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REMARKS ON SOME RECENT FIXED POINT THEOREMS

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ABSTRACT. We obtain fixed and common point theorems generalizing fixed point theorems of W. A. Kirk and T. Suzuki for Banach and Meir–Keeler type asymptotic contractions.

RÉSUMÉ. Nous démontrons des théorèmes de points fixes et de points communs qui généralisent des théorèmes de points fixes du type de Banach et de Meir–Keeler pour les contractions asymptotiques.

1. **Introduction.** The classical Banach contraction theorem is one of the most useful results in fixed point theory. In recent years, a number of generalizations and applications of Banach's theorem have appeared. In 1969, Meir–Keeler [8] obtained the following generalization.

THEOREM 1.1. *Let (X, d) be a complete metric space and T a self-map on X. Assume that for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$$

for all x, $y \in X$. Then T has a unique fixed point.

Cho *et al.* [2], Lim [7], Park and Rhoades [9], Jachymski [3], and others obtained various generalizations of the above theorem. Kirk [6] introduced the following notion of asymptotic contraction on a metric space, and proved a fixed point theorem for such contractions.

DEFINITION 1.1 (Kirk [6]). Let (X, d) be a metric space and T a self-map on X. T is an *asymptotic contraction* on X if there exists a continuous function φ from $[0, \infty)$ into itself and a sequence $\{\varphi_n\}$ of functions from $[0, \infty)$ into itself such that

(K1) $\varphi(0) = 0$;
(K2) $\varphi(r) \leq r$ for $r \in (0, \infty)$;
(K3) $\{\varphi_n\}$ converges to φ uniformly on the range of d; and
(K4) for x, $y \in X$ and $n \in \mathbb{N}$, $d(T^n x, T^n y) \leq \varphi_n\big(d(x, y)\big)$.

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Kirk [6] obtained the following theorem.

THEOREM 1.2. *Let (X, d) be a complete metric space and T a continuous asymptotic contraction on X with $\{\varphi_n\}$ and φ as in Definition* 1.1. *Assume that there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ of x is bounded, and that φ_n is continuous for $n \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover* $\lim_n T^n x = z$ *for all $x \in X$.*

Jachymski and Jòżwic [4] showed that the continuity of T is essential in Theorem 1.2 [4, Ex. 1]. Recently Suzuki [11] introduced the following notion of an *asymptotic contraction of Meir–Keeler type* generalizing the Meir–Keeler contraction and Kirk's asymptotic contraction (*cf.* Definition 1.1).

DEFINITION 1.2. Let (X, d) be a metric space. Then a map T on X is an *asymptotic contraction of Meir–Keeler type* (*ACMK* for short) if there exists a sequence φ_n of functions from $[0, \infty)$ into itself satisfying the following:

(S1) $\limsup \varphi_n(\varepsilon) \le \varepsilon)$ for all $\varepsilon \ge 0$;
(S2) for each $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\varphi_\nu(t) \le \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$;
(S3) $d(T^n x, T^n y) < \varphi_n\big(d(x, y)\big)$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \ne y$.

Inspired by Jachymski and Jòżwic [4, Lemma 4], Suzuki [11] obtained the following result.

THEOREM 1.3. *Let (X, d) be a complete metric space and T an ACMK on X. Assume that T^l is continuous for some $l \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover,* $\lim_n T^n x = z$ *for all $x \in X$.*

Following largely Suzuki [11], we present an extension of the above theorem. Further, with a view to increasing the scope of Theorem 1.3, we introduce a dummy map f in (S3) and obtain a common fixed point theorem for a pair of maps commuting just at a coincidence point of T and f.

