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ON THE CYCLES OF INDEFINITE BINARY QUADRATIC FORMS AND CYCLES OF IDEALS III

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Abstract. Let δ be a real quadratic irrational integer with trace $t = \delta+\overline{\delta}$ and norm $n = \delta.\overline{\delta}$. Then for a real quadratic irrational $\gamma \in \mathbb{Q}(\delta)$, there are rational integers P and Q such that $\gamma = \frac{P+\delta}{Q}$ with $Q|(\delta + P)(\overline{\delta} + P)$. So for each γ, we have an ideal $I_\gamma = [Q, P + \delta]$ and an indefinite quadratic form $F_\gamma(x, y) = Q(x + \delta y)(x + \overline{\delta}y)$ of discriminant $\Delta = t^2 - 4n$. In this work, we derive some properties of I_γ and F_γ for some specific values of δ.

Résumé. Soit δ un entier irrationel quadratique réel de trace $t = \delta+\overline{\delta}$ et norme $n = \delta.\overline{\delta}$. Pour un irrationel quadratique réel $\gamma \in \mathbb{Q}(\delta)$, il existe des entiers rationels P et Q tels que $\gamma = \frac{P+\delta}{Q}$ avec $Q|(\delta + P)(\overline{\delta} + P)$. Ainsi pour chaque γ, on a un idéal $I_\gamma = [Q, P + \delta]$ et une forme quadratique indéfinie $F_\gamma(x, y) = Q(x + \delta y)(x + \overline{\delta}y)$ de discriminant $\Delta = t^2 - 4n$. On déduit quelques propriétés de I_γ et F_γ pour certains valeurs de δ.

1. **Introduction.** A real binary quadratic form (or just a form) F is a polynomial in two variables x, y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2 \tag{1.1}$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by Δ. A quadratic form F of discriminant Δ is called indefinite if $\Delta > 0$, and is called integral if and only if $a, b, c \in \mathbb{Z}$. An indefinite quadratic form $F = (a, b, c)$ of discriminant Δ is said to be reduced if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{1.2}$$

Most properties of quadratic forms can be giving by the aid of extended modular group $\overline{\Gamma}$ (see [10]). Gauss (1777–1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$\begin{aligned} gF(x, y) = {} & (ar^2 + brs + cs^2)x^2 + (2art + bru + bts + 2csu)xy \\ & + (at^2 + btu + cu^2)y^2 \end{aligned} \tag{1.3}$$

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for $g = \left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) = [r; s; t; u] \in \overline{\Gamma}$, that is, gF is gotten from F by making the substitution $x \to rx + tu$ and $y \to sx + uy$. Moreover $\Delta(F) = \Delta(gF)$ for all $g \in \overline{\Gamma}$, that is, the action of $\overline{\Gamma}$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \overline{\Gamma}$. Let F and G be two forms. If there exists a $g \in \overline{\Gamma}$ such that $gF = G$, then F and G are called equivalent. If $\det g = 1$, then F and G are called properly equivalent and if $\det g = -1$, then F and G are called improperly equivalent. A form F is called ambiguous if it is improperly equivalent to itself. An element $g \in \overline{\Gamma}$ is called an automorphism of F if $gF = F$. If $\det g = 1$, then g is called a proper automorphism of F and if $\det g = -1$, then g is called an improper automorphism of F. Let $\mathrm{Aut}(F)^+$ denote the set of proper automorphisms of F and let $\mathrm{Aut}(F)^-$ denote the set of improper automorphisms of F (for further details on binary quadratic forms see [1–3]).

