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ON MULTIPLIERS FOR THE HILBERT SPACE OF A HYPERGROUP

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Abstract. In this note we characterize the multiplier algebra for the Hilbert space of a commutative hypergroup.

Résumé. Dans cette note, nous charactérisons l'algèbre des multiplicateurs pour l'espace hilbertien d'un hypergroupe commutatif.

1. Introduction. Multipliers for hypergroup algebras are of some interest, and several authors have extensively extended some fundamental results on locally compact groups [1], [4], [9], [12] to hypergroups [6], [7], [10]. See also L. Paval [11] for a recent work on multipliers for L^p-spaces. Wendel's theorem, for instance, identifies the multiplier algebra for a hypergroup algebra with the measure algebra; Helson's theorem determines this algebra with a certain set of bounded continuous functions on the dual space when the hypergroup is commutative. In this brief note we characterize the multiplier algebra for the Hilbert space of a commutative hypergroup. In fact, we generalize some results in [4] to the hypergroup case by showing that the multiplier algebra in this case can also be identified with the pseudomeasures on the hypergroup as well as a certain set of bounded functions on the dual space. These results are taken from the author's Ph.D. thesis [2].

2. Preliminaries. Assume $(K, \omega, \sim)$ denotes a locally compact hypergroup with Jewett's axioms [8], where $\omega\colon K \times K \to M^1(K)$, $(x, y) \mapsto \omega(x, y)$, and $\sim\colon K \to K$, $x \mapsto \tilde{x}$, specify the convolution and involution on K. Here $M^1(K)$ stands for all probability measures on K. Throughout, K is a commutative hypergroup, *i.e.,* $\omega(x, y) = \omega(y, x)$ for every $x, y \in K$, with the Haar measure m [13]. The main reference for the following topics is [8].

Let $C_c(K)$, $C_0(K)$, and $C^b(K)$ be the spaces of all continuous functions that have compact support, vanish at infinity, and are bounded on K, respectively. Both $C^b(K)$ and $C_0(K)$ will be topologized by the uniform norm $\|\cdot\|_\infty$, and by Riesz's theorem $C_0(K)^* \cong M(K)$, the space of complex regular Radon measures on K. The translation of $f \in C_c(K)$ at the point $x \in K$, $T_x f$, is defined by $T_x f(y) = \int_K f(t) d\omega(x, y)(t)$ for every $y \in K$.

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Let $(L^p(K), \|\cdot\|_p)$ $(p \geq 1)$ be the usual Banach space. If $p = 1$, $L^1(K)$ is a Banach $*$-algebra, where the convolution and involution of $f, g \in L^1(K)$ are given by $f * g(x) = \int_K f(y) T_{\tilde{y}} g(x) dm(y)$ (m-a.e.) and $f^*(x) = \overline{f(\tilde{x})}$, respectively. The dual of $L^1(K)$ can be identified with the usual Banach space $L^\infty(K)$, and its structure space is homeomorphic to the character space of K, *i.e.,*

$$\mathfrak{X}^b(K) := \big\{\alpha \in C^b(K) : \alpha(e) = 1, \omega(x,y)(\alpha) = \alpha(x)\alpha(y),\ \forall x, y \in K\big\},$$

when equipped with the compact-open topology; $\mathfrak{X}^b(K)$ is a locally compact Hausdorff space. Let $\widehat{K}$ denote the set of all hermitian characters, *i.e.,* $\alpha(\tilde{x}) = \overline{\alpha(x)}$ for every $x \in K$, with the Plancherel measure π on it. In contrast to the group case, $\widehat{K}$ does not, in general, have the dual hypergroup structure and might properly contain $\mathcal{S} = \operatorname{supp} \pi$.

The Fourier–Stieltjes transform of $\mu \in M(K)$, $\widehat{\mu} \in C^b(\widehat{K})$, is

$$\widehat{\mu}(\alpha) := \int_K \overline{\alpha(x)} d\mu(x).$$

Its restriction to $L^1(K)$ is called the Fourier transform, and $\widehat{f} \in C_0(\widehat{K})$ for every $f \in L^1(K)$. The Fourier transform defines an isometric isomorphism of $L^1(K) \cap L^2(K)$ onto $L^1(\mathcal{S}) \cap L^2(\mathcal{S})$ whose extension from $L^2(K)$ onto $L^2(\mathcal{S})$ will be denoted by $\mathcal{P}$.

3. **Multipliers.**

DEFINITION 3.1. Let K be a hypergroup and $B(L^p(K))$ the Banach algebra of bounded linear operators on $L^p(K)$, $1 \leq p < \infty$. Multipliers for $L^p(K)$ are operators $T' \in B(L^p(K))$ such that $T' \circ T_x = T_x \circ T'$ for every $x \in K$. The set of all multipliers for $L^p(K)$ is denoted by $M(L^p(K))$ which is plainly a closed subalgebra of $B(L^p(K))$.

