

FACTORIZATION OF AN INDEFINITE CONVECTION-DIFFUSION OPERATOR

Dedicated to the memory of Abraham Vilgelmovich Strauss

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ABSTRACT. We prove that a certain non-self-adjoint differential operator admits factorization, and we apply this new representation of the operator to explicitly construct its domain. We also show that the operator is J -self-adjoint in a Krein space.

RÉSUMÉ. On montre qu'un certain opérateur non autoadjoint admet une factorisation et, on utilise cette représentation pour construire explicitement son domaine. On montre aussi que cet opérateur est J -autoadjoint dans un espace de Krein.

1. Introduction. The time-evolution of a thin film of viscous fluid on the inner surface of a rotating cylinder can be approximated by the Cauchy problem for the periodic backward-forward heat equation

$$(1.1) \quad y_t + l[y] = 0, \quad y(0, x) = y_0, \quad y(-\pi, t) = y(\pi, t), \quad x \in (-\pi, \pi), \quad t > 0$$

where

$$l[y] = \epsilon(\sin xy_x)_x + y_x, \quad \epsilon > 0.$$

This model was derived by Benilov, O'Brien and Sazonov [2], [3] under assumption that the parameter ϵ , related to the thickness of the film, is sufficiently small, *i.e.*, $\epsilon \ll 1$.

It was first shown numerically by Benilov, O'Brien and Sazonov [2] that the spectrum of the operator L defined by the operation $l[\cdot]$ and periodic boundary conditions $y(-\pi) = y(\pi)$ consists of pure imaginary eigenvalues only. This was quite a surprising result, because it is well known that the Cauchy problem (1.1) is ill-posed at least for the class of finitely smooth functions.

The spectral properties of the operator L were studied rigorously by Davies [5] and by Chugunova and Pelinovsky [4]. Approaching the problem in two different ways, they proved analytically that if the parameter ϵ is within the interval

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$|\epsilon| < 2$, then the operator L admits closure in the Hilbert space $L^2[-\pi, \pi]$. Furthermore, when L is restricted to the orthogonal to a constant subspace it has compact inverse of Hilbert–Schmidt type, and as a consequence its spectrum is discrete with the only accumulation point at infinity.

The numerical conjecture that all eigenvalues are pure imaginary was recently proved by Weir [6]. The elegant proof is based on the continuation of the eigenfunctions into Hardy space on the unit disk and explicit construction of the symmetric operator.

Our goal in this paper is to find the factorization of the operator L and to construct its domain explicitly. As a consequence we prove that the operator L acting in physical space is unitary equivalent to the operator A introduced by Davies [5] acting in the Fourier space. We also prove that the non-self-adjoint differential operator L belongs to the class of J -self-adjoint operators in some Krein space. For basic facts related to Krein spaces see [1].

2. Factorization of the non-self-adjoint operator L . We denote by $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ the domain and the range of linear operator T respectively. The notation \mathcal{L}^2 is used for the standard Lebesgue space of scalar functions defined on the interval $(-\pi, \pi)$. From here on L is the indefinite convection-diffusion operator

$$L: \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad (Ly)(x) = \epsilon \cdot (\sin(x)y'(x))' + y'(x),$$

with domain of all absolutely continuous 2π -periodic functions $y(x)$ such that $(Ly)(x) \in \mathcal{L}^2$.

In addition, we define the operator

$$S: \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad (Sy)(x) = y'(x),$$

where $y'(x) \in \mathcal{L}^2$, $y(-\pi) = y(\pi)$, and the operator

$$M: \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad (My)(x) = \epsilon \cdot (\sin(x)y(x))' + y(x)$$

with domain of all absolutely continuous functions $y(x)$ such that $\sin(x)y(x)$ is absolutely continuous and $(My)(x) \in \mathcal{L}^2$.