Let $\rho(F)$ denotes the normalization (it means that replacing F by its normalization) of $(c, -b, a)$. To be more explicit, we set

$$\rho(F) = (c, -b + 2cr_i, cr_i^2 - br_i + a), \tag{1.4}$$

where

$$r = r(F) = \begin{cases} \mathrm{sign}(c)\left\lfloor \frac{b}{2|c|} \right\rfloor & \text{for } |c| \geq \sqrt{\Delta}, \\ \mathrm{sign}(c)\left\lfloor \frac{b+\sqrt{\Delta}}{2|c|} \right\rfloor & \text{for } |c| < \sqrt{\Delta} \end{cases} \tag{1.5}$$

for $i \geq 0$. The number r is called the reducing number and the form $\rho(F)$ is called the reduction of F. Further if F is reduced, then so is $\rho(F)$. In fact, ρ is a permutation of the set of all reduced indefinite forms. Let $\tau(F) = \tau(a, b, c) = (-a, b, -c)$. Then the cycle of F is the sequence $\big((\tau\rho)^i(G)\big)$ for $i \in \mathbb{Z}$, where $G = (A, B, C)$ is a reduced form with $A > 0$ which is equivalent to F. The cycle of F can be derived by the following theorem.

THEOREM 1.1. *Let $F = (a, b, c)$ be reduced indefinite quadratic form of discriminant Δ. Then the cycle of F is a sequence $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ of length l, where $F_0 = F = (a_0, b_0, c_0)$,*

$$s_i = |s(F_i)| = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor \tag{1.6}$$

and

$$F_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1}) = \big(|c_i|, -b_i + 2s_i|c_i|, -(a_i + b_i s_i + c_i s_i^2)\big) \tag{1.7}$$

for $1 \leq i \leq l - 2$ [1].

Mollin [4] considered the arithmetic of ideals in his book. Let $D \neq 1$ be a square-free integer and let $\Delta = \frac{4D}{r^2}$, where $r = 2$ if $D \equiv 1 \pmod 4$ and $r = 1$, otherwise. If we set $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, then $\mathbb{K}$ is called a quadratic number field of discriminant Δ. Thus there is a one-to-one correspondence between quadratic fields and square-free rational integers $D \neq 1$.

A complex number is an algebraic integer if it is the root of a monic polynomial with coefficients in $\mathbb{Z}$. The set of all algebraic integers in the complex field $\mathbb{C}$ is a ring which we denote by A. Then $A \cap \mathbb{K} = O_\Delta$ is the ring of integers of the quadratic field $\mathbb{K}$ of discriminant Δ. Let $I = [\alpha, \beta]$ denote the $\mathbb{Z}$-module $\alpha\mathbb{Z} \oplus \beta\mathbb{Z}$ for $\alpha, \beta \in \mathbb{K}$, *i.e.*, the additive abelian group, with basis elements α and β consisting of $\{\alpha x + \beta y : x, y \in \mathbb{Z}\}$. Then $O_\Delta = [1, \frac{1+\sqrt{D}}{r}]$. In this case $w_\Delta = \frac{r-1+\sqrt{D}}{r}$ is called the principal surd. Every principal surd $w_\Delta \in O_\Delta$ can be uniquely expressed as $w_\Delta = x\alpha + y\beta$, where $x, y \in \mathbb{Z}$ and $\alpha, \beta \in O_\Delta$. We call α, β an integral basis for $\mathbb{K}$. If $\frac{\alpha\overline{\beta} - \beta\overline{\alpha}}{\sqrt{\Delta}} > 0$, then α and β are called ordered basis elements. Two basis of an ideal are ordered if and only if they are equivalent under an element of $\overline{\Gamma}$. If I has ordered basis elements, then we say that I is simply ordered. If I is ordered, then

$$F(x, y) = \frac{N(\alpha x + \beta y)}{N(I)}$$

is a quadratic form of discriminant Δ (here $N(x)$ denotes the norm of x). In this case we say that F belongs to I and write $I \to F$.

Conversely let us assume that

$$G(x, y) = Ax^2 + Bxy + Cy^2 = d(ax^2 + bxy + cy^2)$$

be a quadratic form, where $d = \pm \gcd(A, B, C)$ and $b^2 - 4ac = \Delta$. If $B^2 - 4AC > 0$, then we get $d > 0$, and if $B^2 - 4AC < 0$, then we choose d such that $a > 0$. If

$$I = [\alpha, \beta] = \begin{cases} [a, \frac{b-\sqrt{\Delta}}{2}] & \text{for } a > 0, \\ [a, \frac{b-\sqrt{\Delta}}{2}]\sqrt{\Delta} & \text{for } a < 0 \text{ and } \Delta > 0, \end{cases}$$

then I is an ordered O_Δ-ideal. Note that if $a > 0$, then I is primitive, and if $a < 0$, then $\frac{I}{\sqrt{\Delta}}$ is primitive. Thus to every form G, there corresponds an ideal I to which G belongs and we write $G \to I$. Hence we have a correspondence between ideals and quadratic forms (for further details see [5–7]).

