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ON THE VOLUME OF UNIT VECTOR FIELDS ON RIEMANNIAN THREE-MANIFOLDS

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ABSTRACT. H. Gluck and W. Ziller [5] proved that the Hopf vector fields, namely, the unit Killing vector fields, are the unique unit vector fields on the unit sphere S^3 that minimize the functional volume. The authors proved this important and famous result by using the method of "calibrated geometries" of Federer and Harvey–Lawson. In this paper, by using a different method, we get an analogue of Gluck and Ziller's theorem for a compact Sasakian three-manifold with Webster scalar curvature $w \geq 1$. Moreover, our method gives a new proof of Gluck and Ziller's theorem. We also extend a theorem of F. Brito [2] about the energy of unit vector fields.

RÉSUMÉ. H. Gluck et W. Ziller [5] prouvèrent que les champs de Hopf, c'est-á-dire, les champs vectoriels unitaires de Killing, sont les seuls champs vectoriels unitaires sur la sphére unitaire S^3 que minimisent le volume fonctionnel. Ils prouvèrent cet résultat important en utilisant la méthode des "géométries calibrées" de Federer et Harvey–Lawson. Dans cet article, en utilisant une méthode différente, nous obtenons l'analogue du théorème de Gluck et Ziller pour une 3-variété compact de Sasaki avec courbure scalaire de Webster $w \geq 1$. En outre, notre méthode donne une nouvelle démonstration du théorème de Gluck et Ziller. Nous aussi étendons un théorème de Brito [2] concernant l'énergie de champs vectoriels unitaires.

1. **Introduction.** Let (M, g) be a compact Riemannian manifold and (T^1M, g_s) its unit tangent sphere bundle equipped with the Sasaki metric g_s. Furthermore, let $\mathfrak{X}^1(M)$ be the set of all smooth unit vector fields on M which we suppose to be non-empty, equivalently, the Euler–Poincaré characteristic of M vanishes. A unit vector field $U \in \mathfrak{X}^1(M)$ determines a map between (M, g) and (T^1M, g_s) and the volume of U is defined (see [5]) as the volume of the submanifold $U(M)$ of (T^1M, g_s). A formula for the volume of U is given by

$$\mathrm{vol}(U) = \int_M \sqrt{\det\big(I + (\nabla U)^t \circ \nabla U\big)}\, v_g,$$

where v_g is the canonical measure on (M, g) and ∇ denotes the Levi–Civita connection. It follows from that formula that $\mathrm{vol}(U) \geq \mathrm{vol}(M)$ and equality holds

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if and only if U is parallel. But a parallel unit vector field doesn't always exist, since such a vector field determines two mutually orthogonal, complementary and totally geodesic foliations. Therefore, Herman Gluck and Wolfgang Ziller posed the following question [5, p. 179]: "If M admits no parallel vector fields, is $\mathrm{vol}(U)$ bounded away from $\mathrm{vol}(M)$ for any $U \in \mathfrak{X}^1(M)$?" In the class of compact Riemannian manifolds which do not admit a parallel vector field, the simplest ones are the unit spheres of odd dimension. Then, they proved the following theorem.

Theorem (Gluck and Ziller [5]). *The unit vector fields of minimum volume on the unit sphere S^3 are precisely the Hopf vector fields, equivalently, the unit Killing vector fields, and no others.*

This much-cited theorem motivated the study of the volume, and later of the energy of unit vector fields, on various Riemannian manifolds, mainly the critical points of these functionals (see the abundant bibliography on the subject for instance in the references of [4]). More recently, F. Brito [2] proved the analogue of Gluck and Ziller's theorem for the energy instead of the volume. He proved the uniqueness part of his theorem by applying the uniqueness part of Gluck and Ziller's theorem. An interesting class of compact Riemannian three-manifolds which do not admit a parallel vector field and in which a distinguished unit Killing vector field appears in a natural way, is given by the class of compact Sasakian three-manifolds where the Reeb vector field is a unit Killing vector field. On the other hand, a Hopf vector field on the unit sphere S^3 is precisely the Reeb vector field of a natural Sasakian structure on S^3. Therefore, it is natural to study the Gluck and Ziller question on a compact Sasakian three-manifold. Gluck and Ziller proved their result by using the method of "calibrated geometries" of Federer and Harvey–Lawson. In this paper, by using a completely different method, we get an analogue of Gluck and Ziller's theorem in the Sasakian case. In fact, we obtain the following.

