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SIGNAL ACQUISITION FROM MEASUREMENTS VIA NON-LINEAR MODELS

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ABSTRACT. We consider the problem of reconstruction of a non-linear finite-parametric model $M = M_p(x)$ with $p = (p_1, \dots, p_r)$ a set of parameters, from a set of measurements $m_j(M)$. In this paper $m_j(M)$ are always the moments $m_j(M) = \int x^j M_p(x)\,dx$. This problem is a central one in signal processing, statistics, and in many other applications.

We concentrate on a direct (and somewhat "naive") approach to the above problem: we simply substitute the model function $M_p(x)$ into the measurements m_j and compute explicitly the resulting "symbolic" expressions of $m_j(M_p)$ in terms of the parameters p. Equating these "symbolic" expressions to the actual measurement results, we produce a system of nonlinear equations in the parameters p, which we then try to solve.

The aim of this paper is to review some recent results in this direction, stressing the algebraic structure of the arising systems and mathematical tools required for their solution.

In particular, we discuss the relation of the reconstruction problem above with recent results on the vanishing problem for generalized polynomial moments and on the Cauchy-type integrals of algebraic functions.

RÉSUMÉ. Nous étudions le problème de reconstruction d'un modèle non-linéaire parametrisé $M = M_p(x)$, aux paramètres $p = (p_1, \dots, p_r)$, à partir d'un ensemble de mesures $m_j(M)$. Dans cet article les $m_j(M)$ sont des moments $m_j(M) = \int x^j M_p(x)\,dx$. Ce problème est central dans le traitement du signal, dans les statistiques et dans bien d'autres domaines.

Nous nous concentrons sur une approche directe (et un peu "naîve") du problème décrit ci-dessus: nous substituons simplement la fonction modèle $M_p(x)$ dans les mesures m_j et calculons explicitement l'expression symbolique résultant de $m_j(M_p)$ en fonction des paramètres p. En comparant ces expressions "symboliques" aux vraix valeurs des mesures, nous produisons un système d'équations non-linéaires en p, que nous essayons de résoudre.

Le but de cet article est d'examiner des résultats récents qui vont dans cette direction, tout en insistant sur la structure algébrique des systèmes qui interviennent et des outils mathématiques nécessaires pour leur solution.

En particulier nous discuterons la relation du problème de reconstruction décrit ci-dessus aux résultats récents sur le problème des zéros des moments polynomiaux généralisés et sur les intégrales du type Cauchy des fonctions algébriques.

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1. Introduction In this paper we consider the following problem: let a finite-parametric family of functions $M = M_p(x),\ x \in \mathbb{R}^m$ be given with $p = (p_1, \dots, p_r)$ a set of parameters. We call $M_p(x)$ a model, and usually we assume that it depends on some of its parameters in a non-linear way (this is always the case with the "geometric" parameters representing the shape and the position of the model).

The problem is how to reconstruct, in a robust and efficient way, the parameters p from a set of "measurements" $m_1(M), \dots, m_l(M)$?

In this paper m_j will be the moments $m_j(M) = \int x^j M_p(x)\,dx$. This assumption is not too restrictive (see [11, 13]).

The above problem is certainly among the central ones in signal processing (non-linear matching), statistics (non-linear regression), and in many other applications. See [11, 13, 18, 27–30, 46] and the references therein.

We concentrate in the present paper on a direct (and somewhat "naive") approach to the above problem: we simply substitute the model function $M_p(x)$ into the measurements m_j and compute explicitly the resulting "symbolic" expressions of $m_j(M_p)$ in terms of the parameters p. Equating these "symbolic" expressions to the actual measurement results, we produce a system of nonlinear equations on the parameters p which we try to solve.

Certainly, the polynomial moments do not present the best choice of measurements for practical applications since the monomials x^j are far from being orthogonal (see, for example, [45]). However, the main features of the arising non-linear systems remain the same for a much wider class of measurements, while their structure is much more transparent for moments.

The aim of this paper is to review some recent results (mostly of [11, 13, 18, 19, 27–30, 46]) in this direction, stressing the algebraic structure of the arising systems and mathematical tools required for their solutions. In particular, we stress the role of the moment generating function.

We start with some initial examples of the models $M_p(x)$ in one dimension: these are polynomials and rational functions. Then we consider linear combinations of δ-functions. The system which appears in this example is typical in many applications. We discuss one of the classical solution methods following [13, 18, 27, 36, 46].

Next we deal with the piecewise-solution of linear differential equations, providing some prerequisites for the reconstruction method described in [24] (this issue). Then we consider piecewise-algebraic functions of one variable. We prove injectivity of the finite moment transform on such functions, and discuss the relation of the reconstruction problem for such functions with recent results on the vanishing problem of generalized polynomial moments [2, 4, 5, 7, 31, 33, 34].

In two dimensions we shortly present results concerning reconstruction of polygons from their complex moments [13, 18, 29], as well as results on reconstruction of "quadrature domains" [19]. Finally we consider the problem of reconstruction of δ-functions along algebraic curves, relating it to the vanishing problem of double moments [1, 2, 9, 21, 22, 35].

We barely touch the classical moment theory, referring the reader to [32] and especially to [19, 20, 37–40, 45], where, in particular, a review of the classical results and methods is given, as applied to the effective reconstruction problem. We also do not discuss the problem of noise resistance in this paper. It is treated in [13, 18, 27, 28, 30].

1.1. Applicability of the "direct substitution" method The key condition for applicability of our approach is the assumption that the signals we work with can be faithfully approximated by *a priori* known "simple" geometric models.