THEOREM 1.2. *If* $I = [a, b + cw_\Delta]$, *then* I *is a non-zero ideal of* O_Δ *if and only if* $c|b, c|a$ *and* $ac|N(b + cw_\Delta)$ [4].

Let δ denote a real quadratic irrational integer with trace $t = \delta + \overline{\delta}$ and norm $n = \delta\overline{\delta}$. Given a real quadratic irrational $\gamma \in \mathbb{Q}(\delta)$, there are rational integers P and Q such that $\gamma = \frac{P+\delta}{Q}$ with $Q|(\delta + P)(\overline{\delta} + P)$. Hence for each

$$\gamma = \frac{P + \delta}{Q} \tag{1.8}$$

there is a corresponding $\mathbb{Z}$-module

$$I_\gamma = [Q,\, P+\delta] \tag{1.9}$$

(in fact, this module is an ideal by Theorem 1.2) and an indefinite quadratic form

$$F_\gamma(x,y) = Q(x+\delta y)(x+\overline{\delta} y) \tag{1.10}$$

of discriminant $\Delta = t^2 - 4n$. The ideal I_γ in (1.9) is said to be reduced if and only if

$$P+\delta > Q \text{ and } -Q < P+\overline{\delta} < 0 \tag{1.11}$$

and is said to be ambiguous if and only if it contains both $\frac{P+\delta}{Q}$ and $\frac{P+\overline{\delta}}{Q}$ so if and only if $\frac{2P}{Q} \in \mathbb{Z}$.

Let $[m_0; \overline{m_1, m_2, \ldots, m_{l-1}}]$ denote continued fraction expansion of γ with period length $l = l(I)$, where

$$m_i = \left\lfloor \frac{P_i+\delta}{Q_i} \right\rfloor, \quad P_{i+1} = m_i Q_i - P_i \quad \text{and} \quad Q_{i+1} = \frac{\delta^2 - P_{i+1}^2}{Q_i} \tag{1.12}$$

for $i \geq 0$. From the continued fraction factoring algorithm we get all reduced ideals equivalent to a given reduced ideal I_γ, *i.e.*, in the continued fraction expansion of γ we have $I_\gamma = I_\gamma^0 \sim I_\gamma^1 \sim \cdots \sim I_\gamma^{l-1}$. Finally $I_\gamma^l = I_\gamma^0$ for a complete cycle of reduced ideals of length $l(I) = l$.

2. Quadratic ideals and quadratic forms.

In [8], [9], and [11], we derived some properties of quadratic irrationals $\gamma = \frac{P+\delta}{Q}$, quadratic ideals I_γ and indefinite quadratic forms F_γ in (1.8), (1.9) and (1.10), respectively. In this section we consider the same problem for some specific values of $\delta = \sqrt{D}$, where $D \neq 1$ is a square-free positive integer. Now let $\gamma = -k + \sqrt{D}$ for an integer $k \geq 1$. Then

$$I_\gamma = [1, -k+\sqrt{D}] \tag{2.1}$$

is a quadratic ideal, and

$$F_\gamma = (1, 2k, k^2 - D) \tag{2.2}$$

is an indefinite binary quadratic form of discriminant $\Delta = 4D$.

2.1. Quadratic ideals. In this subsection, we will consider some properties of γ and I_γ in four cases: $D = k^2+1$, $D = k^2+2$, $D = k^2+2k-1$ and $D = k^2+2k$.

THEOREM 2.1. *Let I_γ be the ideal in* (2.1).

