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THE RANGE OF THE ORBIT OPERATOR AND INVARIANT SUBSPACES

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ABSTRACT. To a bounded linear operator and a vector in the Hilbert space on which it acts we associate a linear map which we call the orbit operator. We prove a number of results linking properties of the range of the orbit operator to the existence of invariant subspaces of the original operator.

RÉSUMÉ. On associe à un opérateur T et un vecteur x dans un espace de Hilbert, un opérateur "d'orbite" $\mathcal{O}_T^{e_i}(x)$, et on démontre des résultats reliant les propriétés de l'image de $O_T^{e_i}(x)$ et des sous-espaces invariants de T.

1. **Introduction and notation.** The invariant subspace problem is the long-standing question of whether every operator on a complex Hilbert space of dimension greater than one has a non-trivial invariant subspace. This question has been answered in the affirmative for the finite dimensional case, the non-separable case and for many classes of operators (*e.g.* compact, normal, etc).

In order to study the invariant subspaces of an operator, T, we will introduce an operator called the orbit operator (denoted by $O_T^{e_i}(x)$). This operator is closely related to an operator studied by Caradus in [1]. We will prove, among other things, that if the spectral radius of T is strictly less than one and $O_T^{e_i}(x)$ does not have dense range, then T has an invariant subspace. This result was announced in [1], but the proof there is incomplete.

We will let $\mathcal{H}$ denote an infinite dimensional, separable, complex Hilbert space with inner product $\langle \cdot , \cdot \rangle$, and T a bounded linear operator acting on it. A vector, $x \in \mathcal{H}$, is said to be T-*cyclic* if $\text{span}\{T^n x\}_{n\geq 0}$ is dense in $\mathcal{H}$, and a subspace $\mathcal{M}$ in $\mathcal{H}$ is an *invariant subspace* if $Tx \in \mathcal{M}$ for all $x \in \mathcal{M}$. The trivial invariant subspaces are $\{0\}$ and $\mathcal{H}$. It is easy to check that T has only the trivial invariant subspaces if and only if each nonzero vector is a cyclic vector of T. We will let $\sigma(T)$ and $r(T)$ denote the *spectrum* and *spectral radius* of T respectively (*i.e.*, $\sigma(T) = \{\lambda \in \mathbb{C} \mid (\lambda I - T) \text{ is not invertible}\}$ and $r(T) = \sup_{\lambda\in\sigma(T)} |\lambda|$). We will follow the notation in [4] when dealing with Hardy spaces and [5] when dealing with contractions. For example, as in [5], if T is a contraction, then we will denote $(I - T^*T)^{\frac{1}{2}}$ by D_T. Also, recall that C_0-*contractions* are contractions

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for which there exists nonzero $\phi \in H^\infty$ such that $\phi(T) = 0$. A contraction T is called a C_{10}-*contraction* when for all $x \neq 0$ the sequence $T^n x$ does not tend to zero in norm and for all x the sequence $(T^*)^n x$ does tend to zero in norm. Throughout, S will denote the unilateral shift.

2. Main results.

Definition 1. Let $\{e_i\}_{i=0}^\infty$ be an orthonormal basis of $\mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$, $x \in \mathcal{H}$ and $D_x = \{y \in \mathcal{H} \mid \sum_n |\langle y, T^n x\rangle|^2 < \infty\}$. We then define a map $O_T^{e_i}(x)\colon D_x \to \mathcal{H}$ via

$$y \longmapsto \sum_{i=0}^{\infty} \langle y, T^i x\rangle e_i$$

We call $O_T^{e_i}(x)$ the orbit operator of T at x.

Remarks on Definition 1. If we assume that $r(T) < 1$, then for each $x \in \mathcal{H}$ the domain of $O_T^{e_i}(x)$ is all of $\mathcal{H}$ (*i.e.*, $D_x = \mathcal{H}$ for all $x \in \mathcal{H}$). Moreover, it is not difficult to show that, in this case, $O_T^{e_i}(x)$ is trace class for each $x \in \mathcal{H}$. An important property of this construction is that

$$\ker\big(O_T^{e_i}(x)\big) = (\overline{\mathrm{span}}\{x, Tx, T^2x, \dots\})^\perp.$$

Hence we have the following proposition.

Proposition 2. *A vector x is T-cyclic if and only if* $\ker\big(O_T^{e_i}(x)\big) = \{0\}$.

$O_T^{e_i}(x)$ is conjugate linear when considered as a map from $\mathcal{H}$ to (possibly unbounded) linear maps on $\mathcal{H}$. That is, $O_T^{e_i}(\lambda x + y) = \bar{\lambda} O_T^{e_i}(x) + O_T^{e_i}(y)$ for all $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. (One should be careful with the domains in the case when these operators are unbounded). This result, along with the fact that $S^* O_T^{e_i}(x) = O_T^{e_i}(Tx) = O_T^{e_i}(x) T^*$, leads to the next proposition.