A natural question is to what extent is this assumption realistic? The answer to this question is twofold:

- In many specific applications the form of the signal is indeed known *a priori.* Besides the wide circle of applications mentioned in [11, 13, 18, 27–30, 46], notice that this is usually the case in visual quality inspection. Similar situations arise in some medical applications where a non-linear parametric model of an important pattern has to be matched to the radiology or ultrasound measurements data.
- A general applicability of our approach in problems involving image acquisition, analysis and processing depends on a possibility to represent general images of the real world via geometric models.

The importance of such a representation in many imaging problems, from still and video-compression to visual search and pattern detection, is well known. Some initial implementations of geometric image "modelization" have been suggested, in particular, [3, 14, 15, 26]. See [15] and the references therein for a general overview and analysis of the performance of edges-based methods in image representation.

However, in general the "geometric" methods currently suffer from an inability to achieve a full visual quality for high resolution photo-realistic images of the real world. *In fact, the mere possibility of faithfully capturing such images with geometric models presents one of the important open problems, sometimes called "the vectorization problem", in image processing.*

Certainly, this current state of affairs makes problematic immediate practical applications of general imaging methods based on a geometric model.

Let us express our strong belief that a full visual quality geometric-model representation of high resolution photo-realistic images is possible. If achieved, it promises a major advance in image compression, capturing, and processing, in particular, via the approach of the present paper.

Recently some "semi-linear" approaches have emerged providing a reliable reconstruction of "simple" (and not necessarily regular) signals from a small number of measurements. In these approaches (see [6, 10] and the references therein) "simplicity" or "compressibility" of a function is understood as a possibility for its accurate sparse representation in a certain (wavelet) basis.

A somewhat more general approach to the notion of a complexity of functions has been suggested [47, 48]: here we take as a complexity measure the rate of semi-algebraic approximation. If the wavelet base is semi-algebraic, "compressible" functions have low semi-algebraic complexity. The same is typically true for functions allowing for a fast approximation by various types of non-linear models.

2. Examples of moment inversion: one variable

In this section we consider some natural examples of the models $M_p(x)$ in one dimension and of their reconstruction from the moments. These are polynomials, rational functions, linear combinations of δ-functions, and the class A_D of piecewise-analytic functions, each piece satisfying a fixed linear differential operator D with rational coefficients. (Piecewise-polynomials belong to A_D for $D = \frac{d^n}{dx^n}$). Then we consider piecewise-algebraic functions.

In this paper we use as one of the main tools in solving the moment inversion problem, the moment generating function $I_g(z)$ defined as

$$I_g(z) = \sum_{k=0}^{\infty} m_k(g) z^k = \int_0^1 \frac{g(t)dt}{1 - zt}. \tag{2.1}$$

2.1. Vetterli's approach In [11, 27, 46] an important class of signals was introduced, possessing a "finite rate of innovation", *i.e.,* a finite number of degrees of freedom per unit of time. Usually such signals are not band-limited, so classical sampling theory does not enable a perfect reconstruction of signals of this type. It has been shown that using an adequate sampling kernel and a sampling rate greater or equal to the rate of innovation, it is possible to reconstruct such signals uniquely [11, 27, 46]. The behavior of the reconstruction in the presence of noise has also been investigated.

The main type of signals for which explicit reconstruction schemes have been proposed include linear combinations of δ-functions and their derivatives, splines, and piecewise polynomials. In spite of a somewhat different setting of the problem, the reconstruction schemes turn out to be mathematically similar to the ones presented below. In fact, moments enter the reconstruction procedure as an intermediate step in [11], and systems very similar to (2.6) and (2.7) below explicitly appear in [11, 27, 46]. It is a remarkable fact (although traced at least to [36]) that exactly the same systems arise in exponential approximation [17], in reconstruction of plane polygons [13,18,29] (see Section 3.1 below), in reconstruction of quadrature domains [19] (see Section 3.2 below), in Padé approximations, and in many other problems.

In [28] the approach of [11, 27, 46] is extended to some classes of parametric non-bandlimited two-dimensional signals. This includes linear combinations of 2D δ-functions, lines, and polygons. Notice that the first problem in its complex setting (where we consider as the allowed measurements only the *complex* moments $\mu_k(f) = \iint z^k f(x,y)\, dxdy$) leads once more to a complex system (2.6).

2.2. Polynomials Let $P(x)$ be a polynomial of degree d, $P(x) = \sum_{j=0}^{d} a_j x^j$. For the k-th moment $m_k(P)$ we have

$$m_k(P) = \int_0^1 \sum_{j=0}^{d} a_j x^{j+k}\, dx = \sum_{j=0}^{d} \frac{a_j}{j+k+1} = \sum_{j=0}^{d} h_{kj} a_j, \tag{2.2}$$

if we put $h_{kj} = \frac{1}{j+k+1}$. Now let a denote the column-vector of the coefficients a_j of the polynomial $P(x)$ and let m denote the column-vector of the moments $m_0(P), \ldots, m_d(P)$. We get the following linear system:

$$Ha = m, \quad H = (h_{kj}). \tag{2.3}$$

Notice that the matrix H is a Hankel matrix: the rows of this matrix are obtained by the shifts of its first row. More specifically, the matrix H belongs to the class of Hilbert-type matrices [23]. In particular, its determinant is nonzero, and system (2.2) has a unique solution. Therefore, we have the following.

Proposition 2.1. *A polynomial $P(x)$ of degree d can be uniquely reconstructed from its first $d+1$ moments $m_0(P), \ldots, m_d(P)$, via solving system* (2.3).