Theorem 1. *Let (M, g, ξ, η) be a compact Sasakian three-manifold with Webster scalar curvature $w \geq 1$. Then, the Reeb vector field ξ minimises the volume, $\mathrm{vol}(\xi) = 2\,\mathrm{vol}(M)$, and the unit vector fields of minimum volume are precisely the unit Killing vector fields eigenvectors of the Ricci operator with eigenvalue 2, and no others.*

We also extend a theorem of F. Brito [2] about the energy of unit vector fields for compact Sasakian three-manifold (*cf.* Section 5, Theorem 3).

Moreover, our method gives a new proof of Gluck and Ziller's theorem for a compact Riemannian three-manifold of constant sectional curvature $c \geq 0$ (*cf.* Section 3).

Remark 1. Regarding the Webster scalar curvature, the main result of Chern and Hamilton [3] says that a compact contact three-manifold (M, η) admits a contact metric g whose Webster scalar curvature w is either > 0 or it is a constant ≤ 0. Now, let (M, g, ξ, η) be a compact Sasakian three-manifold with Webster scalar curvature $w > 0$. Consider the deformation $g_t = tg + (t^2 - t)\eta \otimes \eta$, $\eta_t = t\eta$, $\xi_t = \frac{1}{t}\xi$, where $0 < t \leq w_0 = \inf\{w(p) : p \in M\} > 0$. (g_t, η_t, ξ_t) is also a Sasakian structure with Webster scalar curvature w_t given by (see [1, p. 173]), $w_t = \frac{1}{t}w \geq \frac{1}{t}w_0 \geq 1$. Then, (M, g_t, η_t, ξ_t) is a compact Sasakian three-manifold satisfying the conditions of Theorems 1 and 3.

2. **Preliminaries on contact metric manifolds.** In this section, we collect some basic facts about contact metric manifolds. A $(2n+1)$-dimensional manifold M is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. Given η, there exists a unique vector field ξ, called the *Reeb vector field* or the *characteristic vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Furthermore, a Riemannian metric g is said to be an *associated metric* if there exists a tensor ϕ of type $(1,1)$ such that

$$\eta = g(\xi, \cdot), d\eta(\cdot, \cdot) = g(\cdot, \phi\cdot), \phi^2 = -I^2 + \eta \otimes \xi.$$

(η, g, ξ, ϕ), or (η, g), is called a *contact metric structure* and (M, η, g) a *contact metric manifold*. We denote by ∇ the Levi–Civita connection and by R the corresponding Riemannian curvature tensor of (M, g) with the sign convention

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

Moreover, we denote by Ric the Ricci tensor of type $(0,2)$, by Q the corresponding endomorphism field and by r the scalar curvature.

A contact metric manifold is said to be a *K-contact manifold* if ξ is a Killing vector field. Moreover, the condition $Q\xi = 2n\xi$ characterizes the K-contact manifolds in the class of all contact metric $(2n+1)$-manifolds. A *contact metric manifold* (M, η, g, ξ, ϕ) is said to be a *Sasakian manifold*, or a *normal contact metric manifold*, if the almost complex structure J on $M \times R$ defined by $J(X, f\frac{d}{dt}) = \big(\phi X - f\xi, \eta(X)\frac{d}{dt}\big)$ is integrable. Equivalently, a contact metric structure (ξ, η, ϕ, g) is a Sasakian structure if and only if

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X.$$

Any Sasakian manifold is K-contact and the converse also holds when $n = 1$, *i.e.*, in the 3-dimensional case. Sasakian manifolds have been studied extensively, they may be viewed as odd-dimensional analogues of Kähler manifolds. The unit odd-dimensional sphere is a classical example of Sasakian manifold. For a three-dimensional Sasakian manifold, the Ricci tensor is given by (see Tanno [10], or [1, p. 105])

$$\text{Ric} = ag + b\eta \otimes \eta, \quad \text{where } a, b \text{ are functions,} \tag{2.1}$$

and the *Webster scalar curvature* w is given by (see Chern–Hamilton [3, p. 284] or [1, p. 171])

$$8w = \big(r - \text{Ric}(\xi, \xi) + 4\big) = (r + 2). \tag{2.2}$$

For the unit sphere S^3, the Webster scalar curvature $w = 1$. We refer to [1] for more information and details about contact metric geometry.