Since the translation operator on $L^p(K)$ is continuous [8], applying Fubini's theorem yields the following lemma.

LEMMA 3.2. *Let $T' \in M(L^p(K))$. Then $T'f * g = T'(f * g) = f * T'g$ for every $f, g \in L^1(K) \cap L^p(K)$.*

In the case $p = 1$, the following characterization of the multipliers for $L^p(K)$ is well known [3], [10].

THEOREM 3.3 (Helson–Wendel). *Let K be a hypergroup. Then the following statements are equivalent:*

(i) *$T' \in M(L^1(K))$.*
(ii) *$T'f * g = T'(f * g) = T'g * f$ for every $f, g \in L^1(K)$.*

(iii) *There exists a unique function* $\varphi \in C^b(\mathcal{S})$ *such that* $\widehat{T'g}| = \varphi \cdot \widehat{g}|$ *for every* $g \in L^1(K)$.
(iv) *There exists a unique* $\mu \in M(K)$ *such that* $T'g = \mu * g$ *for every* $g \in L^1(K)$. *In fact,* $M(L^1(K)) \cong M(K)$ *and* $\|\varphi\|_\infty = \|T_\mu\| \leq \|\mu\| = \|T'\|$.

In the sequel we consider the case $p = 2$. Let us begin with the following proposition, which associates every multiplier for $L^2(K)$ with a unique function on $\mathcal{S}$.

Proposition 3.4. *Let* $T' \in M(L^2(K))$. *Then there exists a unique* $\varphi \in L^\infty(\mathcal{S})$ *such that* $\mathcal{P}(T'g) = \varphi\mathcal{P}(g)$, *for every* $g \in L^2(K)$. *Moreover,* $\|T'\| = \|\varphi\|_\infty$.

Proof. According to [5, Proposition 4.14.9], $\mathcal{S}$ is the disjoint union of a locally π-zero set N and a disjoint locally countable family $(C_i)_{i\in I}$ of a compact subset of $\mathcal{S}$. For each $i \in I$ there exists $f_i \in L^1(K) \cap L^2(K)$ such that $\widehat{f}_i|_{C_i} > 0$. Define the function φ on $\mathcal{S}$ by setting $\varphi := \mathcal{P}(T'f_i)/\widehat{f}_i$ on C_i and $\varphi := 0$ on N. Then Lemma 3.2 implies $\mathcal{P}(T'f) = \varphi \cdot \widehat{f}$ a.e. on C_i for every $f \in L^1(K) \cap L^2(K)$; thence, $\mathcal{P}(T'f) = \varphi \cdot \widehat{f}$ a.e. on $\mathcal{S}$, as the family (C_i) is locally countable and $\widehat{f}$ vanishes outside a σ-finite set. The latter holds for every $f \in L^2(K)$, *i.e.,* $\mathcal{P}(T'f) = \varphi\mathcal{P}(f)$ for every $f \in L^2(K)$, in that T' is continuous and $C_c(K)$ is dense in $L^2(K)$. One can easily verify that φ is measurable and independent of the choice of family f_i. Since $\mathcal{P}$ is an isometric isomorphism of $L^2(K)$ onto $L^2(\mathcal{S})$, we may easily deduce that $\varphi \in L^\infty(\mathcal{S})$ and $\|\varphi\|_\infty \leq \|T'\|$. Contrarily suppose that there is a compact subset $C \subset \mathcal{S}$ such that $\pi(C) > 0$, and $|\varphi(\alpha) > \|T'\|$ for π-almost $\alpha \in C$. Let $g \in L^2(K)$ such that $\mathcal{P}(g) = \chi_C$. Then $\|\varphi\chi_C\|_2 > \|T'\|\sqrt{\pi(C)}$ and, on the other hand, we have $\|\varphi\chi_C\|_2 = \|\varphi\mathcal{P}(g)\|_2 = \|\mathcal{P}(T'g)\|_2 = \|T'g\|_2 \leq \|T'\|\|g\|_2 = \|T'\|\sqrt{\pi(C)}$, which is a contradiction. Finally $\|T'g\|_2 = \|\mathcal{P}(T'g)\|_2 = \|\varphi\mathcal{P}(g)\|_2 \leq \|\varphi\|_\infty\|g\|_2$, and we get $\|T'\| \leq \|\varphi\|_\infty$. □