Notice, however, that the smallest eigenvalue $\lambda_{\min}(H)$ behaves asymptotically for $d \to \infty$ as $\lambda_{\min}(H) = K\sqrt{d}\rho^{-4(d+1)}(1 + o(1))$, where $K = 8\pi\sqrt{2\pi}2^{1/4}$ and $\rho = 1 + \sqrt{2}$, (see [23]). Therefore, the inversion of the matrix H becomes problematic for large d.

Notice also that for each fixed polynomial $P(x)$ expression (2.1) defines $m_k(P)$ as a rational function of k.

As for the moment generating functions, we have the following.

Proposition 2.2. $I_P(z) = -\frac{1}{z}\log(1 - \frac{1}{z}) + \hat{P}(\frac{1}{z})$, *with $\hat{P}(s)$ a polynomial of degree $d-1$ in s.*

Proof. We have $P(t) = \tilde{P}(t)(t - \frac{1}{z}) + P(\frac{1}{z})$ where $\tilde{P}(t)$ is a polynomial of degree $d-1$ in t whose coefficients are polynomials of degree $d-1$ in $\frac{1}{z}$. Hence

$$I_P(z) = \int_0^1 \frac{P(t)dt}{1 - zt} = -\frac{1}{z}\int_0^1 \frac{P(\frac{1}{z})dt}{t - \frac{1}{z}} - \int_0^1 \tilde{P}(t)\,dt.$$

Integrating from 0 to 1 now provides the required expression. □

2.3. Rational functions Let $R(x)$ be a rational function of degree d, $R(x) = \frac{P(x)}{Q(x)}, \deg Q = d, \deg P \le d-1$ (we assume that R does not have a "polynomial part"). Thus

$$P(x) = \sum_{j=0}^{d-1} a_j x^j, \quad Q(x) = \sum_{j=0}^{d} b_j x^j.$$

We have $P(x) = Q(x)R(x) = \sum_{j=0}^{d} b_j x^j R(x)$. Hence

$$m_k(P) = \sum_{j=0}^{d} b_j m_{k+j}(R), \; k = 0, 1, \ldots,$$

and using our notations from Section 2.1 above we finally get a system for the unknowns a_j, b_j:

$$\sum_{j=0}^{d-1} h_{kj} a_j = \sum_{j=0}^{d} m_{k+j}(R) b_j, \quad k = 0, 1, \ldots, 2d, \tag{2.4}$$

where, as above, $h_{kj} = \frac{1}{j+k+1}$. We do not analyze here the solvability conditions for (2.4) (*cf.* [24, Lemma 3.3]). Let us notice also that counting the sign changes as in Section 2.6 below shows that a rational function $R(x)$ of degree d can be uniquely reconstructed from its first $4d$ moments $m_0(R), \ldots, m_{4d}(R)$.

To compute the moment generating function $I_R(z)$, let us assume that the roots $\alpha_1, \ldots, \alpha_d$ of Q are all distinct. Then $R(t) = \sum_{i=1}^{d} \frac{A_i}{t-\alpha_i}$, and denoting $\frac{1}{z}$ by w we get

$$\frac{R(t)}{t-w} = \sum_{i=1}^{d} \frac{A_i}{(t-\alpha_i)(t-w)} = \sum_{i=1}^{d} A_i \Big(\frac{1}{(\alpha_i - w)(t-\alpha_i)} - \frac{1}{(\alpha_i - w)(t-w)} \Big).$$

Transforming integral (2.1) as in the proof of Proposition 2.2 and integrating, we finally get the following.

PROPOSITION 2.3. *The moment generating function $I_R(z)$ of a rational function $R(x)$ is given by*

$$I_R(z) = -w \sum_{i=1}^{d} \frac{A_i}{\alpha_i - w} \Big[\log\Big(\frac{1-\alpha_i}{\alpha_i}\Big) - \log\Big(\frac{w-1}{w}\Big) \Big], \quad w = \frac{1}{z}.$$

2.4. Linear combination of δ-functions Let $g(x) = \Sigma_{i=1}^{n} A_i \delta(x - x_i)$. For this function we have

$$m_k(g) = \int_0^1 x^k \Sigma_{i=1}^{n} A_i \delta(x - x_i)\, dx = \Sigma_{i=1}^{n} A_i x_i^k. \tag{2.5}$$

So assuming that we know the moments $m_k(g) = \alpha_k$, $k = 0, 1, \ldots, 2n-1$, we obtain the following system of equations for the parameters A_i and x_i, $i = 1, \ldots, n$, of the function g:

$$\Sigma_{i=1}^{n} A_i x_i^k = \alpha_k, \quad k = 0, 1, \ldots, 2n-1. \tag{2.6}$$

Notice that system (2.6) is linear with respect to the parameters A_i and non-linear with respect to the parameters x_i.

System (2.6) appears in many mathematical and applied problems. First of all, if we want to approximate a given function $f(x)$ by an exponential sum

$$f(x) \approx C_1 e^{a_1 x} + C_2 e^{a_2 x} + \cdots + C_n e^{a_n x},$$

then the coefficients C_i and the values $\mu_i = e^{a_i}$ satisfy a system of the form (2.6) with the right-hand side (the "measurements") being the values of $f(x)$ at the integer points $x = 1, 2, \dots,$ (see [17, §4.9]). The method of solution of (2.6) which we give below is usually called Prony's method [36].

On the other hand, system (2.6), recurrence (2.7), and system (2.8) below form one of the central objects in Padé approximation, (see, in particular, [32] and the references therein).

System (2.6) appears also in error correction codes, in array processing (estimating the direction of signal arrival) and in other applications in signal processing (see, for example, [11, 29] and the references therein).