Notation 3. Given a polynomial, $p(t) = \sum_{j=0}^n a_j t^j$, we will let $\bar{p}(t) = \sum_{j=0}^n \bar{a}_j t^j$.

Proposition 4. *Let p be a polynomial and S be the unilateral shift on the basis $\{e_i\}$. Then*

$$O_T^{e_i}\big(p(T)x\big) = \bar{p}(S^*) O_T^{e_i}(x) = O_T^{e_i}(x) \bar{p}(T^*)$$

Example 5. If $T = S$, the unilateral shift on the basis $\{e_i\}_{i \geq 0}$, then $O_S^{e_i}(x)$ is given by a Toeplitz matrix. In particular, $O_S^{e_i}(x)$ is a compact operator if and only if $x = 0$. In general, if T is a C_{10} contraction, then $O_T^{e_i}(x)$ is a compact operator if and only if $x = 0$. This follows since $T^k x \to 0$ weakly, but if T is C_{10} and $x \neq 0$, then $O_T^{e_i}(x) T^k x$ does not converge to zero in norm.

EXAMPLE 6. If $T = S^*$, the adjoint of the unilateral shift on the basis $\{e_i\}_{i\geq 0}$, then $O_{S^*}^{e_i}(x)$ is given by a Hankel matrix. A result of Kronecker (see [3]) implies that $O_{S^*}^{e_i}(x)$ is a finite rank operator if and only if $(\langle e_i, x\rangle)_{i\geq 0}$ are the coefficients of a rational function.

To prove the next theorem, we will use the following proposition which can be found in [3]. In this proposition, S^* is the backward shift on the Hardy space H^2. As remarked in [3], the assumption on f in this proposition can be restated as the exponential decay of the Taylor coefficients of f.

PROPOSITION 7. *If f is holomorphic in $|z| < R$ for some $R > 1$, then f is either S^*-cyclic or is a rational function and hence S^*-noncyclic.*

THEOREM 8. *If $T \in \mathcal{B}(\mathcal{H})$ with $r(T) < 1$ and there exists $x \in \mathcal{H}$ such that the range of $O_T^{e_i}(x)$ contains a nonzero non-cyclic vector of the backward shift on $\{e_i\}$, then T has a non-trivial invariant subspace.*

PROOF. We may take $\mathcal{H} = H^2$ and $\{e_i\} = \{z^i\}$. To begin, the reader may verify that the condition $r(T) < 1$ implies that the Taylor coefficients of any element of the range of $O_T^{z^i}(x)$ decay exponentially. Assume T has no non-trivial invariant subspaces and that there exists nonzero x and y such that $O_T^{z^i}(x)(y)$ is non-cyclic for S^*. Applying Proposition 7, we conclude that $O_T^{z^i}(x)(y)$ is a rational function and hence that $O_{S^*}^{z^i}\big(O_T^{z^i}(x)(y)\big)$ is a finite rank operator (see Example 6). A calculation leads to

$$O_{S^*}^{z^i}\big(O_T^{z^i}(x)(y)\big) = O_{T^*}^{z^i}(y)O_T^{z^i}(x)^*.$$

Since T has no non-trivial invariant subspaces, T^* also does not. By Proposition 2, $\ker\big(O_{T^*}^{z^i}(y)\big)$ is trivial leading us to conclude that $O_T^{z^i}(x)^*$ is finite rank. Thus, $O_T^{z^i}(x)$ is also finite rank and hence has non-trivial kernel. This contradicts Proposition 2. □

COROLLARY 9. *If $T \in \mathcal{B}(\mathcal{H})$ with $r(T) < 1$ and there exists nonzero $x \in \mathcal{H}$ such that the range of $O_T^{e_i}(x)$ is not dense, then T has a non-trivial invariant subspace.*

PROOF. The fact that the range of $O_T^{e_i}(x)$ is not dense implies that for any $y \in \mathcal{H}$, the set of vectors

$$\big\{O_T^{e_i}(x)\big(p(T^*)y\big) :\ p \text{ is a polynomial}\big\} = \{p(S^*)O_T^{e_i}(x)y :\ p \text{ is a polynomial}\}$$

is also not dense. Hence $O_T^{e_i}(x)(y)$ is a non-cyclic vector of S^*. Theorem 8 then implies the result. □

Remarks on Corollary 9. We can prove Theorem 8 from Corollary 9 by assuming T has no non-trivial invariant subspaces and, for nonzero vectors x and y, considering the equation

$$O^{e_i}_{S^*}\big(O^{e_i}_T(x)(y)\big) = O^{e_i}_{T^*}(y)O^{e_i}_T(x)^*.$$

Based on the corollary, $O^{e_i}_T(x)^*$ has trivial kernel. $O^{e_i}_{T^*}(y)$ also has trivial kernel by the assumption that T has no non-trivial invariant subspaces. Hence, $O^{e_i}_{S^*}\big(O^{e_i}_T(x)(y)\big)$ has trivial kernel, from which the result follows by Proposition 2. This is an instance of a more general result linking the notions of cyclic vectors and quasi-affine transformations.

