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CAUCHY TYPE INTEGRALS AND A D-MOMENT PROBLEM

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ABSTRACT. We consider a Cauchy-type integral $F(z) = \int_\Gamma \frac{g(\xi)\,\mathrm{d}\xi}{\xi - z}$, where g is a piecewise analytic function satisfying an n-th order linear homogeneous differential equation $Ly = \frac{\mathrm{d}^n y}{\mathrm{d}z^n} + c_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \cdots + c_0 y = 0$ with coefficients $c_k \in \mathbb{C}(z)$ rational functions. Our main theorem asserts that the function F satisfies a linear non-homogeneous equation $Ly = R$ with R a rational function. The precise description of R leads to the solution of a vanishing problem and to the solution of a moment-type problem, which we call D-moment problem.

RÉSUMÉ. On considère une integrale du type Cauchy $F(z) = \int_\Gamma \frac{g(\xi)\mathrm{d}\xi}{\xi - z}$, où g est une fonction analytique par morceaux satisfaisant une équation différentielle linéaire homogène d'ordre n, $Ly = \frac{\mathrm{d}^n y}{\mathrm{d}z^n} + c_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \cdots + c_0 y = 0$, aux coefficients $c_k \in \mathbb{C}(z)$ rationnels. Notre théorème principal affirme que la fonction F satisfait une équation linéaire non-homogène $Ly = R$ avec R rationnelle. La description précise de R mène à la solution du problème d'évanescence et à la solution d'un problème du type moment que nous appelons problème de D-moment.

1. **Introduction** The present paper was inspired by a question of a piecewise polynomial moment problem, posed to us by Y. Yomdin, and by the results of F. Pakovich, N. Roytvarf and Y. Yomdin [PRY] and Roytvarf and Yomdin [RY]. We consider a Cauchy-type integral $F(z) = \int_\Gamma \frac{g(\xi)\mathrm{d}\xi}{\xi - z}$, where g is a piecewise analytic function satisfying a linear homogeneous equation $Ly = \frac{\mathrm{d}^n y}{\mathrm{d}z^n} + c_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \cdots + c_0 y = 0$ with coefficients $c_k \in \mathbb{C}(z)$ rational functions. Theorem 2.1 asserts that the function F satisfies a linear non-homogeneous equation $Ly = R$ with R a rational function. The precise description of the singularities of the function R, given in Proposition 2.2, leads to the solution of a certain vanishing problem (Theorem 3.1); and combined with the explicit formulas from Lemma 3.3 leads to the solution of the D-moment problem (Theorem 3.2). Theorem 2.4 provides an explicit linear recurrence relation with rational coefficients for the sequence of moments of a piecewise analytic function g as above. The possibility of such a relation was pointed out by Yomdin. This is a brief version of the paper, the complete one will appear elsewhere. We would like to

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thank Y. Yomdin for bringing the problem to our attention and for stimulating discussions.

2. Main theorems

We start by establishing some notations. Throughout the paper, by a holomorphic function in a domain $D \subset \mathbb{C}$ we mean a univalued analytic function in D. In contrast, an analytic function in D may be, and in our paper usually is, multivalued.

On the complex line $\mathbb{C}^1$ with a fixed coordinate z we consider a linear differential operator $L = \frac{\mathrm{d}^n}{\mathrm{d}z^n} + c_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \cdots + c_0$ with coefficients $c_k \in \mathbb{C}(z)$ rational functions. We denote by $\Omega \subset \mathbb{C}$ the set of poles of all the coefficients c_k. In the complement $\mathbb{C}\backslash\Omega$ the coefficients of the differential operator L are holomorphic functions, and thus in a sufficiently small neighborhood of any point z_0 solutions of the homogeneous differential equation $Ly = 0$ form an n-dimensional vector space.