3. The main proposition.

In [2], F. Brito proved the following proposition.

PROPOSITION. *Let (M, g) be a compact Riemannian three-manifold. Then*

$$\mathrm{vol}(U) \geq \mathrm{vol}(M) + \int_M \sigma_2(U^\perp) v_g = \int_M \Big(1 + \frac{1}{2}\,\mathrm{Ric}(U,U)\Big) v_g \tag{3.1}$$

for all $U \in \mathfrak{X}^1(M)$, where $\sigma_2(U^\perp)$ is the second elementary symmetric function of the second fundamental form of the distribution orthogonal to U.

Nevertheless, the case of the equality in (3.1) it is not considered. In this section we consider that case and prove the following proposition, from which we will get Theorem 1.

MAIN PROPOSITION. *Let (M, g) be a compact Riemannian three-manifold and U a unit vector field eigenvector of the Ricci operator Q with eigenvalue λ. Then the following properties are equivalent:*

(a) *the eigenvalue λ is constant along the integral curves of U and*

$$\mathrm{vol}(U) = \int_M \Big(1 + \frac{1}{2}\,\mathrm{Ric}(U,U)\Big) v_g; \tag{3.2}$$

(b) *U is Killing and $\lambda = const \geq 0$.*

Moreover, if (a) holds, then:

(i) *U is parallel or, equivalently, the distribution $U^\perp$ is integrable and M is locally a product of a two-dimensional integral manifold and the integral curves of U, or*

(ii) *(M, g) is homothetic to a Sasakian manifold, more precisely, $(\bar{g}, \xi, \eta, \varphi)$ is a Sasakian structure, where $\bar{g} = c^2 g$, $\xi = \frac{1}{c}U$, $\eta = \bar{g}(\xi, \cdot)$, $\varphi = -\nabla\xi$ and $c = \sqrt{\lambda/2} > 0$.*

PROOF. Let U be a unit vector field. We consider a local orthonormal basis $\{e_1, e_2, e_3 = U\}$ and put

$$V = \nabla_U U = \sum_i V_i e_i, \quad V_i = g(\nabla_U U, e_i),$$

$$S_{ij} = g(\nabla_{e_i} U, e_j), \quad \|S\|^2 = \sum_{i,j=1}^{3} S_{ij}^2.$$

Notice that $V_3 = S_{33} = 0$ and $S_{i3} = 0$ for any $i = 1, 2, 3$. The volume of U is given by (see [7])

$$\operatorname{vol}(U) = \int_M \Big((1+\sigma_2)^2 + (\|S\|^2 - 2\sigma_2) + \|V\|^2 + (S_{11}V_2 - S_{12}V_1)^2 + (S_{21}V_2 - S_{22}V_2)^2\Big)^{1/2} v_g,$$

where $\sigma_2 := \sigma_2(U^\perp) = \frac{1}{2}\big((\operatorname{div} U)^2 - \operatorname{tr}(\nabla U \circ \nabla U)\big)$ satisfies (see [9, p. 170])

$$\int_M \sigma_2 v_g = \frac{1}{2}\int_M \operatorname{Ric}(U,U) v_g.$$

Since

$$\|S\|^2 = S_{11}^2 + S_{22}^2 + S_{12}^2 + S_{21}^2, \quad (\operatorname{div} U)^2 = \Big(\sum_{i=1}^{3} g(\nabla_{e_i} U, e_i)\Big)^2 = (S_{11} + S_{22})^2,$$

and

$$\operatorname{tr}(\nabla U \circ \nabla U) = \sum_{i,k=1}^{3} g(\nabla_{e_i} U, e_k) g(\nabla_{e_k} U, e_i) = S_{11}^2 + S_{22}^2 + 2S_{12}S_{21},$$

we get

$$(\|S\|^2 - 2\sigma_2) = (S_{11} - S_{22})^2 + (S_{12} + S_{21})^2 \geq 0$$

and hence the inequality (3.1). Consequently,

$$\operatorname{vol}(U) = \int_M \Big(1 + \frac{1}{2}\operatorname{Ric}(U,U)\Big) v_g$$

if and only if

$$\nabla_U U = 0, \quad S_{11} = S_{22} \quad \text{and} \quad S_{12} + S_{21} = 0. \tag{3.3}$$

