\Gamma)$ is called a hypercentral unit.

Hypercentral units has been an object of intensive study in recent years. See, for example, [1], [2], [6], [7], [8], [10], [11], [12], [13].

In [14], Polcino Milies classified finite groups such that the unit group of an integral group ring is nilpotent. The problem of classifying those G for which $\mathcal{U}_1(\mathbb{Z}G)$ is hypercentral was posed by several leading experts in the field. We completely solve this problem as a natural consequence of our research about the hypercentral units of an integral group ring done in [8]. Some of the results done in [8], including those needed for the result just mentioned, will appear in [7].

Hyperbolic groups have been a subject of intensive study in recent years. In Section 3, we deal with the topic of hyperbolic unit groups. In the context of hyperbolic unit groups, Juriaans, Passi and Prasad [9] studied the groups $\mathcal{G}$ whose unit group $\mathcal{U}(\mathbb{Z}\mathcal{G})$ is hyperbolic and classified the polycyclic-by-finite subgroups of $\mathcal{U}(\mathbb{Z}\mathcal{G})$.

We classify the groups G for which the group of units with augmentation one of a modular group algebra, $\mathcal{U}_1(KG)$, is hyperbolic.

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2. Groups with Hypercentral Unit Group. Unless otherwise stated explicitly, G will always denote an arbitrary group G.

Firstly we recall a result proved in [8] which we will need in our investigations. This result will also appear in [7].

LEMMA 2.1. *Let $u \in \mathcal{Z}_n(\Gamma)$ and v an element of finite order in Γ. If $c = [u, v] \neq 1$ then $u^{-1}vu = v^{-1}$, $v^2 \in G \cap \mathcal{Z}_{n+1}(\Gamma) \subseteq \mathcal{Z}_{n+1}(G)$, $o(v) = 2^m$, $m \leq n$, $v^{2^{n-1}}$ is central, and if $n = 2$, then $m = 2$. In particular, elements of Γ that are of finite order and whose order is not a power of 2 commute with $\mathcal{Z}_\infty(\Gamma)$, and $\mathcal{Z}^2_\infty(\Gamma) \subseteq \mathcal{C}_\Gamma\big(T(G)\big)$, where $\mathcal{C}_\Gamma\big(T(G)\big)$ denotes the centralizer of $T(G)$ in Γ. Here $T(G)$ denotes the set of torsion elements of G.*

We need the following result proved in the context of nilpotent unit groups by Sehgal–Zassenhaus [19].

LEMMA 2.2. *Suppose that Γ is hypercentral and let $t, t_1, t_2 \in T = T(G)$, $g \in G$.*

(a) *Every finite subgroup of G is normal in G.*
(b) *If $g^{-1}tg \neq t$ then $g^{-1}tg = t^{-1}$.*
(c) *If t has odd order then $gt = tg$.*
(d) *If $1 \neq t_1$ has odd order, t_2 has even order then T is a central subgroup of G.*

As a consequence of the previous results, we now state the main result in this section.

THEOREM 2.3. *$\Gamma = \mathcal{U}_1(\mathbb{Z}G)$ is hypercentral if and only if G is hypercentral and the torsion subgroup T of G satisfies one of the following conditions:*

(a) *T is central in G;*
(b) *T is an abelian $2-$group and for $g \in G$, $t \in T$*

$$g^{-1}tg = t^{\delta(g)}, \quad \delta(g) = \pm 1;$$

(c) *$T = K_8 \times E_2$, where K_8 denotes the quaternion group of order 8 and E_2 is an elementary abelian 2-group. Moreover, E_2 is central, and conjugation by $g \in G$ induces one of the four inner automorphisms on K_8.*

3. Modular group algebras with hyperbolic group ring units. Let $\mathbb{Z}^2$ denote the free abelian group of rank two. Let p be a rational prime, and let $GF(p^n)$ denote the Galois field with p^n elements. Let $\mathrm{tr.deg}(K)$ denote the transcendence degree of the field K over $GF(p)$, and $\mathcal{U}_1(KG)$ the group of units of KG with augmentation one.

LEMMA 3.1. *Let G be an arbitrary group, K a field with $\mathrm{char}(K) = p > 0$ and $\mathrm{tr.deg}(K) \geq 1$. Suppose that g_0 is a torsion element of G and $p \nmid o(g)$. Then $\mathbb{Z}^2$ embeds in $\mathcal{U}_1(KG)$, and consequently, $\mathcal{U}_1(KG)$ is not hyperbolic.*

In what follows we investigate under what conditions the group of units of a modular group algebra of a finite (non-trivial) group G is hyperbolic.

We denote by $\mathcal{J}(KG)$ the Jacobson Radical of KG. Let $\omega(G)$ represent the augmentation ideal of KG.

LEMMA 3.2. *Suppose that G is a finite (non-trivial) group and K is a field with* $\operatorname{char}(K) = p > 0$ *and* $\operatorname{tr.deg}(K) \geq 1$. *Then $\mathcal{U}_1(KG)$ is not hyperbolic.*

THEOREM 3.3. *Let G be a finite (non-trivial) group and K a field with* $\operatorname{char}(K) = p > 0$. *Under these conditions, $\mathcal{U}_1(KG)$ is hyperbolic if and only if K is finite.*

We state the main result of this section.

THEOREM 3.4. *Let G be an arbitrary group with torsion and K a field with* $\operatorname{char}(K) = p > 0$. *If $\mathcal{U}_1(KG)$ is hyperbolic then K is algebraic over $GF(p)$.*

Our next theorem considers the case in which G is an arbitrary (non-trivial) group and K a field of $\operatorname{char}(K) = p > 0$, under the hypothesis that $\mathcal{U}(KG)$ is hyperbolic.

THEOREM 3.5. *Let G be an arbitrary (non-trivial) group and K a field of* $\operatorname{char}(K) = p > 0$. *If $\mathcal{U}(KG)$ is hyperbolic then K is finite.*

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