THEOREM 1. *If the parameter $\epsilon \in (0, 2)$, then $L = MS$.*

PROOF. Let us consider the operator

$$A: \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad (Ay)(x) = (\sin(x)y(x))'$$

with $\mathfrak{D}(A) = \{y(x) \mid y(x), (Ay)(x) \in \mathcal{L}^2\}$. Then a function $y(x)$ can be written as

$$(2.1) \quad y(x) = \frac{1}{\sin(x)} \cdot \left(c + \int_0^x \theta(t) dt \right), \quad \theta(t) \in \mathcal{L}^2.$$

If $x > 0$ then

$$\left| \int_0^x \theta(t) dt \right| \leq \frac{1}{\sin(x)} \cdot \alpha(x) \cdot x^{1/2},$$

where $\alpha(x) = \left(\int_0^x |\theta(x)|^2 dx \right)^{1/2}$. Since the two summands in (2.1) have different orders of growth as $x \rightarrow 0$, this implies that if $y(x) \in \mathcal{L}^2$ then $c = 0$ and

$$(2.2) \quad y(x) = \frac{1}{\sin(x)} \cdot \int_0^x \theta(t) dt.$$

Moreover,

$$|y(x)| \leq \frac{x^{1/2}}{\sin(x)} \cdot \alpha(x).$$

A small modification of the same reasoning leads to the following estimation for every $x \in (-\pi, \pi)$

$$(2.3) \quad |y(x)| \leq \frac{|x|^{1/2}}{|\sin(x)|} \cdot \alpha(x),$$

with $\alpha(x) = \left| \int_0^x |\theta(x)|^2 dx \right|^{1/2}$.

Alternatively the same function $y(x)$ can be written as

$$(2.4) \quad y(x) = \frac{1}{\sin(x)} \cdot \left(\tilde{c} - \int_x^\pi \theta(t) dt \right), \quad \theta(t) \in \mathcal{L}^2,$$

with the same $\theta(x)$ as in (2.1). Representation (2.4) yields the following relations

$$(2.5) \quad y(x) = \frac{-1}{\sin(x)} \cdot \int_x^\pi \theta(t) dt$$

and

$$(2.6) \quad |y(x)| \leq \frac{(\pi - x)^{1/2}}{\sin(x)} \cdot \beta(x),$$

with $\beta(x) = \left(\int_x^\pi |\theta(x)|^2 dx \right)^{1/2}$.

It follows from (2.2) and (2.5) that

$$(2.7) \quad \int_0^\pi \theta(t) dt = 0.$$

Starting from the point $-\pi$ one can also obtain that

$$(2.8) \quad \begin{aligned} y(x) &= \frac{1}{\sin(x)} \cdot \int_{-\pi}^x \theta(t) dt, \\ |y(x)| &\leq \frac{(-\pi + x)^{1/2}}{|\sin(x)|} \cdot \gamma(x), \end{aligned}$$

with $\gamma(x) = \left(\int_{-\pi}^x |\theta(t)|^2 dt\right)^{1/2}$ and

$$(2.9) \quad \int_{-\pi}^0 \theta(t) dt = 0.$$

Obviously, the natural domain of the operator

$$B: \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad (By)(x) = \sin(x) \cdot y(x)',$$

where $y(x), (By)(x) \in \mathcal{L}^2$ coincides with the domain of the operator A . And as a consequence we obtain that $A^* = -B$.

Now let us define the operator $C: \mathcal{L}^2 \rightarrow \mathcal{L}^2$,

$$(Cy)(x) = -i \cdot \left(\sin(x) \cdot y(x)' - \frac{1}{2} \cos(x)y(x) \right) = -i \cdot \left((By)(x) + \frac{1}{2} \cos(x)y(x) \right),$$

where $y(x), (Cy)(x) \in \mathcal{L}^2$ and

$$D: \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad (Dy)(x) = \left(\frac{\epsilon}{2} \cdot \cos(x) + 1 \right) \cdot y(x).$$

It follows directly from the relation between the operators A and B and inequalities (2.3), (2.6), (2.8) that C is a self-adjoint operator.