In [13, 18, 29], system (2.6) appears in the reconstruction of plane polygons from their complex moments. These results are shortly described in Section 3.2 below.

This system appears also in some perturbation problems in nonlinear model estimation.

We now give a sketch of the proof of solvability of (2.6) and of the solution method, which is, essentially, Prony's method. We follow the lines of [29]. See also the literature on Padé approximation, in particular, [32] and the references therein

Theorem 2.4. *A linear combination $g(x)$ of n δ-functions can be uniquely reconstructed from its first $2n-1$ moments $m_0(g), \dots, m_{2n-1}(g)$, via solving system* (2.6).

Representation (2.5) of the moments immediately implies the following result for the moments generating function $I_g(z)$.

Proposition 2.5. *For $g(x) = \Sigma_{i=1}^n A_i \delta(x - x_i)$, the moments generating function $I_g(z)$ is a rational function with the poles at x_i and with the residues A_i at these poles:*

$$I(z) = \Sigma_{i=1}^n \frac{A_i}{1 - z x_i}.$$

We see that the function $I(z)$ encodes the solution of system (2.6). So to solve this system it remains to find explicitly the rational function $I(z)$ from the first $2n$ of its Taylor coefficients $\alpha_0, \dots, \alpha_{2n-1}$. This is, essentially, the problem of Padé approximation [32].

Now we use the fact that the Taylor coefficients of a rational function of degree n satisfy a linear recurrence relation of the form

$$m_{r+n} = \Sigma_{j=0}^{n-1} C_j m_{r+j}, \quad r = 0, 1, \dots. \tag{2.7}$$

Since we know the first $2n$ Taylor coefficients $\alpha_0, \dots, \alpha_{2n-1}$, we can write a *linear* system on the unknown recursion coefficients C_l:

$$\Sigma_{j=0}^{n-1} C_j \alpha_{j+r} = \alpha_{n+r}, \quad r = 0, 1, \dots, n-1. \tag{2.8}$$

Solving linear system (2.8) with respect to the recurrence coefficients C_j, we find them explicitly. For a solvability of (2.8) see [17, 29, 32]. Now the recurrence relation (2.8) with known coefficients C_l and known initial moments allows us to easily reconstruct the generating function $I_g(z)$ and hence to solve (2.6).

REMARK. Another proof of Theorem 2.4 can be obtained in lines of the proof of Theorem 2.9 below. Indeed, a difference of two linear combinations of n δ-functions can have at most $2n-1$ "sign changes". Then we apply Lemma 2.10.

2.5. Piecewise-solutions of linear ODE's In this paper we do not consider separately the case of piecewise-polynomial functions. See [46] where a method for reconstruction of piecewise-polynomial functions from samplings is suggested (which starts with a reconstruction of linear combinations of δ-functions and of their derivatives). Instead we consider, as a natural generalization of piecewise-polynomial functions, the class A_D of piecewise-analytic functions, each piece satisfying a fixed linear differential operator D with rational coefficients. Such functions are usually called "L-splines" (see [43, 44] and the references therein). For piecewise-polynomial functions of degree d we have $D = \frac{d^{d+1}}{dx^{d+1}}$. Notice that Vetterli's method [46] can be extended also to our class A_D. However, in the present paper we stress another approach to the moment reconstruction problem for the class A_D. It is presented in an accompanying paper in this issue [24], while here we provide a necessary background.

Consider the equation

$$Dy = y^{(k)} + a_{k-1}(x)y^{(k-1)} + \cdots + a_1(x)y' + a_0(x)y = 0 \tag{2.9}$$

with the coefficients $a_{k-1}(x), \ldots, a_0(x)$ real-analytic and regular on $[0,1]$. All the solutions of (2.9) on $[0,1]$ form a linear space L_D with the basis $y_1(x), \ldots, y_k(x)$ being the fundamental set of solutions of (2.9). For $D = \frac{d^k}{dx^k}$ the space L_D consists of all the polynomials of degree at most $k-1$, and we can take

$$\{y_1(x), \ldots, y_k(x)\} = \{1, x, x^2, \ldots, x^{k-1}\}.$$

Now we consider the class A_D of all the piecewise-continuous functions $g(x)$ on $[0,1]$ with the jumps at $x_1, \ldots, x_n \in [0,1]$, such that on each continuity interval $\Delta_i = [x_i, x_{i+1}]$ the function $g(x)$ satisfies $Dg = 0$. We extend $g(x)$ by the identical zero outside the interval $[0,1]$.

We can represent $g(x)$ on the intervals Δ_i in a "polynomial form": $g(x) = \sum_{j=1}^{k} \alpha_{ij} y_j(x)$, where $y_1(x), \ldots, y_k(x)$ is the fundamental set of solutions of (2.9). Alternatively, we can parametrize $g(x)$ on the intervals Δ_i by its initial data at the point x_i. We can further define "splines" of a prescribed smoothness in A_D. The constructions of [46] can be extended to this case.

While up to this point we could restrict our presentation to the real domain, in what follows it will be necessary to extend the consideration to the complex plane.

First we recall shortly some classical facts related to the structure of linear differential equations in the complex domain (see, for example, [35, 42] where these facts are presented in a form convenient for our applications).

Consider the equation

$$Dy = y^{(k)} + a_{k-1}(x)y^{(k-1)} + \cdots + a_1(x)y' + a_0(x)y = 0 \tag{2.10}$$

with the coefficients $a_{k-1}(x), \ldots, a_0(x)$ regular and *univalued* in the complex domain $\Omega = \mathbb{C} \setminus \{x_0, \ldots, x_m\}$. We do not specify at this stage the character of possible singularities of $a_j(x)$ at the points $x_0, \ldots, x_m$.

