C. R. Math. Rep. Acad. Sci. Canada Vol. **29** (3) 2007, pp. 81–86

ON THE PROPERTY SP OF CERTAIN AH ALGEBRAS

TOAN M. HO

Presented by George Elliott, FRSC

Abstract. A certain non-zero projection in a simple AH algebra with diagonal morphisms between the building blocks in its inductive limit decomposition is constructed and used to prove that this algebra has the property SP.

Résumé. On construit une projection convenable dans une certaine algèbre AH simple, et on l'utilise pour montrer que cette algèbre a la propriété SP.

1. **Introduction.** Let X and Y be compact Hausdorff spaces. A $*$-homomorphism $\phi\colon M_m(C(X)) \to M_{nm}(C(Y))$ is called *diagonal* if there are n continuous maps from Y to X, namely $\lambda_1, \lambda_2, \dots, \lambda_n$, such that

$$\phi(f) = \begin{pmatrix} f\circ\lambda_1 & 0 & \cdots & 0 \\ 0 & f\circ\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f\circ\lambda_n \end{pmatrix}$$

for all $f \in M_m(C(X))$. The family of maps $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is called the *eigenvalue pattern* of ϕ and is denoted by $\mathrm{ep}(\phi)$. This definition can be extended to $*$-homomorphisms

$$\phi\colon \bigoplus_{i=1}^{n} M_{n_i}(C(X_i)) \longrightarrow \bigoplus_{j=1}^{m} M_{m_j}(C(Y_j))$$

by requiring that each partial map $\phi^{ij}\colon M_{n_i}(C(X_i)) \to M_{m_j}(C(Y_j))$ induced by ϕ be diagonal.

In this paper, we will study some properties of the class of simple C*-algebras obtained as limits of inductive systems (A_i, ϕ_i) where $A_i = \bigoplus_{t=1}^{k_i} M_{n_{it}}(C(X_{it}))$, and each ϕ_i is diagonal. We will also assume that all the base spaces X_{it} are connected, compact Hausdorff spaces. This class contains some interesting algebras such as the examples of Jesper Villadsen [6] and Toms [5]. These examples have been not classified by the Elliott invariant so far, even though the classification program of Elliott, the goal of which is to classify amenable C^*-algebras

Received by the editors on August 2, 2007.
AMS Subject Classification: 46L99.
Keywords: inductive limit, property SP, AH algebra.

by their K-theoretical data, has been successful for many classes of C^*-algebras, in particular for simple AH algebras with (very) slow dimension growth.

A property which is weaker than real rank zero is the property SP. This property has been studied by Rørdam [4], Huaxin Lin [3], and others. Rørdam [4] constructed a simple AH algebra with the property SP, but real rank not equal to zero. However, one can see that many simple AH algebras with the property SP do not have real rank zero by using either Theorem 3.2 or Theorem 3.5 below. In this paper, we will construct a non-zero projection in an algebra in the class under consideration and use it to prove Theorem 3.2. The existence of a projection in a simple AH algebra with slow dimension growth but no assumption of the maps between the building blocks in the limit, which was constructed in [2], can be also used to prove that such an algebra has the property SP (Theorem 3.5).

2. A Rank One Projection.

Let X be a connected, compact Hausdorff space throughout.

Lemma 2.1. *Let h be a self-adjoint, diagonal element of $M_n(C(X))$. If h is nowhere zero, then there exists a rank one projection p in $M_n(C(X))$ such that the range of $p(x)$ is a subspace of the range of $h(x)$ for every x in X. In particular, if $a \in M_n(C(X))$ and $ah = 0 = ha$, then $ap = 0 = pa$.*

Proof. Suppose that

$$h = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n \end{pmatrix},$$

where $f_1, f_2, \ldots, f_n$ are real-valued, continuous functions on X with

$$\sum_{i=1}^{n} f_i^2(x) > 0$$

for all $x \in X$. Let $v(x)$ denote the vector $(f_1(x), f_2(x), \ldots, f_n(x))^T$ for every x in X. Let $p(x)$ denote the projection on $\mathrm{Span}(v(x))$ (in $\mathbb{C}^n$) for every x in X. That is, $p = \frac{1}{|v|^2} vv^T$. Then p is a projection in $M_n(C(X))$ and satisfies the requirements. $\square$

If $a \in M_n(C(X))$ is invertible, then so is $a(x)$ (*i.e.*, $\det(a(x)) \neq 0$, for every $x \in X$). The converse is still true.