Now, assume that U is a unit vector field which is an eigenvector of the Ricci operator: $QU = \lambda U$. Then we prove (a) $\Rightarrow$ (b), that is, $U(\lambda) = 0$ and (3.3) imply that U is Killing and λ is a nonnegative constant. Since

$$(\mathcal{L}_U g)(e_i, e_j) = S_{ij} + S_{ji},$$

where $\mathcal{L}_U$ denotes the Lie derivative, we have to show that $S_{11} = S_{22} = 0$. Put

$$f_1 := S_{11} = S_{22}, \quad f_2 := S_{12} = -S_{21}, \quad \alpha = g(\nabla_U e_1, e_2)$$
$$\beta = g(\nabla_{e_1} e_2, e_1) \quad \text{and} \quad \gamma = g(\nabla_{e_2} e_2, e_1).$$

Then we have the following table of covariant derivatives:

$$\begin{aligned}
&\nabla_{e_1}U = f_1e_1 + f_2e_2, \qquad \nabla_{e_2}U = -f_2e_1 + f_1e_2, \qquad \nabla_U U = 0,\\
&\nabla_{e_1}e_1 = -f_1U - \beta e_2, \qquad \nabla_{e_2}e_1 = f_2U - \gamma e_2, \qquad \nabla_U e_1 = \alpha e_2,\\
&\nabla_{e_1}e_2 = -f_2U + \beta e_1, \qquad \nabla_{e_2}e_2 = -f_1U + \gamma e_1, \qquad \nabla_U e_2 = -\alpha e_1,\\
&[e_1,U] = f_1e_1 + (f_2-\alpha)e_2, \quad [e_2,U] = (\alpha - f_2)e_1 + f_1e_2,\\
&[e_1,e_2] = -2f_2U + \beta e_1 + \gamma e_2.
\end{aligned}$$

By using the table of covariant derivatives, we get:

$$\begin{aligned}
R(e_1,U)U &= \nabla_U(f_1e_1+f_2e_2) + f_1\nabla_{e_1}U + (f_2-\alpha)\nabla_{e_2}U\\
&= \big(U(f_1)+f_1^2-f_2^2\big)e_1 + \big(U(f_2)+2f_1f_2\big)e_2;
\end{aligned}\tag{3.4}$$

$$\begin{aligned}
R(e_2,U)U &= \nabla_U(-f_2e_1+f_1e_2) + (\alpha-f_2)\nabla_{e_1}U + f_1\nabla_{e_2}U\\
&= -\big(U(f_2)+2f_1f_2\big)e_1 + \big(U(f_1)+f_1^2-f_2^2\big)e_2;
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
R(e_1,e_2)U &= -\nabla_{e_1}(-f_2e_1+f_1e_2) + \nabla_{e_2}(f_1e_1+f_2e_2) + \beta\nabla_{e_1}U + \gamma\nabla_{e_2}U\\
&= \big(e_1(f_2)+e_2(f_1)\big)e_1 + \big(-e_1(f_1)+e_2(f_2)\big)e_2.
\end{aligned}\tag{3.6}$$

Since

$$\lambda = g(QU,U) = \mathrm{Ric}(U,U) = R(e_1,U,e_1,U) + R(e_2,U,e_2,U),$$

from (3.4) and (3.5), we obtain

$$U(f_1) + f_1^2 - f_2^2 = -\frac{\lambda}{2}. \tag{3.7}$$

Since

$$R(e_1,e_2,U,e_2) = \mathrm{Ric}(e_1,U) = 0 \quad \text{and} \quad R(e_1,e_2,U,e_1) = -\mathrm{Ric}(e_2,U) = 0,$$

from (3.6), we obtain

$$e_1(f_2) + e_2(f_1) = 0 \quad \text{and} \quad e_1(f_1) - e_2(f_2) = 0. \tag{3.8}$$