Let $\Gamma \subset \mathbb{C}\backslash\Omega$ be a smooth realization of an abstract oriented finite graph $\tilde{\Gamma}$ without loops. We recall that the graph $\tilde{\Gamma}$ is a collection of its vertices V and its edges E; its orientation is given by the choice of the initial, $\mathrm{in}(e)$ and the end, $\mathrm{end}(e)$ vertices for every edge e. By a smooth realization we mean the following:

(i) a finite collection of pairwise distinct points z_v in $\mathbb{C}\backslash\Omega$ indexed by the vertices of the graph $\tilde{\Gamma}$;
(ii) a finite collection of smooth embeddings $\gamma_e\colon [0,1] \mapsto \mathbb{C}\backslash\Omega$ indexed by the edges of the graph $\tilde{\Gamma}$ with $\gamma_e(0) = z_{\mathrm{in}(e)}$ and $\gamma_e(1) = z_{\mathrm{end}(e)}$;
(iii) a natural orientation from $\gamma_e(0)$ to $\gamma_e(1)$ on each curve $\gamma_e([0,1])$.

Now assume that a function $g_{e,\mathrm{in}(e)}$ is a solution of the homogeneous n-th order differential equation $Ly = 0$ and is defined in a small neighborhood of the point $\gamma_e(0) = z_{\mathrm{in}(e)}$. Let us denote by g_e the analytic continuation of the germ $g_{e,\mathrm{in}(e)}$ along the curve $\gamma_e([0,1])$. It is convenient to think of g_e as a density of the mass distribution along the curve $\gamma_e([0,1])$. Given a germ $g_{e,\mathrm{in}(e)}$ for every edge of the graph, the number $m_n = \int_\Gamma z^n g_e \,\mathrm{d}z := \sum_{\gamma_e|e\in E} \int_{\gamma_e([0,1])} z^n g_e \,\mathrm{d}z$ is called the n-th moment of the system of masses (Γ, g_e). We denote by G_v the function in a neighborhood of a vertex z_v defined as $\sum \pm g_{e,in(e)}$, where the summation goes over all the edges that contain the vertex v; the sign is chosen to be "+" if the vertex v is the initial for the edge e and "−" otherwise. Clearly, for any vertex z_v the function G_v is a solution of the differential equation $Ly = 0$. Finally, we denote by $\vec{\mathrm{G}}_v$ the vector $(G_v|_{z=z_v}, \frac{\mathrm{d}G}{\mathrm{d}z}|_{z=z_v}, \ldots, \frac{\mathrm{d}^{n-1}G}{\mathrm{d}z^{n-1}}|_{z=z_v})$ of the initial values at the point z_v.

Now we are ready to formulate our first main result.

THEOREM 2.1. *The Cauchy-type integral* $-\int_\Gamma \frac{g(\xi)\mathrm{d}\xi}{\xi - z}$ *in* $\mathbb{C}\backslash\Gamma$ *allows for an analytic continuation* $F(z)$ *in* $\mathbb{C}\backslash\Omega$ *with the following properties:*

(i) *The germ of the original branch of the function* F *at infinity is* $\sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}}$.
(ii) *The function* F *is a solution of an* n*-th order non-homogeneous equation* $LF = R$ *with* R *a rational function.*

(iii) *In a neighborhood of a vertex* z_v, $F(z) = G_v(z)\ln(z - z_v) + h(z)$ *with* h *holomorphic in a neighborhood of* z_v.

Denote by $\operatorname{ord}_a f$ the order of the pole of a rational function f at a point a. The next proposition describes poles of the rational function $R = LF$ from the previous theorem.

PROPOSITION 2.2. *Poles of the function* R *are located in the union of the set* Ω *and the set of the points* z_v. *Moreover:*

(i) *For every point* $a \in \Omega$ *the function* R *has a pole of order at most*

$$\max_{k=0,\dots,n-1} \operatorname{ord}_a c_k.$$

(ii) *For every point* z_v *the function* R *has a pole of order at most* n; *there is a lower triangular unipotent (with* 1 *on the main diagonal) matrix* M *of rational functions that depends on the operator* L *only such that* $\mathrm{M}\,\vec{\mathrm{G}}_v$ *evaluated at* z_v *is* $(a_n, \dots, a_1)$, *where* $\frac{a_n}{(z-z_v)^n} + \cdots + \frac{a_1}{(z-z_v)}$ *is the principal part of the Laurent series of the function* R *at the point* z_v.