2. Main results.

THEOREM 2.1. *Let (X, d) be a complete metric space and T a map satisfying the following conditions:*

(A1) $\limsup_n \varphi_n(\varepsilon) \le \varepsilon$ *for all $\varepsilon \ge 0$;*
(A2) *for each $\varepsilon > 0$ there exists $\delta > 0$ and $\mu \in \mathbb{N}$ such that $\varphi_\mu(t) \le \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$;*
(A3) $d(T^n x, T^n y) < \varphi_n\big(M(x, y)\big)$ *for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \ne y$, where* $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

If T^k is continuous for some $k \in \mathbb{N}$ then T has a unique fixed point $z \in X$. Moreover, $\lim_n T^n x = z$ *for all $x \in X$.*

Proof. Pick x_0 in X. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$, $n = 1, 2, \ldots$. First we show that

$$\lim_{n\to\infty} d(T^n x_0, T^n x_1) = 0 \quad \text{for all } x_0, x_1 \in X. \tag{2.1}$$

It initially holds if $x_0 = x_1$. In the other case of $x_0 \neq x_1$, we assume that $\alpha := \limsup_n d(T^n x_0, T^n x_1) > 0$. From the condition (A2), we can choose $\mu_1 \in \mathbb{N}$ satisfying $\varphi_{\mu_1}\big(d(x_0, x_1)\big) \leq d(x_0, x_1)$. By (A3) and (A1),

$$d(T^{\mu_1} x_0, T^{\mu_1} x_1) < \varphi_{\mu_1}\big(M(x_0, x_1)\big) \leq M(x_0, x_1). \tag{2.2}$$

Then proceeding as in Suzuki [11],

$$\begin{aligned} \alpha &:= \limsup_n d(T^n o T^{\mu_1} x_0, T^n o T^{\mu_1} x_1) \\ &\leq \limsup_n \varphi_n\big(M(T^{\mu_1} x_0, T^{\mu_1} x_1)\big) \leq M(T^{\mu_1} x_0, T^{\mu_1} x_1) \\ &= \max\{d(T^{\mu_1} x_0, T^{\mu_1} x_1), d(T^{\mu_1} x_0, T^{\mu_1 + 1} x_0), d(T^{\mu_1} x_1, T^{\mu_1 + 1} x_1)\} \\ &= \max\{d(T^{\mu_1} x_0, T^{\mu_1} x_1), d(T^{\mu_1} x_0, T^{\mu_1} x_1), d(T^{\mu_1} x_1, T^{\mu_1} x_2)\} \end{aligned} \tag{2.3}$$

Notice that

$$\begin{aligned} &d(T^{\mu_1} x_1, T^{\mu_1} x_2) \\ &\quad = d(TT^{\mu_1} x_0, TT^{\mu_1} x_1) < \varphi_1\big(M(T^{\mu_1} x_0, T^{\mu_1} x_1)\big) \leq M(T^{\mu_1} x_0, T^{\mu_1} x_1) \\ &\quad = \max\{d(T^{\mu_1} x_0, T^{\mu_1} x_1), d(T^{\mu_1} x_0, T^{\mu_1} x_1), d(T^{\mu_1} x_1, T^{\mu_1} x_2)\} \\ &\quad = d(T^{\mu_1} x_1, T^{\mu_1} x_2), \end{aligned}$$

a contradiction. Therefore (2.3) yields $M(T^{\mu_1} x_0, T^{\mu_1} x_1) = d(T^{\mu_1} x_0, T^{\mu_1} x_1)$. From (2.2),

$$\begin{aligned} d(T^{\mu_1} x_0, T^{\mu_1} x_1) < \varphi_{\mu_1}\big(M(x_0, x_1)\big) &\leq \big(M(x_0, x_1)\big) \\ &= \max\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1)\} \\ &= \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1). \end{aligned}$$

So, $\alpha < d(x_0, x_1)$. Using (2.3) and (A1) and proceeding as above, we have

$$d(T^{\mu_1 + k} x_0, T^{\mu_1 + k} x_1) < \varphi_{\mu_1}\big(M(T^k x_0, T^k x_1)\big) \leq M(T^k x_0, T^k x_1),$$

and

$$\begin{aligned} \alpha &:= \limsup_n d(T^n o T^{\mu_1 + k} x_0, T^n o T^{\mu_1 + k} x_1) \\ &\leq \limsup_n \varphi_{\mu_1}\big(M(T^{\mu_1 + k} x_0, T^{\mu_1 + k} x_1)\big) < M(T^k x_0, T^k x_1) \\ &= \max\{d(T^k x_0, T^k x_1), d(T^k x_0, T^{k+1} x_0), d(T^k x_1, T^{k+1} x_1)\} \\ &= \max\{d(T^k x_0, T^k x_1), d(T^k x_1, T^k x_2)\} = d(T^k x_0, T^k x_1). \end{aligned}$$

Thus we obtain $\alpha < d(T^k x_0, T^k x_1)$ for all $k \in \mathbb{N} \cup \{0\}$. Hence $\{d(T^n x_0, T^n x_1)\}$ converges to α.