(1) *If $D = k^2+1$, then the continued fraction expansion of γ is $[0;\overline{2k}]$ and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [1, k+\sqrt{D}]$ of length* 2*;*

(2) *If $D = k^2+2$, then the continued fraction expansion of γ is $[0;\overline{k, 2k}]$ and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [2, k+\sqrt{D}] \sim I_\gamma^2 = [1, k+\sqrt{D}]$ of length* 3*;*

(3) *If $D = k^2+2k-1$, then the continued fraction expansion of γ is $[0;\overline{1, k-1, 1, 2k}]$ and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [2k-1, k+\sqrt{D}] \sim I_\gamma^2 = [2, k-1+\sqrt{D}] \sim I_\gamma^3 = [2k-1, k-1+\sqrt{D}] \sim I_\gamma^4 = [1, k+\sqrt{D}]$ of length* 5*;*

(4) *If $D = k^2+2k$, then the continued fraction expansion of γ is $[0;\overline{1, 2k}]$ and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [2k, k+\sqrt{D}] \sim I_\gamma^2 = [1, k+\sqrt{D}]$ of length* 3.

PROOF. (1) Let $D = k^2+1$. Then $I_\gamma = I_\gamma^0 = [1, -k+\sqrt{k^2+1}]$. So we have from (1.12) $m_0 = 0$ and hence $P_1 = k$ and $Q_1 = 1$. For $i = 1$, we have $m_1 = 2k$ and hence $P_2 = k = P_1$ and $Q_2 = 1 = Q_1$. For $i = 2$, we have $m_2 = 2k = m_1$. Therefore the continued fraction expansion of γ is $[0;\overline{2k}]$ and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [1, k+\sqrt{D}]$ of length 2.

(2) Let $D = k^2+2$. Then $I_\gamma = I_\gamma^0 = [1, -k+\sqrt{k^2+2}]$. We have $m_0 = 0$ and so $P_1 = k$ and $Q_1 = 2$. For $i = 1$, we have $m_1 = k$ and hence $P_2 = k$ and $Q_2 = 1$. For $i = 2$, we have $m_2 = 2k$ and hence $P_3 = k = P_1$ and $Q_3 = 2 = Q_1$. For $i = 3$, we have $m_3 = k = m_1$. Therefore the continued fraction expansion of γ is $[0;\overline{k, 2k}]$, and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [2, k+\sqrt{D}] \sim I_\gamma^2 = [1, k+\sqrt{D}]$ of length 3.

(3) Let $D = k^2+2k-1$. Then $I_\gamma = I_\gamma^0 = [1, -k+\sqrt{k^2+2k-1}]$. Hence we have $m_0 = 0$, and so $P_1 = k$ and $Q_1 = 2k-1$. For $i = 1$, we have $m_1 = 1$ and hence $P_2 = k-1$ and $Q_2 = 2$. For $i = 2$, we have $m_2 = k-1$ and hence $P_3 = k-1$ and $Q_3 = 2k-1$. For $i = 3$, we have $m_3 = 1$ and hence $P_4 = k$ and $Q_4 = 1$. For $i = 4$, we have $m_4 = 2k$ and hence $P_5 = k = P_1$ and $Q_5 = 2k-1 = Q_1$. For $i = 5$, we have $m_5 = 1 = m_1$. Therefore the continued fraction expansion of γ is $[0;\overline{1, k-1, 1, 2k}]$ and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [2k-1, k+\sqrt{D}] \sim I_\gamma^2 = [2, k-1+\sqrt{D}] \sim I_\gamma^3 = [2k-1, k-1+\sqrt{D}] \sim I_\gamma^4 = [1, k+\sqrt{D}]$ of length 5.

(4) Let $D = k^2+2k$. Then $I_\gamma = I_\gamma^0 = [1, -k+\sqrt{k^2+2k}]$ and hence $m_0 = 0$ and $P_1 = k$ and $Q_1 = 2k$. For $i = 1$, we have $m_1 = 1$ and hence $P_2 = k$ and $Q_2 = 1$. For $i = 2$, we have $m_2 = 2k$ and hence $P_3 = k = P_1$ and $Q_3 = 2k = Q_1$. For $i = 3$, we have $m_3 = 1 = m_1$. Therefore the continued fraction expansion of γ is $[0;\overline{1, 2k}]$ and the cycle of I_γ is $I_\gamma^0 = [1, -k+\sqrt{D}] \sim I_\gamma^1 = [2k, k+\sqrt{D}] \sim I_\gamma^2 = [1, k+\sqrt{D}]$ of length 3. □

EXAMPLE 2.1. Let $k = 6$.