Proofs of Theorem 2.1 and Proposition 2.2 are based on a well-known fact about Cauchy-type integrals. For convenience we will state it below (Lemma 2.3). Let a, b be two points on the complex line and $\gamma\colon [0,1] \mapsto \mathbb{C}$ be a smooth embedding such that $\gamma(0) = a$, $\gamma(1) = b$. Consider a Cauchy-type integral $I(z) = \int_{\gamma([0,1])} \frac{g(\xi)\mathrm{d}\xi}{\xi - z}$, where the curve $\gamma([0,1])$ is oriented from point a to b, and g is a function holomorphic in a small neighborhood of the curve. The curve $\gamma([0,1])$ divides a sufficiently small neighborhood U of a point $c \in \gamma((0,1))$ into the "right side" U_+ and the "left side" U_-. We denote by I_+ and I_- the function I restricted to U_+ and the function F restricted to U_-, respectively.

LEMMA 2.3. *Functions* I, I_+, *and* I_- *satisfy the following properties.*

(i) *The function* $I(z) = -\sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}}$ *in the neighborhood of infinity, where* $m_k = \int_\gamma z^k g(z)\,\mathrm{d}z$.

(ii) *Functions* I_+, I_- *can be extended to holomorphic functions* $\tilde{I}_+$, $\tilde{I}_-$ *on the whole neighborhood* U; *moreover, the following identity holds* $\tilde{I}_+ - \tilde{I}_- = g$.

(iii) *Near point* a *(resp. point* b*) the function* $I(z) = -g(z)\ln(z - a) + h_a(z)$ *(resp.* $I(z) = g(z)\ln(z - b) + h_b(z)$*) with* h_a *(resp.* h_b*) holomorphic in a neighborhood of point* a *(resp. point* b*).*

Sketch of proofs of Theorem 2.1 *and Proposition* 2.2. Since

$$-\frac{g(\xi)\mathrm{d}\xi}{\xi - z} = \frac{g(\xi)\mathrm{d}\xi}{z(1 - \xi/z)} = \sum_{k=0}^{\infty} \frac{g(\xi)\xi^k \mathrm{d}\xi}{z^{k+1}}$$

for $|z| \gg 0$, we have $F(z) = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}}$ at infinity. This proves Theorem 2.1(i).

To verify Theorem 2.1(ii) and Proposition 2.2, it is sufficient to consider the case when the graph $\tilde{\Gamma}$ has only one edge e. Denote by Ω' the union of the set Ω and the set $\{z_{\mathrm{in}(e)}, z_{\mathrm{end}(e)}\}$.

Next we prove Theorem 2.1(ii) and Proposition 2.2. We first verify that the function F is analytic in $\mathbb{C}\backslash\Omega'$, and simultaneously we will show that the function LF is holomorphic in $\mathbb{C}\backslash\Omega'$.

Pick a point $a \in \mathbb{C}\backslash\Omega'$. Take a smooth closed path $\alpha\colon [0,1] \mapsto \mathbb{C}\backslash\Omega'$ ($\alpha(0) = \alpha(1) = a$) that transversely intersects the curve $\gamma_e([0,1])$ in a finite number of points $z_1, \dots, z_s$ at times $0 < t_1 < \cdots < t_s < 1$ with $\alpha(t_i) = z_i$. We denote by g_i, $i = 1, \dots, s$, the germ of the function g_e at the point z_i. According to the second part of Lemma 2.3, there is an analytic continuation $\tilde{F}$ of the function F along the curve $\alpha([0,1])$ and, moreover, $\tilde{F}(z) - F(z) = \sum_{i=1}^{s} \pm\tilde{g}_i(z)$, where $\tilde{g}_i(z)$ is the analytic continuation of the germ g_i along the curve $\alpha([t_i, 1])$. In particular, $L\tilde{F} = LF + L(\pm\tilde{g}_i(z)) = LF$.

According to Lemma 2.3(iii), the function F and therefore the function LF grow slowly when approaching points $z_{\mathrm{in}(e)}, z_{\mathrm{end}(e)}$. Near a point $a \in \Omega$ the function F is holomorphic, so the function LF has at most a pole of order $\max_{k=0,\dots,n-1} \operatorname{ord}_a c_k$. At infinity the function F is holomorphic, so the function LF has at most a pole. Hence, the function LF is rational. This proves Proposition 2.2(i).

Theorem 2.1(iii) directly follows from Lemma 2.3(iii) and the definition of the functions G_v.