The following proposition (see, for example, [42]) characterizes multivalued analytic functions which are solutions of a certain equation of the form (2.10).

Proposition 2.6. *Any solution $y(x)$ of* (2.10) *is a regular multivalued function in Ω, satisfying the following additional property* (F): *For any point $w \in \Omega$ the linear subspace L_w spanned by all the branches of $y(x)$ at w in the space $\mathcal{O}(w)$ of all the analytic germs at w, has dimension at most k.*

Any regular multivalued function $v(x)$ in Ω with the property (F) *satisfies a certain equation of the form* (2.10) *of order at most k with all the coefficients regular and univalued in the domain Ω.*

Let us recall that for a given function $g(x)$ on $[0, 1]$, the moment generating function $I_g(z) = \sum_{k=0}^{\infty} m_k(g)z^k$ is given by the Cauchy-type integral

$$I_g(z) = \int_0^1 \frac{g(t)dt}{1 - zt} = w \int_0^1 \frac{g(t)dt}{w - t}, \qquad w = \frac{1}{z}.$$

Now one of the basic classical facts about Cauchy-type integrals is that if g (on each its continuity intervals) satisfies a certain equation of the form (2.10), then $I_g(z)$ satisfies another equation of this form. A proof (in a specific case which we need in the present paper) can be found in [35, 42]. In these papers also specific ramification properties of $I_g(z)$ are studied for g algebraic.

Now, in the accompanying paper in this issue [24], the functions $g(x)$ from the class A_D are considered. A *non-homogeneous* equation of the form (2.10) for $I_g(z)$ is presented explicitly, and on this basis a reconstruction procedure is suggested.

2.6. Piecewise-algebraic functions Exact reconstruction of piecewise-algebraic (*i.e.,* semi-algebraic) functions can be considered as one of the ultimate goals of our approach. If we extend this class SA of semi-algebraic functions to $SA(\psi_1, \ldots, \psi_l)$, adding a finite number of fixed "models" $\psi_1, \ldots, \psi_l$ and allowing for all the elementary operations and for solving equations, we shall probably cover all the examples of interest. In particular, such extensions include linear combinations of shifts and dilations of $\psi_1, \ldots, \psi_l$, an important class appearing in reconstruction of signals with finite innovation rate [11, 27, 46], and in wavelet theory. Extensions of this sort are also closely related to what appears in theory of o-minimal structures, see, for example, [12]. Because of the "finiteness results" in this theory we can hope that the "finite moments determinacy" of semi-algebraic functions (Theorem 2.9 below) can be extended to at least some important classes $SA(\psi_1, \dots, \psi_l)$.

Let us recall that $g(x)$ is an algebraic function (as usual, restricted to $[0,1]$) if $y = g(x)$ satisfies an equation

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) = 0, \tag{2.11}$$

where $a_n(x), \dots, a_0(x)$ are polynomials in x of degree m. By definition, $d = m+n$ is the degree $\deg g$ of g.

We shall need the following simple properties of algebraic functions:

(i) The number of zeroes of an algebraic function $g(x)$ defined by (2.11) does not exceed m (and so it does not exceed $\deg g = m + n$).

(ii) A sum $g(x) = g_1(x) + g_2(x)$ of two algebraic functions of degrees d_1 and d_2 is an algebraic function, with $\deg g \le \eta(d_1, d_2)$.

We consider piecewise-algebraic functions on $[0,1]$. Let such a function $g(x)$ be represented by the algebraic functions $g_q(x)$ of the degrees d_q, respectively, on the intervals $\Delta_q = [x_q, x_{q+1}]$, $q = 0, \dots, r$, of the partition of $[0,1]$ by $x_0 = 0 < x_1 < \cdots < x_r < x_{r+1} = 1$. We define the *combinatorial complexity, (or the degree)* $\sigma(g)$ of g as follows:

Definition 2.7. The combinatorial complexity $\sigma(g)$ is the sum $\sum_{q=1}^{r} d_q + r$. See [47, 48].

The specific choice of this expression is motivated by the following simple observation: *the number of sign changes of a piecewise-algebraic function g on $[0,1]$ does not exceed $\sigma(g)$.* This follows directly from property (i) above.

We also need the following lemma.

Lemma 2.8. *Let g_1, g_2 be piecewise-algebraic functions with $\sigma(g_j) \le d$, $j = 1, 2$. Then for $g = g_1 \pm g_2$ the combinatorial complexity $\sigma(g)$ satisfies $\sigma(g) \le \kappa(d) = 2d(\eta(d,d) + 1)$, where $\eta(d,d)$ is given by property* (ii) *above.*

Proof. Observe that g has at most $2d$ jumps, and on each continuity interval its degree is bounded by $\eta(d,d)$. □

Now we can show that piecewise-algebraic functions are uniquely defined by their few moments. At this stage, we do not touch the question of how such a function can be actually reconstructed from the moments data, postponing this problem till Section 2.6.1.

Theorem 2.9. *A piecewise-algebraic function of a given combinatorial complexity d is uniquely defined by its first $\kappa(d)$ moments.*

Proof. Assume, contrary to the statement of the theorem, that there are functions g_1 and g_2 of complexity at most d, with exactly the same moments up to order $s = \kappa(d)$. Hence for the difference $g = g_2 - g_1 \neq 0$ we have the vanishing of the moments up to s: $m_j(g) = 0$, $j = 0, 1, \ldots, s$. By Lemma 2.8 we have for the combinatorial complexity of g the bound $\sigma(g) \leq s$. Consequently, the number of sign changes of g does not exceed s.