Lemma 2.2. *Let a be a non-invertible element of $M_n(C(X))$ and ϵ be any positive number. There exist a non-empty open subset U of X, an element b in $M_n(C(X))$, and permutation matrices u, v in M_n, such that all the functions in the first row and the first column of b vanish on U and $\|uav - b\| < \epsilon$. Furthermore, if a is positive, then b can be chosen to be positive and v to be equal to u^*.*

PROOF. Since a is not invertible, there is a point, say $x_0 \in X$, such that $\det(a(x_0)) = 0$. There are permutation matrices u and v in M_n such that the first row and the first column of $ua(x_0)v$ are zero (if $a(x_0)$ is self-adjoint, then we may choose $v = u^*$). Let us denote by (a_{ij}) the matrix uav in $M_n(C(X))$. Set

$$U = \bigcap_{i=1}^{n} a_{1i}^{-1}(J) \cap a_{i1}^{-1}(J), \text{ where } J = \Big(\frac{-\epsilon}{4n}, \frac{-\epsilon}{4n}\Big).$$

Using Urysohn's lemma, choose a continuous function, denoted by δ, from X to $[0,1]$ which vanishes on $\overline{U}$ and is equal to 1 on the complement of

$$\bigcap_{i=1}^{n} a_{1i}^{-1}\Big(\frac{-\epsilon}{2n}, \frac{-\epsilon}{2n}\Big) \cap a_{i1}^{-1}\Big(\frac{-\epsilon}{2n}, \frac{-\epsilon}{2n}\Big).$$

Denote by b_{ij} the function δa_{ij} whenever either $i = 1$ or $j = 1$, and a_{ij} otherwise. The matrix $b = (b_{ij})$ is in $M_n(C(X))$, and satisfies the requirements. □

LEMMA 2.3. *Let ϵ be a positive number and A be a simple inductive limit of $A_i = M_{n_i}(C(X_i))$ with injective and diagonal morphisms $\phi_i \colon A_i \to A_{i+1}$. Suppose that a is a non-invertible element of A_i for some positive integer i. Then there exist permutation matrices u, v in M_{n_i}, an integer $j \geq i$, an element b of A_j and a rank one projection p in A_j, such that $\|\phi_{ij}(uav) - b\| < \epsilon$, and $bp = 0 = pb$. Furthermore, if a is positive, b may also be chosen to be positive and v to be equal to u^*.*

PROOF. By Lemma 2.2, there exist a non-empty open subset U of X_i, an element b' in A_i and permutation matrices u, v in M_{n_i} (with $u^* = v$ and b' positive if a is positive) such that $\|uav - b'\| < \epsilon$, and $b'(x)e_{11} = 0 = e_{11}b'(x)$, $\forall x \in U$, where $(e_{ij}, 1 \leq i, j \leq n)$ is the canonical system of matrix units of M_{n_i}. Take a point $x_0 \in U$ and a continuous function f from X_i to $[0,1]$ such that $f(x_0) = 1$ and f vanishes outside U. Set $h' = e_{11} \otimes f$. Then h' is a non-zero, self-adjoint, diagonal element of A_i and $h'b' = 0 = b'h'$. Since A is simple, by [2, Proposition 2.1], there is an integer $j \geq i$ such that $h = \phi_{ij}(h')$ is nowhere zero. Let b be $\phi_{ij}(b')$. Then $\|b - \phi_{ij}(uav)\| < \epsilon$ and $bh = 0 = hb$. By Lemma 2.1, there exists a rank one projection p in A_j such that $bp = 0 = pb$. □

3. Application.

The following is an application of the existence of rank one projections discussed in Section 2.