For the proof of Proposition 2.2(ii) we need the following calculation, which can be easily proved by induction. For any $k \in \mathbb{N}$:

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left[g(z)ln(z-z_0)\right] = g(z_0)\Big(\frac{1}{z-z_0}\Big)^{(k-1)} + g'(z_0)\Big(\frac{1}{z-z_0}\Big)^{(k-2)}$$

$$+ g''(z_0)\Big(\frac{1}{z-z_0}\Big)^{(k-3)} + \cdots + g^{(k-1)}(z_0)\Big(\frac{1}{z-z_0}\Big) + g^{(k)}(z)ln(z-z_0) + h(z),$$

with h and g holomorphic functions in a neighborhood of a point $z_0 \in \mathbb{C}$.

According to Theorem 2.1(iii), to find the Laurent series for the function $R = LF$ at the point z_v, we consider $\left[\frac{\mathrm{d}^n}{\mathrm{d}z^n} + c_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \cdots + c_0\right](G_v(z)ln(z-z_v))$. Applying the above calculation, we derive Proposition 2.2(ii). □

Remark 1. Under some natural assumptions on the growth of the functions g_e, the first two parts of Theorem 2.1 remain valid even if some vertices of the graph Γ belong to the set Ω. We will employ this remark in the complete version of this paper.

Remark 2. As follows from the proof of Proposition 2.2, the entries of the matrix M are certain linear combinations of the coefficients $c_{n-1}, \dots, c_0$ and its derivatives.

According to Theorem 2.1, the Laurent series $L(\sum_{i=0}^{\infty} \frac{m_i}{z^{i+1}})$, which is equal to $\sum_{i=-l}^{\infty} \frac{M_i}{z^{i+1}}$ for some $M_i \in \mathbb{C}$ and $l \in \mathbb{Z}$, is a Laurent series of a rational function. In particular, it implies that the sequence $\{M_i\}_{i=-l}^{\infty}$ satisfies a linear recurrence relation with constant coefficients. Since each M_i is a certain linear combination of the initial moments m_i, one may expect that the sequence $\{m_i\}_{i=0}^{\infty}$ satisfies a linear recurrence relation with rational coefficients (for the precise meaning, see Theorem 2.4 below).

Having multiplied the operator L by the common denominator of its coefficients, we may assume that $L = p_n \frac{\mathrm{d}^n}{\mathrm{d}z^n} + p_{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \cdots + p_0$, where $p_i \in \mathbb{C}[z]$ are polynomials. Consider the formal adjoint operator $L^* = \sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}z^k}(p_k)$ The operator L^* is an n-th order linear differential operator with polynomial coefficients and $L^* = (-1)^n p_n \frac{\mathrm{d}^n}{\mathrm{d}z} + \cdots$. In particular, L^* is not identically equal to zero.

Theorem 2.4. *The sequence $\{m_i\}_{i=0}^{\infty}$ satisfies the following recurrence relation*

$$q_{N-1}(r+N-1)m_{r+N-1} + q_{N-2}(r+N-2)m_{r+N-2} + \cdots + q_0(r)m_r = 0,$$

where $q_0, \ldots, q_{N-1}$ are polynomials of the complex variable z.

Proof. Let $Q(z) = ((z-z_1)\cdots(z-z_s))^n$, where s is a number of vertices of the graph $\tilde{\Gamma}$ and z_i are corresponding points in $\mathbb{C}\backslash\Omega$. By Stokes' formula (applied several times) $\int_\Gamma L^*(z^r Q) g_e \,\mathrm{d}z = \int_\Gamma (z^r Q) L(g_e) \,\mathrm{d}z = 0$. Notice that for every $k, l, r \in \mathbb{N}$, $z^k \frac{\mathrm{d}}{\mathrm{d}z^l}(z^r) = r(r-1)\cdots(r-l+1)z^{r+k-l} = q_{kl}(r+k-1)z^{r+k-l}$, where $q_{kl}(z) = (z-k+1)(z-k+2)\cdots(z-k+l-1)$ is a polynomial of the complex variable z. By the linearity, $L^*(z^r Q) = \sum_{i=0}^{N} q_i(r+i) z^{r+i}$, where N is some integer and q_i $(i = 1, \ldots N)$ are some polynomials. □

3. Applications

In this section we adhere to the previous notations. Recall that $\Gamma \subset \mathbb{C}\backslash\Omega$ is an aggregate of a finite number of points z_v and smooth oriented paths between them. Points of Γ correspond to the vertices of an abstract oriented finite graph $\tilde{\Gamma}$, paths correspond to the (oriented) edges; Ω is a singular set of a linear n-th order differential operator L with rational coefficients. Abusing the notation, we will refer to Γ as to a graph. We will also call the points z_v and the paths between them vertices and edges of the graph Γ, respectively.