Since $0 < \alpha < d(x_0, x_1) < \infty$, there exists $\delta_2 > 0$ and $\mu_2 \in \mathbb{N}$ such that

$$\varphi_{\mu_2}(t) \leq \alpha \quad \text{for all } t \in [\alpha, \alpha + \delta_2].$$

We choose $\mu_3 \in \mathbb{N}$ with $d(T^{\mu_3} x_0, T^{\mu_3} x_1) < \alpha + \delta_2$. Then we have

$$d(T^{\mu_2} o T^{\mu_3} x_0, T^{\mu_2} o T^{\mu_3} x_1) < \varphi_{\mu_2}\big(M(T^{\mu_3} x_0, T^{\mu_3} x_1)\big) \leq \alpha,$$

which is a contradiction. This proves (2.1).

Let $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$, $n \in \mathbb{N}$. From (2.1), $\lim_n (u_n, u_{n+1}) = 0$. Now we show that

$$\lim_{n \to \infty} \sup_{m > n} d(u_n, u_m) = 0. \tag{2.4}$$

Let $\varepsilon > 0$ be fixed. Then there exists $\delta \in (0, \epsilon)$ and $\mu_4 \in \mathbb{N}$ such that $\varphi_{\mu_4}(t) \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta_4]$. For such δ_4 there exists $\mu_5 \in \mathbb{N}$ such that $d(u_n, u_{n+1}) < \delta/\mu_4$ for every $n \geq \mu_5$. We assume that there exist $k, m \in \mathbb{N}$ with $m > k \geq \mu_5$ and $d(u_l, u_m) > 2\epsilon$. Then we put $l = \min\{j \in \mathbb{N} : k < j, \varepsilon + \delta_4 \leq d(u_l, u_j)\}$. Since

$$2\delta_4 < \varepsilon + \delta_4 \leq d(u_k, u_l) \leq \sum_{j=k}^{l-1} d(u_j, u_{j+1} \leq \sum_{j=k}^{l-1} \frac{\delta_4}{\mu_4} = (l-k)\frac{\delta_4}{\mu_4},$$

we have $2\mu_4 < l - k$. Hence $k < l - \mu_4$. Now

$$\begin{aligned} d(u_k, u_{l-\mu_4}) &\geq d(u_k, u_l) - d(u_{l-\mu_4}, u_l) \\ &\geq d(u_k, u_l) - \sum_{j=0}^{\mu_4 - 1} d(u_{l-j-1}, u_{l-j}) \geq \varepsilon + \delta_4 - \mu_4 \frac{\delta_4}{\mu_4} = \varepsilon. \end{aligned}$$

Since $\varepsilon \leq d(u_k, u_{l-\mu_4}) < \varepsilon + \delta$,

$$d(u_{k+\mu_4}, u_l) = d(T^{\mu_4} u_k, T^{\mu_4} u_{l-\mu_4}) < \varphi\big(M(u_k, u_{l-\mu_4})\big) \leq \varepsilon.$$

Therefore

$$d(u_k, u_l) \leq \sum_{j=1}^{\mu_4} d(u_{k+j-1}, u_{k+j}) + d(u_{k+\mu_4}, u_l) < \mu_4 \frac{\delta_4}{\mu_4} + \varepsilon = \delta_4 + \varepsilon.$$

This contradicts the definition of l. Therefore $m > n \geq \mu_5$ implies $d(u_n, u_m) \leq 2\varepsilon$, and (2.4) holds. So $\{u_n\}$ is Cauchy sequence.