Now, by using (3.7), (3.8) and the condition $U(\lambda) = 0$, we get

$$\begin{aligned}
\sum_{i=1}^{3} e_i\, e_i(f_1) &= U\Big(f_2^2 - f_1^2 - \frac{\lambda}{2}\Big) + e_1\, e_2(f_2) - e_2\, e_1(f_2)\\
&= -2f_1\, U(f_1) + 2f_2\, U(f_2) + [e_1,e_2](f_2)\\
&= -2f_1\, U(f_1) + \beta\, e_1(f_2) + \gamma\, e_2(f_2).
\end{aligned}$$

Moreover,

$$(\nabla_U U)(f_1) = 0, \quad (\nabla_{e_1} e_1)(f_1) = -f_1 U(f_1) - \beta\, e_2(f_1),$$
$$(\nabla_{e_2} e_2)(f_1) = -f_1 U(f_1) + \gamma\, e_1(f_1).$$

Then, if Δ is the Laplace–Beltrami operator acting on functions, we get

$$\Delta(f_1) = -\operatorname{tr} \nabla^2 f_1 = -\sum_{i=1}^{3} \big(e_i e_i(f_1) - (\nabla_{e_i} e_i)(f_1)\big) = 0,$$

that is, f_1 is a harmonic map. Since f_1 is a smooth function globally defined on M and M is compact and connected, then f_1 must be a constant. On the other hand f_1 is a divergence ($\operatorname{div} U = 2f_1$), therefore $f_1 = 0$ and hence U is Killing. Moreover, it follows from (3.7) and (3.8) that f_2 is constant and $\lambda = 2f_2^2 = \text{constant} \geq 0$. Conversely, if U is Killing and $\lambda = \text{const} \geq 0$, we easily get (3.3) and hence (3.2).

Now, we show the second part of proposition. Let (M, g) be a compact Riemannian three-manifold and U a unit vector field eigenvector of the Ricci operator Q with eigenvalue λ, satisfying (a). Suppose $f_2 \leq 0$ and put $c = -f_2 = \sqrt{\lambda/2}$ (if $f_2 \geq 0$, we put $c = f_2$). By the table of covariant derivatives, we have

$$\nabla_{e_1} U = -c\, e_2, \quad \nabla_{e_2} U = c\, e_1, \quad \nabla_U U = 0.$$

If $f_2 = 0$, we have that U is parallel and hence we get (i). If $f_2 < 0$, we consider the tensors

$$\bar{g} = c^2\, g, \quad \xi = \frac{1}{c} U, \quad \eta = c\, g(U, \cdot) \quad \text{and} \quad \varphi = -\nabla \xi,$$

where ∇ is the Levi–Civita connection of g and $\bar{g}$. These tensors satisfy the following properties:

$$\eta(\xi) = c\, g(U, \xi) = 1. \tag{3.9}$$

$$\varphi(e_1) = -\nabla_{e_1} \xi = -\frac{1}{c} \nabla_{e_1} U = e_2, \quad \varphi(e_2) = -\nabla_{e_2} \xi = -\frac{1}{c} \nabla_{e_2} U = -e_1$$

and hence

$$\varphi^2(X) = -X \quad \forall X \in \ker \eta.$$

Then we get

$$\varphi^2 = -I + \eta \otimes \xi. \tag{3.10}$$

Moreover, since $\varphi = -\nabla \xi$ is skew-symmetric, we obtain

$$\begin{aligned} 2(\mathrm{d}\,\eta)(X, Y) &= X\eta(Y) - Y\eta(X) - \eta([X, Y]) \\ &= X\bar{g}(\xi, Y) - Y\bar{g}(\xi, X) - \bar{g}(\xi, [X, Y]) = 2\bar{g}(X, -\nabla_Y \xi) \\ &= 2\bar{g}(X, \varphi Y) \end{aligned} \tag{3.11}$$

and

$$\bar{g}(\varphi X, \varphi Y) = -\bar{g}(\varphi^2 X, Y) = -\bar{g}\big(-X + \eta(X)\xi, Y\big) = \bar{g}(X, Y) - \eta(X)\eta(Y). \tag{3.12}$$

Due to properties (3.9), (3.10), (3.11), and (3.12), the tensors $(\bar{g}, \xi, \eta, \varphi)$ define a contact metric structure on M [1, p. 36]). Moreover, such structure is Sasakian because ξ is Killing and M is three-dimensional. □

As an immediate consequence of the Main Proposition, we get the Gluck and Ziller's theorem that can reformulate in the following way.