(1) If $D = 6^2 + 1 = 37$, then the continued fraction expansion of $\gamma = -6 + \sqrt{37}$ is $[0; \overline{12}\,]$ and the cycle of I_γ is $I_\gamma^0 = [1, -6 + \sqrt{37}] \sim I_\gamma^1 = [1, 6 + \sqrt{37}]$.

(2) If $D = 6^2 + 2 = 38$, then the continued fraction expansion of $\gamma = -6 + \sqrt{38}$ is $[0; \overline{6, 12}\,]$ and the cycle of I_γ is $I_\gamma^0 = [1, -6 + \sqrt{38}] \sim I_\gamma^1 = [2, 6 + \sqrt{38}] \sim I_\gamma^2 = [1, 6 + \sqrt{38}]$.

(3) If $D = 6^2 + 2.6 - 1 = 47$, then the continued fraction expansion of $\gamma = -6 + \sqrt{47}$ is $[0; \overline{1, 5, 1, 12}\,]$ and the cycle of I_γ is $I_\gamma^0 = [1, -6 + \sqrt{47}] \sim I_\gamma^1 = [11, 6 + \sqrt{47}] \sim I_\gamma^2 = [2, 5 + \sqrt{47}] \sim I_\gamma^3 = [11, 5 + \sqrt{47}] \sim I_\gamma^4 = [1, 6 + \sqrt{47}]$.

(4) If $D = 6^2 + 2.6 = 48$, then the continued fraction expansion of $\gamma = -6 + \sqrt{48}$ is $[0; \overline{1, 12}\,]$ and the cycle of I_γ is $I_\gamma^0 = [1, -6 + \sqrt{48}] \sim I_\gamma^1 = [12, 6 + \sqrt{48}] \sim I_\gamma^2 = [1, 6 + \sqrt{48}]$.

THEOREM 2.2. *The ideals I_γ in Theorem 2.1 are not reduced.*

PROOF. Let $D = k^2 + 1$. Then $I_\gamma = [1, -k + \sqrt{k^2 + 1}]$. Recall that $k \geq 1$, so $2k > 0$. Hence

$$2k + k^2 + 1 > k^2 + 1 \Leftrightarrow (k+1)^2 > k^2 + 1 \Leftrightarrow k + 1 > \sqrt{k+1} \Leftrightarrow 1 > -k + \sqrt{k+1}$$

which is in contradiction to (1.11). So I_γ is not reduced. The other cases can be dealt with similarly. □

2.2. Quadratic forms. In this subsection, we will consider some properties of indefinite binary quadratic forms F_γ. First we consider their reducibility.

THEOREM 2.3. *F_γ is reduced if and only if $k^2 < D < k^2 + 2k + 1$.*

PROOF. Let F_γ be reduced. Then by (1.2) we have

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta} \Leftrightarrow |\sqrt{4D} - 2| < 2k < \sqrt{4D} \Leftrightarrow \sqrt{D} - 1 < k < \sqrt{D}.$$

Hence it is clear that $k^2 < D$ and $D < (k+1)^2$. So $k^2 < D < k^2 + 2k + 1$.

Conversely let $k^2 < D < k^2 + 2k + 1$. Then $k < \sqrt{D}$ and $\sqrt{D} < k + 1$. So

$$|\sqrt{4D} - 2| < 2k < \sqrt{4D} \Leftrightarrow \left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}$$

and hence F_γ is reduced by (1.2). □

Now we can give the following theorem concerning the cycles of F_γ in four cases: $D = k^2 + 1$, $D = k^2 + 2$, $D = k^2 + 2k - 1$ and $D = k^2 + 2k$. Note that for these values of D, F_γ is reduced by Theorem 2.3.

THEOREM 2.4. *Let F_γ be the quadratic form in* (2.2).

(1) *If $D = k^2 + 1$, then the cycle of F_γ is $F_\gamma^0 = (1, 2k, -1)$ of length* 1.