Since X is complete, there exists $z \in X$ such that $\{u_n\}$ converges to z. Now from the continuity of T^k,

$$z = \lim_{n \to \infty} T^{k+n} z = \lim_{n \to \infty} T^k o T^n z = T^k (\lim_{n \to \infty} T^n z) = T^k z.$$

Thus z is a fixed point of T^k. Recall that (2.1) is true for any arbitrary point $x_0 \in X$. So, by (2.1), we have

$$\lim_{n\to\infty} d(T^{nk+1}x_0, Tz) = \lim_{n\to\infty} d(T^{nk+1}x_0, T^{nk+1}z) = \lim_{n\to\infty} d(T^n x_0, T^n z) = 0.$$

This yields $Tz = z$. If w is another fixed point of T then by (2.1),

$$d(z, w) = \lim_{n\to\infty} d(T^n w, T^n z) = 0,$$

and $z = w$. This completes the proof.

We remark that the contracting condition (S3) of Suzuki's Theorem 1.3 is included in the condition (A3) of Theorem 2.1. So Theorem 2.1 is a generalization of Theorem 1.3.

Now we extend the scope of Theorem 1.3 by introducing a new map f in Theorem 1.3. The idea of the following theorem comes essentially from Jungck [5] and Singh and Pant [10].

Theorem 2.2. *Let Y be an arbitrary set, (X, d) a metric space. Let and $T, f\colon Y \to X$ such that $T(Y) \subseteq f(Y)$ and*

(B1) $\limsup_n \varphi_n(\varepsilon) \le \varepsilon$ *for all* $\varepsilon \ge 0$*;*
(B2) *for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *and* $\mu \in \mathbb{N}$ *such that* $\varphi_\mu(t) < \varepsilon$ *for all* $t \in [\varepsilon, \varepsilon + \delta]$*;*
(B3) $d(T^n x, T^n y) < \varphi_n\big(d(fx, fy)\big)$ *for all* $n \in \mathbb{N}$ *and* $x, y \in X$ *with* $x \ne y$*.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute just at a point.

Proof. Pick $x_0 \in Y$. Define a sequence $\{y_n\}$ by $y_n = Tx_n = fx_{n+1}$, $n = 0, 1, 2, \ldots$. We can do so since the range of f contains the range of T. First we show that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Assume that $\alpha := \limsup_n d(Tx_n, Tx_{n+1}) > 0$. From the condition (B2), we can choose $\mu_1 \in \mathbb{N}$ satisfying $\varphi_{\mu_1}\big(d(x_0, x_1)\big) \le d(x_0, x_1)$. By (B3) and (B2),

$$d(Tx_{\mu_1}, Tx_{\mu_1+1}) < \varphi_{\mu_1}\big(d(fx_0, fx_1)\big) \le d(fx_0, fx_1). \tag{2.5}$$

Then, proceeding as in the proof of Theorem 2.1,

$$\begin{aligned} \alpha &:= \lim_{n\to\infty} \sup d(Tx_{\mu_1+n}, Tx_{\mu_1+n+1}) \\ &\le \lim_{n\to\infty} \sup \varphi_n\big(d(Tx_{\mu_1}, Tx_{\mu_1+1})\big) \\ &\le d(Tx_{\mu_1}, Tx_{\mu_1+1}) < \varphi_{\mu_1} d(fx_0, fx_1) \le d(fx_0, fx_1). \end{aligned}$$

Using (2.5) and (B2) and proceeding as before, we have $d(Tx_{\mu_1+k}, Tx_{\mu_1+k+1}) < \varphi_{\mu_1}(d(Tx_k, Tx_{k+1}) \leq d(fx_k, fx_{k+1})$, and

$$\begin{aligned}\alpha &:= \lim_{n\to\infty} \sup d(Tx_{\mu_1+n+k}, Tx_{\mu_1+n+k+1}) \\ &\leq \lim_{n\to\infty} \sup \varphi_n\big(d(Tx_{\mu_1+k}, Tx_{\mu_1+k+1})\big) \\ &\leq d(Tx_{\mu_1}, Tx_{\mu_1+1}) < \varphi_{\mu_1} d(fx_k, fx_{k+1}) \\ &\leq d(fx_k, fx_{k+1}) = d(Tx_{k-1}, Tx_k).\end{aligned}$$

Thus we obtain $\alpha < d(Tx_{k-1}, Tx_k)$ for all $k \in \mathbb{N} \cup \{0\}$. Hence $\{d(Tx_n, Tx_{n+1})\}$ converges to α.