The next trick comes from classical moment theory.

LEMMA 2.10. *If the number of the sign changes and zeroes of $g(x) \neq 0$ do not exceed s, then some of its first s moments $m_j(g)$, $j = 0, 1, \ldots, s$ do not vanish.*

PROOF. We can assume that g changes its sign at certain points $t_1, \ldots, t_l$, $l \leq s$, and preserves the sign between these points. Let us construct a polynomial $Q(t)$ of degree l with exactly the same sign pattern as g: $Q(t) = \pm(x - t_1)(x - t_2)\cdots(x - t_l)$. Write Q as $Q(x) = \sum_1^l \alpha_j x^j$. We have $g(x)Q(x) > 0$ everywhere, except $t_1, \ldots, t_l$ and possibly some other isolated points. Therefore $\int_0^1 g(x)Q(x) > 0$. On the other hand, this integral can be expressed as a linear combination of the moments: $\int_0^1 g(x)Q(x) = \sum_1^l \alpha_j \int_0^1 x^j g(x)dx = \sum_1^l \alpha_j m_j(g)$. Hence some of the moments of g up to $l \leq s$-th do not vanish. This proves Lemma 2.10. □

To complete the proof of Theorem 2.9 it remains to notice that the difference $g = g_2 - g_1$ is nonzero on at least one of its continuity intervals. □

2.6.1. Explicit moment inversion for algebraic functions As far as an explicit inversion of the moment transform of algebraic functions is concerned, we are not aware of any general approach to this problem. Piecewise-algebraic functions belong to the class A_D, as defined in Section 2.3 above. However, the problem is that we do not know *a priori* the differential operator D which annihilates a given algebraic function g. (The form of D is known, but not the coefficients of the rational entries of D). This fact seems to prevent a direct application of the method of [24] to piecewise-algebraic functions.

Let us analyze in more detail one special case. Assume that the algebraic curve $y = g(x)$ is a rational one. This means that it allows for a rational parametrization $x = P(t)$, $y = Q(t)$. The moments $m_k(g)$ given by $m_k(g) = \int_0^1 x^k g(x)\,dx$, $k = 0, 1, \ldots$, now can be expressed as

$$m_k(g) = \int_a^b P^k(t)Q(t)p(t)\,dt, \tag{2.12}$$

where p denotes the derivative of P and $0 = P(a)$, $1 = P(b)$. Moments of this form naturally appear in a relation with some classical problems in qualitative theory of ODE's (see [2, 4, 5, 7, 33–35]).

Our problem can be reformulated now as the problem of explicitly finding P and Q from knowing a certain number of the moments m_k in (2.12).

Of course, in general we cannot expect this system of nonlinear equations to have a unique solution. Indeed, while the function $y = g(x)$ is determined by its moments in a unique way, the *rational parametrization* P, Q of this curve in general is not unique. In particular, let $W(t)$ be a rational function satisfying $W(0) = 0$, $W(1) = 1$. Substituting $W(t)$ into P and Q we get another rational parametrization of our curve:

$$x = \hat{P}(t), \quad y = \hat{Q}(t), \quad \text{with } \hat{P}(t) = P(W(t)), \hat{Q}(t) = Q(W(t)). \tag{2.13}$$

Consequently, we can ask the following question: are all the solutions of (2.12) related one to another via a composition transform (2.13)?

If the answer to this question is positive, we can restrict our parametrizations P, Q to be "mutually prime in composition sense" (see [41]) and thus to obtain uniqueness of the reconstruction.

More generally, the "inversion problem" for system (2.12) is to characterize all the solutions of system (2.12) and to provide an effective way to find these solutions.

A special case of the inversion problem, in which definite results have been recently obtained, is the "moment vanishing problem", *i.e.,* to characterize all the pairs P, Q for which the moments m_k defined by (2.12) vanish.

The moment vanishing problem plays a central role in study of the center conditions for the Abel differential equation (see [2, 5, 7, 33–35]). In fact, it provides an infinitesimal version of the Poincaré center-focus problem for the Abel equation. In spite of a very classical setting (we ask for conditions of orthogonality of pQ to all the powers of P!) this problem has been solved (for P and Q polynomials) only very recently [31]. Let us describe the solution.

We say that P and Q satisfy a "composition condition" if there are polynomials $\tilde{P}(w)$ and $\tilde{Q}(w)$, and a polynomial $W(x)$, satisfying $W(0) = W(1)$, such that

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)). \tag{2.14}$$

Composition condition (2.14) can be easily shown to imply the vanishing of all the moments (2.12). In many cases it is also a necessary one, but not always. The examples of P, Q annihilating the moments (2.12) but not satisfying (2.14) can be obtained as follows (see [33]): if P has two right composition factors $W_1(x)$ and $W_2(x)$, then P and $Q = W_1 + W_2$ will annihilate the moments (2.12) because of a linearity with respect to Q. For some P we can find W_1 and W_2 which are mutually prime in composition algebra (see [41]). Then typically P and $Q = W_1 + W_2$ will have no common right composition factors [33]. The result of [31] claims that this is essentially the only possibility.

THEOREM 2.11 ([31]). *All the moments* (2.12) *vanish if and only if Q is a sum of Q_j, $j = 1, \ldots, l$, such that for each j the polynomials P and Q_j satisfy composition condition* (2.14).

One can expect that the methods developed in [2, 5, 7, 31, 33–35] can help in further analyzing the reconstruction problem for semi-algebraic functions in one and more variables. See, in particular, Section 3 below.