(2) *If* $D = k^2 + 2$, *then the cycle of* F_γ *is* $F_\gamma^0 = (1, 2k, -2) \sim F_\gamma^1 = (2, 2k, -1)$ *of length* 2.
(3) *If* $D = k^2 + 2k - 1$, *then the cycle of* F_γ *is* $F_\gamma^0 = (1, 2k, -2k+1) \sim F_\gamma^1 = (2k-1, 2k-2, -2) \sim F_\gamma^2 = (2, 2k-2, -2k+1) \sim F_\gamma^3 = (2k-1, 2k, -1)$ *of length* 4.
(4) *If* $D = k^2 + 2k$, *then the cycle of* F_γ *is* $F_\gamma^0 = (1, 2k, -2k) \sim F_\gamma^1 = (2k, 2k, -1)$ *of length* 2.

Proof. (1) Let $D = k^2 + 1$. Then $F_\gamma = F_\gamma^0 = (1, 2k, -1)$. Applying (1.6), we get $s_0 = 2k$ and hence by (1.7), we get $F_\gamma^1 = (a_1, b_1, c_1) = (1, 2k, -1) = F_\gamma^0$. Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (1, 2k, -1)$ of length 1.

(2) Let $D = k^2 + 2$. Then $F_\gamma = F_\gamma^0 = (1, 2k, -2)$ and hence $s_0 = k$. So $F_\gamma^1 = (a_1, b_1, c_1) = (2, 2k, -1)$. Similarly we find that $s_1 = 2k$ and $F_\gamma^2 = (a_2, b_2, c_2) = (1, 2k, -2) = F_\gamma^0$. Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (1, 2k, -2) \sim F_\gamma^1 = (2, 2k, -1)$ of length 2.

(3) Let $k^2 + 2k - 1$. Then $F_\gamma = F_\gamma^0 = (1, 2k, -2k+1)$ and hence $s_0 = 1$. So $F_\gamma^1 = (a_1, b_1, c_1) = (2k-1, 2k-2, -2)$. Similarly we can obtain $s_1 = k - 1$ and $F_\gamma^2 = (a_2, b_2, c_2) = (2, 2k-2, 1-2k)$. Also $s_2 = 1$ and $F_\gamma^3 = (a_3, b_3, c_3) = (2k-1, 2k, -1)$. Finally $s_3 = 2k$ and $F_\gamma^4 = (a_4, b_4, c_4) = (1, 2k, 1-2k) = F_\gamma^0$. Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (1, 2k, -2k+1) \sim F_\gamma^1 = (2k-1, 2k-2, -2) \sim F_\gamma^2 = (2, 2k-2, -2k+1) \sim F_\gamma^3 = (2k-1, 2k, -1)$ of length 4.

(4) Let $D = k^2 + 2k$. Then $F_\gamma = F_\gamma^0 = (1, 2k, -2k)$ and hence $s_0 = 1$. So $F_\gamma^1 = (a_1, b_1, c_1) = (2k, 2k, -1)$. For $i = 1$ we have $s_1 = 2k$ and hence $F_\gamma^2 = (a_2, b_2, c_2) = (1, 2k, -2k) = F_\gamma^0$. Therefore the cycle of F_γ is completed and is $F_\gamma^0 = (1, 2k, -2k) \sim F_\gamma^1 = (2k, 2k, -1)$ of length 2. □

Example 2.2. Let $k = 7$.
(1) If $D = 7^2 + 1 = 50$, then the cycle of $F_\gamma = (1, 14, -1)$ is $F_\gamma^0 = (1, 14, -1)$ of length 1.
(2) If $D = 7^2 + 2 = 51$, then the cycle of $F_\gamma = (1, 14, -2)$ is $F_\gamma^0 = (1, 14, -2) \sim F_\gamma^1 = (2, 14, -1)$ of length 2.
(3) If $D = 7^2 + 2.7 - 1 = 62$, then the cycle of $F_\gamma = (1, 14, -13)$ is $F_\gamma^0 = (1, 14, -13) \sim F_\gamma^1 = (13, 12, -2) \sim F_\gamma^2 = (2, 12, -13) \sim F_\gamma^3 = (13, 14, -1)$ of length 4.
(4) If $D = 7^2 + 2.7 = 63$, then the cycle of $F_\gamma = (1, 14, -14)$ is $F_\gamma^0 = (1, 14, -14) \sim F_\gamma^1 = (14, 14, -1)$ of length 2.