Since $0 < \alpha < d(fx_0, fx_1) < \infty$, there exists $\delta_2 > 0$ and $\mu_2 \in \mathbb{N}$ such that

$$\varphi_{\mu_2}(t) \leq \alpha \quad \text{for all } t \in [\alpha, \alpha + \delta_2].$$

We choose $\mu_3 \in \mathbb{N}$ with $d(Tx_{\mu_3}, Tx_{\mu_3+1}) < \alpha + \delta_2$. Then we have

$$d(Tx_{\mu_2+\mu_3}, Tx_{\mu_2+\mu_3+1}) < \varphi_{\mu_2} d(Tx_{\mu_3}, Tx_{\mu_3+1}) \leq \alpha,$$

a contradiction. This proves that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose $\{y_n\}$ is not Cauchy. Then there exists $\beta > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $n \leq m_k < n_k$,

$$d(y_{m_k}, y_{n_k}) \geq \beta \quad \text{and} \quad d(y_{m_k}, y_{n_k-1}) < \beta.$$

By the triangle inequality,

$$d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}).$$

Making $k \to \infty$, $d(y_{m_k}, y_{n_k}) < \beta$. Thus $d(y_{m_k}, y_{n_k}) \to \beta$ as $k \to \infty$. By (B2),

$$\begin{aligned}d(y_{m_k+n}, y_{n_k+n}) &= d(Tx_{m_k+n}, Tx_{n_k+n}) \\ &< \varphi_n\big(d(fx_{m_k}, fx_{n_k})\big) = \varphi_n\big(d(y_{m_k-1}, y_{n_k-1})\big).\end{aligned}$$

Making $k \to \infty$, $\beta \leq \varphi_n(\beta) < \beta$, a contradiction and the sequence $\{y_n\}$ is Cauchy.

Suppose $f(Y)$ is complete. Then $\{y_n\}$ being contained in $f(Y)$ has a limit in $f(Y)$. Call it z. Let $u \in f^{-1}z$. Then $fu = z$. Using (B2),

$$d(Tu, Tx_n) \leq \varphi\big(d(fu, fx_n)\big).$$

Making $n \to \infty$, $d(Tu, z) \leq \varphi(0) < 0$. Therefore $Tu = z = fu$.

If $Y = X$ and the pair (T, f) commutes at u then $Tfu = fTu$ and $TTu = Tfu = fTu = ffu$. In view of (B2), it follows that

$$d(Tu, TTu) < \varphi\big(d(fu, Tu)\big) = \varphi(0) < 0.$$

So $TTu = Tu$ and $fTu = TTu = Tu = z$.

In case $T(Y)$ is a complete subspace of X, then the sequence $\{y_n\}$ converges in $f(Y)$ since $T(Y) \subseteq f(Y)$. So the previous proof works. The unicity of the common fixed point follows easily.

The following examples show the superiority of Theorem 2.2 over Theorems 1.2 and 1.3.

Example 1. Let $X = [1, \infty)$ with the usual metric d. Let $T\colon X \to X$ such that $Tx = x$ and $\varphi_n(t) = \frac{9t}{10}$ for all nonnegative t (or any other choice of φ_n with $\varphi_n(t) < t$).

It can be easily seen that Theorems 1.2 and 1.3 are not applicable to this map T. If we take $fx = x^2$ for all $x \in X$ then T and f satisfy all the hypotheses of Theorem 2.2, and $T1 = f1 = 1$.

Example 2. Let $X = [0, \infty)$ with the usual metric d. Let $T\colon X \to X$ be such that

$$Tx = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Notice that $d(T^n x, T^n y) = |x - y|$ and $\varphi_n\big(d(x, y)\big) = \varphi_n(|x - y|) < |x - y|$ for distinct x, y. So (S3) of Theorem 1.3 can not be satisfied. However, if we take $f\colon X \to X$ such that $fx = 2x$ then T and f satisfy all the hypotheses of Theorem 2.2, and $T0 = f0 = 0$.

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