We address the following two questions.

- Given a system of masses g_e distributed on the graph Γ. How many first moments should vanish so that all the moments vanish? See Theorem 3.1.
- D-moment problem: Given a (generic) system of masses distributed on the graph Γ, find vertices of the graph Γ and masses given the finite number of moments. Theorem 3.2 provides a partial answer.

Let us multiply the operator L by the common denominator of its coefficients and denote the result by $\tilde{L}$. The operator $\tilde{L} = p_n \frac{\mathrm{d}^n}{\mathrm{d}z^n} + p_{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} + \cdots + p_0$, where p_i are polynomials of the complex variable z. We will call the number $c(L, \Gamma) = dn + \max_{i=1,\dots n} \deg(p_i)$, where d is the number of vertices in the graph Γ, the *complexity* of the system of masses (Γ, g_e).

THEOREM 3.1. *If the first $c(L, \Gamma)$ moments of the system of masses (Γ, g_e) vanish, then all the moments vanish.*

THEOREM 3.2. *For a (generic) system of masses (Γ, g_e) there is a rational function $W = \frac{P}{Q}$ with the following properties.*

(i) *The function W has poles only at vertices of the graph Γ.*
(ii) *Every coefficient b of polynomials P and Q has the form $W_b(m_1, \dots, m_{c(\Gamma,L)})$, where W_b is a polynomial in $c(\Gamma, L)$ variables with complex coefficients. The polynomials W_b depend only on the operator L and the number of vertices.*
(iii) *For every vertex z_v the function W has a pole of order n; there is a polynomial matrix M that depends only on the operator $\tilde{L}$ such that the matrix M evaluated at z_v is non-degenerate and $\mathrm{M}\vec{\mathrm{G}}_v$ evaluated at z_v is $(a_n, \dots, a_1)$, where $\frac{a_n}{(z-z_i)^n} + \cdots + \frac{a_1}{(z-z_i)}$ is the principle part of the Laurent series of the function W at the point z_v.*

We would like to illustrate our proof of Theorem 3.2 together with its application to the D-moment problem with the following example.

Example (piecewise polynomial moment problem). Given the differential operator $L = \frac{\mathrm{d}^n}{\mathrm{d}z^n}$, where $n \in \mathbb{N}$, any solution of the equation $Ly = 0$ is a polynomial of degree smaller than n. Assume that points $x_1 < \cdots < x_{k+1}$ on the real line are vertices of the graph Γ, segments $[x_1, x_2], \dots, [x_k, x_{k+1}]$ are edges, and polynomials $P_1, \dots, P_k \in \mathbb{C}[z]$ of degree smaller than n are masses on the intervals $[x_1, x_2], \dots, [x_k, x_{k+1}]$, respectively. According to Theorem 2.1, the function

$$R = -\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Big(\int_{x_1}^{x_2} \frac{P_1(\xi)\mathrm{d}\xi}{\xi - z} + \cdots + \int_{x_k}^{x_{k+1}} \frac{P_k(\xi)\mathrm{d}\xi}{\xi - z}\Big)$$

is rational and is equal to $\frac{\mathrm{d}^n}{\mathrm{d}z^n}\big(\sum_{j=0}^{\infty} \frac{m_j}{z^{j+1}}\big)$ in a neighborhood of infinity, where m_j are moments. It has poles only at the points $x_1 < \cdots < x_{k+1}$. At the point x_i $(1 \le i \le k+1)$ the principle part of the function R is equal to (see the proof of Proposition 2.2(ii)):

$$G_i(x_i)\Big(\frac{1}{z - x_i}\Big)^{(n-1)} + G_i'(x_i)\Big(\frac{1}{z - x_i}\Big)^{(n-2)} + G_i''(x_i)\Big(\frac{1}{z - x_i}\Big)^{(n-3)} + \cdots + G_i^{(n-1)}(x_i)\Big(\frac{1}{z - x_i}\Big)$$