Now we consider the proper and improper automorphisms of F_γ.

Theorem 2.5. *Let* F_γ *be the quadratic form in (2.2).*
(1) *If* $D = k^2 + 1$, *then*

$$\#\operatorname{Aut}(F_\gamma)^+ = \begin{cases} 6 & \text{if } k = 1, 2, \\ 2 & \text{if } k > 2 \end{cases}$$

and

$$\#\,\mathrm{Aut}(F_\gamma)^- = \begin{cases} 8 & \textit{if } k = 1, \\ 4 & \textit{if } k > 1. \end{cases}$$

(2) *If* $D = k^2 + 2$, *then*

$$\#\,\mathrm{Aut}(F_\gamma)^+ = \begin{cases} 10 & \textit{if } k = 1, \\ 6 & \textit{if } k = 2, 3, \\ 2 & \textit{if } k > 3, \end{cases}$$

and

$$\#\,\mathrm{Aut}(F_\gamma)^- = \begin{cases} 10 & \textit{if } k = 1, \\ 6 & \textit{if } k = 2, \\ 4 & \textit{if } k > 2. \end{cases}$$

(3) *If* $D = k^2 + 2k - 1$, *then*

$$\#\,\mathrm{Aut}(F_\gamma)^+ = \begin{cases} 6 & \textit{if } k = 1, 2, \\ 2 & \textit{if } k > 2, \end{cases}$$

and

$$\#\,\mathrm{Aut}(F_\gamma)^- = \begin{cases} 8 & \textit{if } k = 1, \\ 4 & \textit{if } k > 1. \end{cases}$$

(4) *If* $D = k^2 + 2k$, *then*

$$\#\,\mathrm{Aut}(F_\gamma)^+ = \#\,\mathrm{Aut}(F_\gamma)^- = \begin{cases} 10 & \textit{if } k = 1, \\ 6 & \textit{if } k > 1. \end{cases}$$

Proof. (1): Let $k = 1$. Then $F_\gamma = (1, 2, -1)$. The system of equations

$$\begin{aligned} r^2 + 2rs - s^2 &= 1, \\ 2rt + 2ru + 2ts - 2su &= 2, \\ t^2 + 2tu - u^2 &= -1, \end{aligned}$$

has a solution for $g = \pm[5; -2; -2; 1], \pm[1; 2; 2; 5], \pm[1; 0; 0; 1]$. So

$$\mathrm{Aut}(F_\gamma)^+ = \{\pm[5; -2; -2; 1], \pm[1; 2; 2; 5], \pm[1; 0; 0; 1]\}$$

and hence $\#\,\mathrm{Aut}(F_\gamma)^+ = 6$. Similarly we find that

$$\mathrm{Aut}(F_\gamma)^+ = \{\pm[17; -4; -4; 1], \pm[1; 4; 4; 17], \pm[1; 0; 0; 1]\}$$

for $k = 2$ and $\mathrm{Aut}(F_\gamma)^+ = \{\pm[1; 0; 0; 1]\}$ for all $k > 2$.

Now we consider the improper automorphisms. The system of equations

$$\begin{aligned} r^2+2rs-s^2&=1,\\ 2rt+2ru+2ts-2su&=2,\\ t^2+2tu-u^2&=-1, \end{aligned}$$

has a solution for

$$g=\pm[5;12;-2;-5],\pm[5;-2;12;-5],\pm[1;2;0;-1],\pm[1;0;2;-1],$$

so

$$\mathrm{Aut}(F_\gamma)^-=\{\pm[5;12;-2;-5],\pm[5;-2;12;-5],\pm[1;2;0;-1],\pm[1;0;2;-1]\}$$

and hence $\#\,\mathrm{Aut}(F_\gamma)^-=8$. Similarly we find that

$$\mathrm{Aut}(F_\gamma)^-=\{\pm[1;2k;0;-1],\pm[1;0;2k;-1]\}$$

for every $k>1$.