In [1], it is remarked that if T is a C_0-contraction, then the range of the orbit operator is not dense. Hence, Corollary 9 implies the existence of invariant subspaces for C_0-contractions with spectral radius strictly less than one. It is well known that all C_0-contractions have non-trivial invariant subspaces (see [5, p. 133]). If we assume this result, then we can extend the above results to the case when $\|T\| \leq 1$. First, we need a few facts since, in general, $O^{e_i}_T(x)$ need not be bounded for such an operator T.

These facts are if $\|T\| \leq 1$, then $O^{e_i}_T(x)$ is a closed operator, and if T is a completely non-unitary contraction, then it is densely defined. The first can be proved directly, while the second fact follows from the next proposition whose proof is left as an exercise.

Proposition 10. *Let $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq 1$. For each $k \in \mathbb{N}$, both $O^{e_i}_T(x)D_{T^k}$ and $O^{e_i}_T(x)D_{(T^*)^k}$ are bounded.*

Theorem 11. *Let $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq 1$. If there exists nonzero $x \in \mathcal{H}$ such that the range of $O^{e_i}_T(x)$ contains a nonzero non-cyclic vector of the backward shift on $\{e_i\}$, then T has a non-trivial invariant subspace. In particular, if there exists nonzero $x \in \mathcal{H}$ such that the range of $O^{e_i}_T(x)$ is not dense, then T has a non-trivial invariant subspace.*

Proof. Assume that T has no non-trivial invariant subspaces. By standard arguments, we have that T is completely non-unitary. Let x be a nonzero vector. By Proposition 2, the kernel of $O^{e_i}_T(x)$ is trivial. Let $y_0 \in D_x$ (D_x is the domain of $O^{e_i}_T(x)$) be a nonzero vector. We will show $z = O^{e_i}_T(x)(y_0)$ is cyclic under S^* from which the result will follow. Suppose it is not cyclic. That is, there exists $f \in H^\infty$ with Fourier coefficients $(a_n)_{n\geq 0}$ such that, if we let $a = \sum_n a_n e_n$, then

$$\langle a, (S^*)^n z\rangle = 0 \text{ for each } n$$

(see [3, p. 43, Remark 2.2.2]). Equivalently, $O^{e_i}_S(a)(z) = 0$. It can be checked that $O^{e_i}_S(a) = \bar{f}(S^*)$ (where $\bar{f}$ is defined as in Notation 3 and $\bar{f}(S^*)$ is defined using the H^∞ functional calculus). Using properties of the H^∞ functional calculus and Proposition 4, we have that

$$0 = \bar{f}(S^*)z = \bar{f}(S^*)O^{e_i}_T(x)(y_0) = O^{e_i}_T\big(f(T)x\big)(y_0).$$

Hence, $O_T^{e_i}\big(f(T)x\big)$ has non-trivial kernel. By Proposition 2, $f(T)x = 0$, so that $O_T^{e_i}\big(f(T)x\big)(y) = 0$ for all y. Thus, for all $y \in D_x$,

$$0 = \bar{f}(S^*)O_T^{e_i}(x)(y) = O_T^{e_i}(x)\big(\bar{f}(T^*)y\big).$$

Since $O_T^{e_i}(x)$ has trivial kernel, this implies that $\bar{f}(T^*) = 0$ (since the domain of $O_T^{e_i}(x)$ is dense). It follows that T^* is a C_0-contraction, which is a contradiction. The proof of the second statement is the same as the proof given for Corollary 9. □

Another interesting property of the orbit operator is its connection with Rota's theorem (see [6, p. 54]). In the standard proof of Rota's theorem, the Hilbert space $\bigoplus_{i=1}^{\infty} \mathcal{H}$ is considered. This Hilbert space can be naturally identified with the Hilbert space of Hilbert–Schmidt operators. If we apply this identification in the proof of Rota's theorem, it is easy to check that the operator we obtain is the adjoint of the orbit operator. Rota's theorem can then be inferred from the property that $O_T^{e_i}(Tx)^* = O_T^{e_i}(x)^*S$. Other results closely related to Rota's can also be viewed in terms of the orbit operator. For example, in Theorem 3.29 of [6], the operator in question is $D_T O_T^{e_i}(x)^*$. This paper is based on the author's Master's thesis [2].

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