with $G_1 = P_1, G_2 = P_2 - P_1, G_3 = P_3 - P_2, \dots, G_k = P_k - P_{k-1}, G_{k+1} = -P_k$. For a generic k-tuple of polynomials $P_1, \dots, P_k$ the function R has poles of order precisely n at the $(k+1)$ points $x_1, \dots, x_{k+1}$. Lemma 3.3 below provides the explicit formulas for the rational function R in terms of the first $2n(k+1)$ coefficients in its Taylor expansion at infinity. Note that the first $2n(k+1)$ coefficients in $\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\sum_{j=0}^{\infty} \frac{m_j}{z^{j+1}}\right)$ are determined by the first $2n(k+1)-n$ moments m_j.

Sketch of the proof of Theorem 3.1. According to Theorem 2.1, the function $\tilde{L}F$, where $F(z) = \int_\Gamma \frac{g_e(\xi)\mathrm{d}\xi}{\xi - z}$, is rational. It has poles of order at most n at d vertices of the graph Γ. It also has a zero at infinity of order at least $nd+1$ as the condition of vanishing the first $c(L,\Gamma)$ moments implies. Thus $\tilde{L}F \equiv 0$.

In particular, for any vertex z_v the function $G_v = 0$ (see Proposition 2.2(iii)), which implies that F is holomorphic (see Theorem 2.1(iii)) on the whole complex line. At infinity the function F is also holomorphic and is equal to zero. We conclude that $F \equiv 0$ and thus all the moments are equal to zero. □

Sketch of the proof of Theorem 3.2. We claim that the function $W = \tilde{L}F - P_\infty(\tilde{L}F)$, where $P_\infty(\tilde{L}F)$ is the principle part of the function $\tilde{L}F$ at infinity, satisfies the conditions of Theorem 3.2. Indeed, Theorem 3.2(i) and (iii) easily follow from Theorem 2.1. Explicit formulas for polynomials P and Q follow from the lemma below. □

Let F be a rational function given as a quotient P/Q of two relatively prime polynomials P and Q with complex coefficients. We assume that Q is a monic polynomial of degree n ($n \in \mathbb{N}$) and P is a polynomial of degree strictly less than n. In other words, $P = b_{n-1}z^{n-1} + \dots + b_0$ and $Q = z^n + a_{n-1}z^{n-1} + \dots + a_0$, where a_i, b_j are complex numbers. Consider the Taylor series of the function F at infinity: $F = \sum_{k=1}^{\infty} \frac{s_k}{z^k}$. Denote

$$\Delta = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{pmatrix}, \quad \Delta(z) = \begin{pmatrix} s_1 & s_2 & \cdots & s_n & s_{n+1} \\ s_2 & s_3 & \cdots & s_{n+1} & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots & \\ s_n & s_{n+1} & \cdots & s_{2n-1} & s_{2n} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}.$$

LEMMA 3.3.

(i) *The determinant of the matrix Δ is not zero.*

(ii) *Polynomial $Q(z)$ is equal to* $\det \Delta(z)/\det \Delta$.

(iii) *If $(b_{n-1}, \dots, b_0)$ and $(1, a_{n-1}, \dots, a_1)$ are vectors constructed from coefficients of polynomials P and Q respectively, then*

$$\begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{pmatrix} = \begin{pmatrix} s_1 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_1 \end{pmatrix} \begin{pmatrix} 1 \\ a_{n-1} \\ \vdots \\ a_1 \end{pmatrix}$$

Remark 3. Interestingly, moments $m_1, \dots, m_{c(\Gamma,L)}$ are not independent. For instance, if $L = \frac{\mathrm{d}^n}{\mathrm{d}z^n}$, Γ is a segment on the real line, and g is a polynomial (with complex coefficients) of degree at most $n-1$, then our formulas require $3n$ first moments to reconstruct the polynomial and the end points of the segment. However, one can show that already the first $2n$ moments determine the same data uniquely.

References

[**PRY**] F. Pakovich, N. Roytvarf, and Y. Yomdin, *Cauchy type integrals of algebraic functions and a tangential center-focus problem for Abel equations.* Israel J. Math. **144**(2004), 221–291.

[**RY**] N. Roytvarf and Y. Yomdin, *Analytic continuation of Cauchy-type integrals.* Funct. Differ. Equ. **12**(2005), no. 3-4, 375–388.

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