The Property SP. We begin this section by recalling the definition of the property SP.

DEFINITION 3.1. A C^*-algebra A is said to have the property SP if every non-zero hereditary C^*-subalgebra has a non-zero projection.

It is well known that every hereditary subalgebra of a C^*-algebra with real rank zero contains an approximate unit consisting of projections. In particular, such a C^*-algebra has the property SP. The converse does not hold. Counterexamples can be obtained from either Theorem 3.5 or Theorem 3.2.

THEOREM 3.2. *Let $A = \varinjlim(A_i, \phi_i)$ be a simple C^*-algebra, where*

$$A_i = \bigoplus_{i=1}^{k_i} C(X_{it}) \otimes M_{n_{it}},\ i = 1, 2, 3, \ldots .$$

If the morphism from A_i to A_{i+1} in the inductive limit is diagonal for every i, then A has the property SP.

To prove Theorem 3.2, we need the following lemma.

LEMMA 3.3. *Let $B = \varinjlim(B_i, \phi_i)$ be the inductive limit of a sequence of separable C^*-algebras. Suppose that for any positive element a in B_i, there is a positive integer $j \geq i$ and a projection p in B_j such that the image of p in B (denoted by $\phi_{j\infty}(p)$) is non-zero and $\|\phi_{ij}(a)p - p\| < 1/8$. Then B has the property SP.*

PROOF. Let H be a non-zero hereditary subalgebra of B. Since B is separable, there is a non-zero positive element x in B which generates H. We want to show that $H = \mathrm{Her}(x)$ contains a non-zero projection. Since $\mathrm{Her}(x)$ is the same as $\mathrm{Her}(x/\|x\|)$, we may assume that $\|x\| = 1$. There is an a in B_i, for some i, such that $\|x - \phi_{i\infty}(a)\| < \frac{1}{8}$. By the hypothesis, there is an integer $j \geq i$ and a projection p in B_j such that $\phi_{j\infty}(p)$ is non-zero and $\|\phi_{ij}(a)p - p\| < 1/8$. Hence, $\|x\phi_{j\infty}(p) - \phi_{j\infty}(p)\| < 1/4$. This implies that

$$\|x\phi_{j\infty}(p)x - \phi_{j\infty}(p)\| < \|\phi_{j\infty}(p) - x\phi_{j\infty}(p)\| + \|x\phi_{j\infty}(p) - x\phi_{j\infty}(p)x\| < \tfrac{1}{4}.$$

So $\frac{1}{2}$ is not in, and is less than the supremum of, the spectrum of $x\phi_{j\infty}(p)x$. Therefore, $\chi_{[\frac{1}{2},\infty]}(x\phi_{j\infty}(p)x)$, the spectrum projection of $x\phi_{j\infty}(p)x$ on $[\frac{1}{2}, \infty]$, is a non-zero projection in A. Since, $x\phi_{j\infty}(p)x \in \mathrm{Her}(x)$, this projection is in $\mathrm{Her}(x)$. □

We also need the following fact.

PROPOSITION 3.4. *Let ϵ be any positive number and a, b be two positive elements of C^*-algebras A, B, respectively. If a non-zero projection p in A satisfies $\|ap - p\| < \epsilon$, then $\|(a + b)p - p\| < \epsilon$ in $A \oplus B$.*

PROOF. The remark is obvious, since $pb = 0$. □

PROOF OF THEOREM 3.2. We may assume that all the maps ϕ_i are injective. Let a be a non-zero positive element of A_i and ϵ be any positive number. We want to find a non-zero projection as in the statement of Lemma 3.3. By