3. Functions of two variables

Also in two dimensions exact reconstruction of semi-algebraic functions (and of their extension to $SA(\psi_1, \dots, \psi_l)$) can be considered as one of the ultimate goals of our approach.

3.1. Reconstruction of polygons from complex moments In [13,18,29] the problem of reconstruction of a planar polygon from its complex moments is considered. The complex moments of a function $f(x, y)$ are defined as

$$\mu_k(f) = \iint z^k f(x, y)\, dxdy, \quad k = 0, 1, \dots,\ z = x + iy.$$

Complex moments can be expressed as certain specific linear combinations of the real double moments $m_{kl}(f)$.

For a plane subset A, its complex moments $\mu_k(A)$ are defined by $\mu_k(A) = \mu_k(\chi_A)$, where χ_A is the characteristic function of A.

Let P be a closed n-sided planar polygon with the vertices z_i, $i = 1, \dots, n$. The reconstruction method of [29] is based on the following result of [8].

THEOREM 3.1. *There exists a set of n coefficients a_i, $i = 1, \dots, n$, depending only on the vertices z_i, such that for any analytic function $\phi(z)$ on P we have*

$$\iint_P \phi''(z)\, dxdy = \sum_{i=1}^{n} a_i \phi(z_i).$$

The coefficients a_j, $j = 1, \dots, n$ are given as

$$a_j = \frac{1}{2}\left(\frac{\bar z_{j-1} - \bar z_j}{z_{j-1} - z_j} - \frac{\bar z_j - \bar z_{j+1}}{z_j - z_{j+1}}\right)$$

.

Applying this formula to $\phi(z) = z^k$ we get

$$k(k-1)\mu_{k-2}(\chi_P) = \sum_{i=1}^{n} a_i z_i^k, \quad k = 0, 1, \dots, \tag{3.1}$$

where we put $\mu_{-2} = \mu_{-1} = 0$. So on the left-hand side we have shifted moments of P.

If we ignore the fact that a_j can be expressed through z_i and consider both a_j and z_i as unknowns, we get from (3.1) a system of equations

$$\sum_{i=1}^{n} a_i z_i^k = \nu_k, \quad k = 0, 1, \dots, \tag{3.2}$$

where ν_k denotes the "measurement" $k(k-1)\mu_{k-2}(P)$. System (3.2) is identical to system (2.6) which appears in the reconstruction of linear combination of δ-functions. One of the solution methods suggested in [29] is the Prony method described in Section 2.4 above. Another approach is based on matrix pencils. In [13, 18] an important question was investigated concerning polygon reconstruction from noisy data.

3.2. Quadrature domains We introduce, following [19], a slightly different sequence of double moments: for a function $g(z)=g(x+iy)$ the moments $\tilde{m}_{kl}(g)$ are defined by

$$\tilde{m}_{kl}(g)=\iint z^k\bar{z}^l g(z)\,dxdy,\quad k,l\in\mathbb{N}.$$

One defines the moment generating function $I_g(v,w)=\sum_{k,l=0}^{\infty}\tilde{m}_{kl}(g)v^kw^l$ and the "exponential transform"

$$\begin{aligned}\tilde{I}_g(v,w)&=1-\exp\Big(-\frac{1}{\pi}I_g(v,w)\Big)\\&=\exp\Big(-\frac{1}{\pi}\iint_{\Omega}\frac{g(z)dxdy}{(z-v)(\bar{z}-w)}\Big):=\sum_{k,l=0}^{\infty}b_{kl}(g)v^kw^l.\end{aligned}$$

Now (classical) quadrature domains in $\mathbb{C}$ are defined as follows.

Definition 3.2. A quadrature domain $\Omega\subset\mathbb{C}$ is a bounded domain with the property that there exist points $z_1,\dots,z_m\in\Omega$ and coefficients c_{ij}, $i=1,\dots,m, j=0,\dots,s_i-1$, so that for all analytic integrable functions $f(z)$ in Ω we have

$$\iint_{\Omega}f(x+iy)\,dxdy=\sum_{i=1}^{m}\sum_{j=0}^{s_i-1}c_{ij}f^{(j)}(z_i).\tag{3.3}$$

$N=s_1+\cdots+s_m$ is called the order of the quadrature domain Ω.

The simplest example is provided by the disk $D_R(0)$ of radius R centered at $0\in\mathbb{C}$: $\iint_{D_R(0)}f(x+iy)\,dxdy=\pi R^2f(0)$. The results of Davis ([8]; Theorem 3.1 above) give another example in this spirit.

The following result ([19,20], Theorem 3.1) provides a necessary and sufficient condition for $\Omega\subset\mathbb{C}$ to be a quadrature domain: let $\tilde{I}_\Omega(v,w)=\tilde{I}_{\chi\Omega}(v,w)$ be the exponential transform of Ω.

Theorem 3.3. *Ω is a quadrature domain if and only if there exists a polynomial $p(z)$ with the property that the function $\tilde{q}(z,\bar{w})=p(z)\bar{p}(w)\tilde{I}_\Omega(z,\bar{w})$ is a polynomial at infinity (denoted by $q(z,\bar{w})$). In that case, by choosing $p(z)$ of minimal degree, the domain Ω is given by $\Omega=\{z\in\mathbb{C},\ q(z,\bar{z})<0\}$. Moreover, the polynomial $p(z)$ in this case is given by $p(z)=\prod_{i=1}^{m}(z-z_i)^{s_i}$, where z_i are the quadrature nodes of Ω.*

Now, the algorithm in [19] for reconstruction of a quadrature domain from its moments consists of the following steps:

(i) Given the moments $\tilde{m}_{kl}(\Omega) = \tilde{m}_{kl}(\chi_\Omega)$ up to a certain order, construct the (truncated) exponential transform $\tilde{I}(v,w) = \sum_{k,l=0}^{\infty} b_{kl} v^k w^l$.