We now restrict the parameter ϵ to the interval $0 < \epsilon < 2$. This implies that D has the bounded inverse, and hence that $D^{-1/2} \cdot C \cdot D^{-1/2}$ is a self-adjoint operator. Therefore the operator defined as

$$(i \cdot \epsilon \cdot D^{-1/2} \cdot C \cdot D^{-1/2} + I)$$

also has the bounded inverse, and the same is true for the operator

$$M = (i \cdot \epsilon \cdot C + D) = \epsilon \cdot A + I.$$

Finally, let us show that the subspace $\{\text{const}\}^\perp \cap \mathcal{L}^2$ is invariant under M and M^{-1} . Indeed, if $y(x)$ belongs to the domain of M , then $y(x)$ has representation (2.2), and hence

$$(My)(x) = y(x) + \epsilon \cdot \theta(x).$$

It follows by (2.7) and (2.9) that

$$\int_{-\pi}^{\pi} (My)(x) dx = \int_{-\pi}^{\pi} y(x) dx,$$

so $(My)(x) \in \{\text{const}\}^\perp \cap \mathcal{L}^2$ if and only if $y(x) \in \{\text{const}\}^\perp \cap \mathcal{L}^2$.

Our main goal is to show that $L = MS$. It is easy to check that a function $y(t)$ belongs to the domain (within \mathcal{L}^2) of L only if

$$(2.10) \quad y'(x) = \left(\tan(|x|/2) \right)^{-1/\epsilon} \sin^{-1}(x) \cdot \left(c + \int_0^x \left(\tan(|t|/2) \right)^{1/\epsilon} \phi(t) dt \right),$$

where $\phi(t) \in \mathcal{L}^2$. Let $x > 0$. Then

$$\begin{aligned} \left| \int_0^x (\tan(t/2))^{1/\epsilon} \phi(t) dt \right| \\ \leq \left(\int_0^x t^{2/\epsilon} dt \right)^{(1/2)} \left(\int_0^x \left| \phi(t) \left(\frac{\tan(t/2)}{t} \right)^{1/\epsilon} \right|^2 dt \right)^{(1/2)}. \end{aligned}$$

The latter estimation yields

$$\left| \int_0^x (\tan(t/2))^{1/\epsilon} \phi(t) dt \right| \leq x^{1/2+1/\epsilon} \cdot \alpha(x),$$

where $\alpha(x) \rightarrow 0$ if $x \rightarrow 0$. Then

$$\left| (\tan(x/2))^{-1/\epsilon} \sin^{-1}(x) \left(\int_0^x (\tan(t/2))^{1/\epsilon} \phi(t) dt \right) \right| \leq x^{-1/2} \cdot \beta(x),$$

where $\beta(x) \rightarrow 0$ if $x \rightarrow 0$. If $c = 0$, we obtain that

$$\begin{aligned} |y(x) - y(0)| &= \left| \int_0^x (\tan(\tau/2))^{-1/\epsilon} \sin^{-1}(\tau) \left(\int_0^\tau (\tan(t/2))^{1/\epsilon} \phi(t) dt \right) d\tau \right| \\ &\leq \gamma(x) x^{1/2}, \end{aligned}$$

where $\gamma(x) \rightarrow 0$ if $x \rightarrow 0$. Thus, if $c = 0$, $y(x)$ is continuous at zero. At the same time if $c \neq 0$, $y(x)$ contains an additional summand of the order $x^{-1/\epsilon}$ that is out of \mathcal{L}^2 for $\epsilon \leq 2$. Thus, $c = 0$ in (2.10) and so

$$(2.11) \quad y'(x) = (\tan(|x|/2))^{-1/\epsilon} \sin^{-1}(x) \int_0^x (\tan(|t|/2))^{1/\epsilon} \phi(t) dt.$$

This representation was derived under the hypothesis that $x > 0$, but it is clear that it is valid for every $x \in (-\pi, \pi)$.