Proposition 3.4, we may assume that A_j has only one summand for every $j \geq i$, that is, $A_j = M_{n_j}(C(X_j))$. With the notation $a' = \frac{a}{\|a\|}$, $1 - a'$ is positive and not invertible in A_i. By Lemma 2.3, there exist a permutation matrix u in M_{n_i}, an integer $j \geq i$, an element b in A_j and a rank one projection p in A_j such that $\|\phi_{ij}(u(1-a')u^*) - b\| < \epsilon$, and $bp = 0 = pb$. Hence, $\|p\phi_{ij}(u(1-a')u^*)\| < \epsilon$, and so $\|p(1 - \phi_{ij}(ua'u^*))\| < \epsilon$. Replacing a' by $\frac{a}{\|a\|}$, we obtain

$$\|q\phi_{ij}(a) - q\| < \|a\|\epsilon,$$

where $q = \phi_{ij}(u^*)p\phi_{ij}(u)$. By Lemma 3.3, A has the property SP if we choose $\epsilon < \frac{1}{8||a||}$, which is applicable. □

Without the assumption of diagonal morphisms in some inductive limit decomposition of the algebra, we might need the assumption of dimension growth of these algebras as the following theorem.

Theorem 3.5. *If B is a simple AH algebra with slow dimension growth (the $*$-homomorphisms between the building blocks in its inductive limit decompositions need not be diagonal), then B has the property SP.*

Notation 3.6. Let us recall the following notation from [1]. If B is a C^*-algebra and a, b are positive elements of B, then we write $a \prec b$ if there is a positive element $c \in B$ such that $ac = a$ and $bc = c$. In particular, if this relation holds, then $ab = a$.

Proof of Theorem 3.5. Let a be any non-zero positive element of B_i. Then we can find a non-zero positive element h in B_i such that $h \prec a$. By [1, Lemma F], there is an integer $j \geq i$ such that $\operatorname{rank} \phi_{ij}^t(h)(x) \geq \dim X_{jt} + 1$, for every $t = 1, 2, \ldots, k_j$ and all $x \in X_{jt}$. By [1, Lemma C], there is a projection p in B_j such that $p \prec \phi_{ij}(h)$ and $\dim p \geq \operatorname{rank} \phi_{ij}^t(h)(x) - \frac{1}{2}(\dim X_{jt} + 1) > 0$. Thus, p is non-zero and $p\phi_{ij}(a) = p$. By Lemma 3.3, B has the property SP. □

Note that in the proof of Theorem 3.5, we use the existence of the projection which was constructed in [1] and the construction depends on the dimension of the spectra of the building blocks in the inductive limit decomposition of a simple AH algebra, while it is not the case in our construction of the rank one projection in Section 2 and that is why we have no assumption of the dimensions in the statement of Theorem 3.2.

Acknowledgements. The author would like to thank Professor George Elliott for his suggestion of this problem and for many discussions. Special thanks are due to Professor Andrew Toms for his comments and reading the script. Thanks are also due to Dr. Zhuang Niu, Dr. Ivanescu Cristian and Dr. Hanfeng Li for many helpful discussions.

References

1. B. Blackadar, M. Dădărlat, and M. Rørdam, *The real rank of inductive limit C^*-algebras.* Math. Scand. **69** (1991), no. 2, 211–216.
111–126.

2. M. Dădărlat, G. Nagy, A. Nemethi, and C. Pasnicu, *Reduction of topological stable rank in inductive limits of C^*-algebras.* Pacific J. Math. **153** (1992), no. 2, 267–276.

3. H. Lin, *An Introduction to the Classification of Amenable C^*-Algebras.*, World Scientific Publishing, River Edge, NJ, 2001.

4. M. Rørdam, *The real rank of certain simple C^*-algebras.* In: Advances in Operator Algebras and Mathematical Physics. Theta Ser. Adv. Math. 5, Theta, Bucharest, 2005, pp. 197–206.

5. A. Toms, *An infinite family of non-isomorphic C^*-algebras with identical K-theory.* To appear in Trans. Amer. Math. Soc.

6. J. Villadsen, *Simple C^*-algebras with perforation.* J. Funct. Anal. **154** (1998), no. 1, 110–116.

Department of Mathematics and Statistics
York University
Toronto, ON
M3J 1P3
e-mail: toan@mathstat.yorku.ca