THEOREM 2. *Let (M, g) be a compact Riemannian three-manifold of constant sectional curvature $c \geq 0$. Then the unit vector fields of minimum volume on M (equal to $(1 + c)\,\mathrm{vol}(M)$) are precisely the unit Killing vector fields, and no others.*

4. **Proof of Theorem 1.** Let $(M, g, \eta, \xi, \varphi)$ a compact Sasakian three-manifold. Then the Reeb vector field ξ is a Killing unit vector field eigenvector of the Ricci operator with $Q\xi = 2\xi$. Moreover, in such case, the Ricci operator is given, see (2.1), by

$$Q = a\, I + b\, \eta \otimes \xi, \quad \text{where } a, b \text{ are functions.} \tag{4.1}$$

Since $a + b = \mathrm{Ric}(\xi, \xi) = 2$, $r = \mathrm{tr}\, Q = 3a + b$, by using the Webster scalar curvature w defined by (2.2), we obtain $a = 2(2w-1)$ and $b = 4(1-w)$. Besides, the eigenvalues of Q are $\lambda_1 = 2 = \mathrm{Ric}(\xi, \xi)$, $\lambda_2 = \lambda_3 = 2(2w - 1)$. Thus, if $w \geq 1$, we obtain $\mathrm{Ric}(U, U) \geq 2$ for any $U \in \mathfrak{X}^1(M)$. We have to show that

$$\mathrm{vol}(U) \geq \mathrm{vol}(\xi) = 2\,\mathrm{vol}(M) \quad \forall U \in \mathfrak{X}^1(M),$$

and

$$\mathrm{vol}(U) = \mathrm{vol}(\xi) \quad \text{if and only if } U \text{ is Killing and } QU = 2U.$$

Since ξ is Killing and $Q\xi = 2\xi$, the Main Proposition gives $\mathrm{vol}(\xi) = 2\,\mathrm{vol}(M)$. By using the condition $\mathrm{Ric}(U, U) \geq 2 = \mathrm{Ric}(\xi, \xi)$ and the inequality (3.1), we get

$$\mathrm{vol}(U) \geq \int_M \Big(1 + \frac{1}{2}\,\mathrm{Ric}(U, U)\Big) v_g \geq \mathrm{vol}(\xi).$$

If U is a unit Killing vector field and $QU = 2U$, by the Main Proposition, we have $\mathrm{vol}(U) = \mathrm{vol}(\xi)$. Now, let U be a unit vector field such that $\mathrm{vol}(U) = \mathrm{vol}(\xi)$, that is,

$$\int_M \big(\mathrm{Ric}(U, U) - 2\big) v_g = 0 \quad \text{with } \mathrm{Ric}(U, U) \geq 2.$$

Then $\mathrm{Ric}(U,U) = 2 = \mathrm{Ric}(\xi,\xi)$ and hence $\big(a + b\eta^2(U)\big) = (a+b)$, that is,

$$\big(\eta^2(U) - 1\big)b = 0. \tag{4.2}$$

Since $\mathrm{vol}(U) = 2\,\mathrm{vol}(M) = \int_M \big(1 + \frac{1}{2}\,\mathrm{Ric}(U,U)\big)v_g$, we get (3.3). Then, as in the proof of the Main Proposition, U is Killing iff the function $f_1 := \frac{1}{2}\,\mathrm{div}\,U = 0$. Now, let $\mathcal{A}_1$ be the open subset of M where the function $b \neq 0$, and let $\mathcal{A}_2$ be the open subset of points $p \in M$ such that $b = 0$ in a neighborhood of p. Then $\mathcal{A}_1 \cup \mathcal{A}_2$ is an open dense subset of M. On $\mathcal{A}_1$, using (4.2), we have $\eta^2(U) = 1$, from which we get $U = \pm\xi$, and thus U is Killing and $QU = 2U$. In particular, $f_1 = 0$ on $\mathcal{A}_1$. On $\mathcal{A}_2$, using (4.1), we have that the metric g is Einstein and thus $QU = 2U$. Then, by the proof of the Main Proposition, we get that f_1 is a harmonic function on $\mathcal{A}_2$. Therefore, f_1 is a harmonic function on $\mathcal{A}_1 \cup \mathcal{A}_2$ and thus on M. Since M is compact and connected, f_1 must be a constant and thus $f_1 = 0$. Thus, U is a unit Killing vector field with $QU = 2U$. □