Now let us assume that there is $y(x)$ such that

$$(2.12) \quad y(x) \in \mathfrak{D}(L) \quad \text{but} \quad y(x) \notin \mathfrak{D}(MS).$$

Then there are two options:

- $(Ly)(x) \notin \mathfrak{R}(MS) = \{\text{const}\}^\perp \cap \mathcal{L}^2$;
- $(Ly)(x) \in \mathfrak{R}(MS)$.

The first option means that $\mathfrak{R}(L) = \mathcal{L}^2$ and without loss of generality one can assume that $(Ly)(x) \equiv 1$. Then by (2.11)

$$y'(x) = (\tan(|x|/2))^{-1/\epsilon} \sin^{-1}(x) \cdot \int_0^x (\tan(|t|/2))^{1/\epsilon} dt.$$

Thus, $y'(x) > 0$ for every $x \in (-\pi, \pi)$. The latter is impossible for absolutely continuous 2π -periodic function. It is a contradiction!

Now let us consider the second option. If $y(x)$ satisfies (2.12) and $(Ly)(x) \in \mathfrak{R}(MS)$, then there is $z(x) \in \mathfrak{D}(MS)$ such that $(Lz)(x) = (Ly)(x)$. The latter yields $(L(z - y))(x) \equiv 0$. Thus, by virtue of (2.11), $y(x) = z(x) + \text{const}$ that is impossible thanks to (2.12). \square

COROLLARY 1. *L is a closed operator with the non-empty resolvent set and its resolvent has the Sturm–Liouville property.*

Let us define by L_0 the operator that represents the restriction of L on the set of all 2π -periodic smooth functions and by \bar{L}_0 its closure.

THEOREM 2. *$L = \bar{L}_0$. The operator L is J -self-adjoint in the Krein space with indefinite metric J defined as $J(f(x)) = f(\pi - x)$.*

PROOF. It is evident that $\bar{L}_0 \subseteq L$. In the paper of Davies [5] was shown that \bar{L}_0 has the non-empty resolvent set. If $\bar{L}_0 \neq L$ then every $\lambda \in \rho(\bar{L}_0)$ belongs to $\sigma_p(L)$ but it is impossible due to Corollary 1.

The adjoint operator is defined by the operation

$$(L^*y)(x) = \epsilon \cdot (\sin(x)y'(x))' - y'(x)$$

on the same domain as the operator L . The last statement of the theorem follows immediately from the equality $L = JL^*J$. \square

REMARK 1. The J -self-adjoint operator L being restricted to the subspace orthogonal to a constant has a compact inverse that implies that there exist two, non-positive and non-negative with respect to the metric J , maximal invariant under the operator L subspaces.

3. Domain of the operator L . As it was proved in the previous section, the operator

$$M: \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad (My)(x) = \epsilon \cdot (\sin(x)y(x))' + y(x)$$

with domain all absolutely continuous functions $y(x)$ such that $(My)(x) \in \mathcal{L}^2$ has bounded inverse. Hence, there is a constant p_1 such that

$$(3.1) \quad \|\epsilon \cdot (\sin(x)y(x))' + y(x)\|_{L^2} \geq p_1 \cdot \|y(x)\|_{L^2}$$

for every $y(x) \in \mathfrak{D}(M)$. Next, the operator L is closed, so its domain is closed with respect to the norm of the graphic of L , *i.e.*, for the norm

$$(3.2) \quad \|y(x)\|_g := \{\|y(x)\|_{L^2}^2 + \|\epsilon \cdot (\sin(x)y'(x))' + y'(x)\|_{L^2}^2\}^{1/2}.$$

Our aim is to show that for $y(x) \in \mathfrak{D}(L) \cap \{\text{const}\}^\perp$ the norm (3.2) is equivalent to the following norm

$$\|y(x)\|_m := \{\|y'(x)\|_{L^2}^2 + \|\sin(x) \cdot y'(x)\|_{L^2}^2 + \|(\sin(x) \cdot y'(x))'\|_{L^2}^2\}^{1/2}.$$