(2)–(4): With the same argument we find that if $D=k^2+2$, then

$$\mathrm{Aut}(F_\gamma)^+\begin{cases}\{\pm[11;-4;-8;3],\pm[3;4;8;11],\pm[3;-1;-2;1],\\ \qquad\qquad\pm[1;2;2;3],\pm[1;0;0;1]\,\} & \text{for } k=1,\\ \{\pm[9;-2;-4;1],\pm[1;2;4;9],\pm[1;0;0;1]\} & \text{for } k=2,\\ \{\pm[19;-3;-6;1],\pm[1;3;6;19],\pm[1;0;0;1]\} & \text{for } k=3,\\ \{\pm[1;0;0;1]\} & \text{for } k>3,\end{cases}$$

and

$$\mathrm{Aut}(F_\gamma)^-=\begin{cases}\{\pm[11;15;-8;-11],\pm[3;4;-2;-3],\pm[3;-1;8;-3]\\ \qquad\qquad\pm[1;1;0;-1],\pm[1;0;2;-1]\,\} & \text{for } k=1,\\ \{\pm[9;20;-4;-9],\pm[1;2;0;-1],\pm[1;0;4;-1]\} & \text{for } k=2,\\ \{\pm[1;k;0;-1],\pm[1;0;2k;-1]\} & \text{for } k>2.\end{cases}$$

If $D=k^2+2k-1$, then

$$\mathrm{Aut}(F_\gamma)^+=\begin{cases}\{\pm[5;-2;-2;1],\pm[1;2;2;5],\pm[1;0;0;1]\} & \text{for } k=1,\\ \{\pm[14;-3;-9;2],\pm[2;3;9;14],\pm[1;0;0;1]\} & \text{for } k=2,\\ \{\pm[1;0;0;1]\} & \text{for } k>2,\end{cases}$$

and

$$\mathrm{Aut}(F_\gamma)^-=\begin{cases}\{\pm[5;12;-2;-5],\pm[5;-2;12;-5],\\ \qquad\qquad\pm[1;2;0;-1],\pm[1;0;2;-1]\} & \text{for } k=1,\\ \{\pm[k;k+1;1-k;-k],\pm[1;0;2k;-1]\} & \text{for } k>1.\end{cases}$$

Finally if $D = k^2 + 2k$, then

$$\mathrm{Aut}(F_\gamma)^+ = \begin{cases} \{\pm[11;-4;-8;3], \pm[3;4;8;11], \pm[3;-1;2;1], \\ \qquad \pm[1;1;2;3], \pm[1;0;0;1]\} & \text{for } k = 1, \\ \{\pm[2k+1;-1;-2k;1], \pm[1;1;2k;2k+1], \\ \qquad \pm[1;0;0;1]\} & \text{for } k > 1, \end{cases}$$

and

$$\mathrm{Aut}(F_\gamma)^- = \begin{cases} \{\pm[11;15;-8;-11], \pm[3;4;-2;-3], \pm[3;-1;8;-3], \\ \qquad \pm[1;1;0;-1], \pm[1;0;2;-1]\} & \text{for } k = 1, \\ \{\pm[2k+1;2k+2;-2k;-2k-1], \pm[1;1;0;-1], \\ \qquad \pm[1;0;2k;-1]\} & \text{for } k > 1. \end{cases}$$

□

From Theorem 2.5, we can obtain the following result.

THEOREM 2.6. *F_γ is ambiguous for $D = k^2+1$, $D = k^2+2$, $D = k^2+2k-1$ and $D = k^2 + 2k$.*

PROOF. We proved in Theorem 2.5 that the set of improper automorphisms of F_γ is non-empty, that is, $\mathrm{Aut}(F_\gamma)^- \neq \varnothing$. So there exists at least one element $g \in \overline{\Gamma}$ with $\det g = -1$ such that $gF_\gamma = F_\gamma$, that is, F_γ is improperly equivalent to itself and hence is ambiguous. □

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