5. **The energy of unit vector fields.** Let (M,g) be a compact Riemannian manifold of dimension n. The energy of a unit vector field $U \in \mathfrak{X}^1(M)$ is defined as the energy of the map $U\colon (M,g) \to (T^1M, g_s)$. It follows that the energy $E(U)$ is given (see [12])

$$E(U) = \frac{1}{2}\int_M \|dU\|^2 v_g = \frac{n}{2}\,\mathrm{vol}(M,g) + \frac{1}{2}\int_M \|\nabla U\|^2 v_g.$$

$E(U)$ is equal, up to constants, to $B(U) = \int_M \|\nabla U\|^2 v_g$ which is known as the total bending of U (see Wiegmink [11]). In [2] F. Brito proved the following:

Theorem ([2]). *The unit vector fields of minimum energy on the unit sphere S^3 are precisely the Hopf vector fields and no others.*

If (M,g) is a compact Riemannian three-manifold, Brito also shows that

$$\int_M \|\nabla U\|^2 v_g \geq \int_M \mathrm{Ric}(U,U) v_g \quad \forall U \in \mathfrak{X}^1(S^3). \tag{5.1}$$

Now, examining the proof of (5.1), and by using the notations introduced in Section 3, we find that

$$\begin{aligned} \|\nabla U\|^2 &= \sum_{i=1}^{3} \|\nabla_{e_i} U\|^2 \geq \sum_{i=1}^{2} \|\nabla_{e_i} U\|^2 = S_{11}^2 + S_{22}^2 + S_{12}^2 + S_{21}^2 \\ &= 2\sigma_2 + (S_{11} - S_{22})^2 + (S_{12} + S_{21})^2 \geq 2\sigma_2, \end{aligned}$$

where the equality holds if and only if $\nabla_U U = 0$, $S_{11} = S_{22}$, $S_{12} + S_{21} = 0$. Since $2\int_M \sigma_2 v_g = \int_M \mathrm{Ric}(U,U) v_g$, we get that (see (3.3))

$$\int_M \|\nabla U\|^2 v_g = \int_M \mathrm{Ric}(U,U) v_g$$

if and only if $\mathrm{vol}(U) = \mathrm{vol}(M) + \frac{1}{2}\int_M \mathrm{Ric}(U,U)v_g$. Therefore, the Main Proposition holds also for the energy by replacing the equation (3.2) by the following:

$$E(U) = \int_M \Big(\frac{3}{2} + \frac{1}{2}\mathrm{Ric}(U,U)\Big)v_g.$$

Consequently, as in the volume case, we obtain an analogue of Brito's theorem for a compact Sasakian three-manifold.

Theorem 3. *Let (M, g, ξ, η) be a compact Sasakian three-manifold with Webster scalar curvature $w \geq 1$. Then, the Reeb vector field ξ minimises the energy, $E(\xi) = \frac{5}{2}\mathrm{vol}(M)$, and the unit vector fields of minimum energy are precisely the unit Killing vector fields eigenvectors of the Ricci operator with eigenvalue 2, and no others.*

Remark 2. (1) Brito [2] proved the uniqueness part of his theorem by applying the uniqueness part of Gluck and Ziller's theorem. As a consequence of the main proposition, we get (by a direct proof) Brito's theorem in the following form. *Let (M, g) be a compact Riemannian three-manifold of constant sectional curvature $c \geq 0$. Then, the unit vector fields of minimum energy on M (equal to $(\frac{3}{2} + c)\,\mathrm{vol}(M)$) are precisely the unit Killing vector fields, and no others.*

(2) In [8] the authors proved that the Reeb vector field ξ of a compact Sasakian $(2n+1)$-manifold, $n > 1$, of constant sectional curvature $+1$, minimises the energy if and only if M is not simply connected.

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