Indeed, for every $y(x) \in \mathfrak{D}(L) \cap \{\text{const}\}^\perp$ we have

$$\|y(x)\|_g \leq \{\|y'(x)\|_{L^2}^2 + 2\|\epsilon \cdot (\sin(x)y'(x))'\|_{L^2} + 2 \cdot \|y'(x)\|_{L^2}^2\}^{1/2} \leq p_2 \cdot \|y(x)\|_m,$$

where $p_2 = \max\{3, 2\epsilon\}$. On the other hand, taking into account (3.1), we have

$$\begin{aligned} \|y(x)\|_g &\geq \|\epsilon \cdot (\sin(x)y'(x))' + y'(x)\|_{L^2} \\ &\geq \frac{p_1}{2} \cdot \|y'(x)\|_{L^2} + \frac{1}{2} \cdot \|\epsilon \cdot (\sin(x)y'(x))' + y'(x)\|_{L^2} \\ &\geq 3p_3 \cdot \|y'(x)\|_{L^2} + p_3 \cdot \|\epsilon \cdot (\sin(x)y'(x))' + y'(x)\|_{L^2} \\ &\geq p_3 \cdot \left(\|y'(x)\|_{L^2} + \|\sin(x)y'(x)\|_{L^2} + \|\epsilon \cdot (\sin(x) \cdot y'(x))'\|_{L^2} \right), \end{aligned}$$

where $p_3 = \min\{\frac{p_1}{6}, \frac{1}{2}\}$.

It follows from above that the domain of L is the linear sub-manifold H of the Sobolev space $H^1(-\pi, \pi)$:

$$\mathfrak{D}(L) = H : f \in H^1(-\pi, \pi), \quad f(\pi) = f(-\pi), \quad \sin(x)f' \in H^1(-\pi, \pi)$$

and is a Hilbert space with the norm defined as:

$$\|f\|^2 = \|f'\|_{L^2}^2 + \|\sin(x)f'(x)\|_{H^1}^2.$$

REMARK 2. As a consequence of Theorem 1, the tridiagonal matrix operator \mathcal{A} in Fourier space [4].

$$\mathcal{A} = \begin{bmatrix} 1 & \epsilon & 0 & 0 & \dots \\ -\epsilon & 2 & 3\epsilon & 0 & \dots \\ 0 & -3\epsilon & 3 & 6\epsilon & \dots \\ 0 & 0 & -6\epsilon & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

admits the same type of factorization as in Theorem 1:

$$\mathcal{A} = \mathcal{BC} = \begin{bmatrix} 1 & \epsilon/2 & 0 & 0 & \dots \\ -\epsilon & 1 & \epsilon & 0 & \dots \\ 0 & -3/2\epsilon & 1 & 3/2\epsilon & \dots \\ 0 & 0 & -2\epsilon & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

There, \mathcal{B} has bounded inverse and \mathcal{C} has inverse of the Hilbert–Schmidt type. The operator \mathcal{A} is \mathcal{J} -self-adjoint with $\mathcal{J} = \text{diag}(-1, 1, -1, 1, \dots)$. Thus, the natural domain $\mathfrak{D}(\mathcal{A})$ is the Hilbert space of all number sequences $\{f_n\}_1^\infty$ with the norm

$$\|\{f_n\}_1^\infty\| := \left\{ \sum_{n=1}^{\infty} n^2 (|f_n|^2 + |(n+1)f_{n+1} - (n-1)f_{n-1}|^2) \right\}^{1/2}.$$

In the latter formula we put for definiteness $f_0 = 0$. Note also that there exist two, non-positive and non-negative with respect to the indefinite metric defined by \mathcal{J} , maximal invariant under the operator \mathcal{A} , subspaces.

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