(ii) Identify the minimal integer N such that $\det(b_{k,l})_{k,l=0}^{N} = 0$. Then there are coefficients α_j, $j = 0,\ldots,N-1$, such that for $B = (b_{k,l})_{k,l=0}^{N}$ and $\alpha = (\alpha_1,\ldots,\alpha_{N-1},1)^T$ we have

$$B\alpha = 0. \tag{3.4}$$

We solve this system with respect to α. Then the polynomial $p(z)$ defined above is given by $p(z) = z^N + \alpha_{N-1} z^{N-1} + \cdots + \alpha_0$.

(iii) Construct the function

$$R_\Omega(z,\bar{w}) = p(z)\bar{p}(w)\exp\Big(-\frac{1}{\pi}\sum_{k,l=0}^{N-1}\tilde{m}_{kl}(\Omega)\frac{1}{z^{k+1}}\frac{1}{\bar{w}^{l+1}}\Big)$$

and identify $q(z,\bar{w})$ as the part of $R_\Omega(z,\bar{w})$ which does not contain negative powers of z and $\bar{w}$. Then the domain Ω is given by

$$\Omega = \{z \in \mathbb{C},\ q(z,\bar{z}) < 0\}.$$

Remark. Let us substitute into the definition of the quadrature domain (formula (3.3) above) $f(z) = z^k$. Assuming that all the quadrature nodes z_i are simple, we get for the complex moments $\tilde{m}_{k,0}(\Omega) = m_k(\Omega)$ the expression

$$m_k(\Omega) = \sum_{i=1}^{m} c_i z_i^k,$$

which is identical to (3.2) in reconstruction of planar polygons. So we can reconstruct the quadrature nodes z_i and the coefficients c_i from the complex moments only, and we get once more a complex system which is identical to (2.6). Allowing quadrature nodes z_i of an arbitrary order, we get a system corresponding to a linear combination of δ-functions and their derivatives (*cf.* [28, 46]).

Notice that system (3.4) that appears in step (ii) of the reconstruction algorithm above is very similar to system (2.8) in the solution process of (2.6).

3.3. δ-functions along algebraic curves As we have mentioned above, a natural class of functions $f(x,y)$ of two variables, for which we can hope for an explicit reconstruction from a finite number of the moments $m_{kl}(f) = \iint x^k y^l f(x,y)\,dxdy$, $k,l = 0,1,\ldots,$ consists of semi-algebraic functions. Those are piecewise-algebraic functions with the continuity pieces bounded by piecewise-algebraic curves. Among semi-algebraic functions are piecewise-polynomial functions with the continuity pieces bounded by spline curves, a very natural and convenient object in constructive approximation.

Most of the methods presented in Section 2 for functions of one variable are applicable also in the case of two variables. In particular, generalizing the approach of [46] we can differentiate piecewise-polynomial functions a sufficient number of times and finally get a combination of weighted δ functions along the partition curves. See also [28].

In this paper we restrict ourselves to a discussion of only one example. Assume that $f(x,y)$ is a δ-function δ_S along a rational curve S, *i.e.,* for any $\psi(x,y)$ we have $\iint f\psi\,dxdy = \int_S \psi(x,y)\,dx$. Let

$$x = P(t), \quad y = Q(t), \quad t \in [0,1]$$

be a rational parametrization of S. The moments now can be expressed as

$$m_{kl}(f) = \int_0^1 P^k(t)Q^l(t)p(t)\,dt,$$

where p denotes the derivative P' of P. This system is an extension of system (2.12): here we are allowed to use all the double moments, while in (2.12) only the moments m_{k1} are available.

Also here we notice that a rational parametrization P, Q of the curve S in general is not unique: for any rational function $W(t)$ satisfying $W(0) = 0$, $W(1) = 1$ we get another rational parametrization of our curve:

$$x = \hat{P}(t), \quad y = \hat{Q}(t), \quad \text{with } \hat{P}(t) = P(W(t)), \hat{Q}(t) = Q(W(t)). \tag{3.5}$$

Consequently, we can reiterate the question in Section 2.6 with better chances for a positive answer: are all the solutions of (3.5) related one to another via a composition transform (2.13)?

For the "Moment vanishing problem" for (3.5) a definite answer has been obtained in [35]: *composition condition* (2.14) *is necessary and sufficient for the moments vanishing.*

Let us now assume that the curve S is closed and that it can be parametrized by $x = P(t)$, $y = Q(t)$, with t in the unit circle S^1. The study of the double moments of this form brings us naturally to the recent work of G. Henkin [9, 21, 22]. Indeed, the vanishing condition for the moments $m_{kl}(f)$ is given by Wermer's theorem [1]: $m_{kl}(f) \equiv 0$ if and only if S bounds a complex 2-chain in $\mathbb{C}^2$. See [2] for a simple interpretation of Wermer's condition in the case of rational P, Q. In general, if the moments $m_{kl}(f)$ do not vanish identically, then the local germ of the complex analytic curve $\hat{S}$ generated by S in $\mathbb{C}^2$ does not "close up" inside $\mathbb{C}^2$. Henkin's work [9, 21, 22], in particular, analyzes various possibilities of this sort in terms of the "moments generating function". We expect that a proper interpretation of the results of [9, 21, 22] can help also in understanding of the moment inversion problem.

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