In order to obtain other characterizations of multipliers for $L^2(K)$, we shall define pseudomeasures on K. Let $A(K) = \{\check{\varphi} : \varphi \in L^1(\mathcal{S})\}$ and define the norm of $\check{\varphi} \in A(K)$ by $\|\check{\varphi}\|_A := \|\varphi\|_1$. Obviously $(A(K), \|\cdot\|_A)$ is a Banach space, but in general, due to the lack of the dual hypergroup structure on $\widehat{K}$, $A(K)$ is not an algebra with the pointwise multiplication. The dual space of $A(K)$ is denoted by $P(K)$ and $\|\sigma\|_P = \sup\{|\sigma(\check{\varphi})| : \|\check{\varphi}\|_A \leq 1\}$, for every $\sigma \in P(K)$. The elements of $P(K)$ are called pseudomeasures on K. If $\mu \in M(K)$ and $\varphi \in L^1(\mathcal{S})$, by the Fourier inversion theorem [3] we have

$$\int_K \check{\varphi}(\tilde{x})d\mu(x) = \int_K \int_{\mathcal{S}} \varphi(\alpha)\overline{\alpha(x)}d\pi(\alpha)d\mu(x) = \int_{\mathcal{S}} \widehat{\mu}(\alpha)\varphi(\alpha)d\pi(\alpha),$$

and hence

$$\Big|\int_K \check{\varphi}(\tilde{x})d\mu(x)\Big| \leq \|\widehat{\mu}\|_\infty\|\varphi\|_1 = \|\widehat{\mu}\|_\infty\|\check{\varphi}\|_A.$$

Therefore each measure $\mu \in M(K)$ can be considered as a pseudomeasure on K, and $\|\mu\|_P \leq \|\widehat{\mu}\|_\infty \leq \|\mu\|$. The Banach space $L^1(\mathcal{S})$ is isometrically isomorphic to $A(K)$ via the mapping $\varphi \to \check{\varphi}$, and so $P(K)$ is isometrically isomorphic to $L^\infty(\mathcal{S})$ via the mapping $\Psi\colon P(K) \to L^\infty(\mathcal{S})$, where for each $\sigma \in P(K)$ the element $\Psi(\sigma) \in L^\infty(\mathcal{S})$ is uniquely determined by

$$\int_{\mathcal{S}} \varphi(\alpha)\overline{\Psi(\sigma)(\alpha)}d\pi(\alpha) = \sigma(\check{\varphi}) \text{ for every } \varphi \in L^1(\mathcal{S}).$$

In particular $\|\mu\|_P = \|\widehat{\mu}\|_\infty$; observe that if $\mu \in M(K)$ and $\mu \neq 0$, then there exists $\alpha \in \mathcal{S}$ such that $\widehat{\mu}(\alpha) \neq 0$, so by the latter equality we have $\|\mu\|_P \neq 0$. Applying the pointwise multiplication in $L^\infty(\mathcal{S})$, we may define for $\sigma_1, \sigma_2 \in P(K)$ the pseudomeasure $\sigma_1 * \sigma_2$ by $\sigma_1 * \sigma_2 = \Psi^{-1}(\Psi(\sigma_1)\Psi(\sigma_2))$ and we call $\sigma_1 * \sigma_2$ the convolution of the pseudomeasures σ_1 and σ_2. The Fourier transform of the pseudomeasure $\sigma \in P(K)$ is considered $\Psi(\sigma) \in L^\infty(\mathcal{S})$. In this way $\Psi\colon P(K) \to L^\infty(\mathcal{S})$ is an isometric algebra isomorphism of $P(K)$ onto $L^\infty(\mathcal{S})$.

It is worthwhile to remark that, since we have

$$\int_{\mathcal{S}} \varphi(\alpha)\overline{\widehat{\mu}(\alpha)}d\pi(\alpha) = \int_K \check{\varphi}(x)d\overline{\mu}(x),$$

for $\mu \in M(K)$ and every $\varphi \in L^1(\mathcal{S})$, the Fourier transform of $\overline{\mu} \in M(K)$, seen as a pseudomeasure, coincides with the Fourier–Stieltjes transform of $\mu \in M(K)$. We shall also mention that a pseudomeasure $\sigma \in P(K)$ belongs to $L^2(K)$ if there exists $g \in L^2(K)$ such that

$$\sigma(\check{\varphi}) = \int_K \check{\varphi}(x)\overline{g(x)}dm(x) \text{ for every } \varphi \in L^1(\mathcal{S}) \cap L^2(\mathcal{S}).$$

In this case the function g is determined uniquely, as $\{\check{\varphi} = \mathcal{P}^{-1}(\varphi) : \varphi \in L^1(\mathcal{S}) \cap L^2(\mathcal{S})\}$ is dense in $L^2(K)$. If $\sigma \in P(K)$ such that $\Psi(\sigma) \in L^2(\mathcal{S})$, then σ belongs to $L^2(K)$. Indeed, setting $g = \mathcal{P}^{-1}(\Psi(\sigma)) \in L^2(K)$, we have

$$\sigma(\check{\varphi}) = \int_{\mathcal{S}} \varphi(\alpha)\overline{\Psi(\sigma)(\alpha)}d\pi(\alpha) = \int_K \check{\varphi}(x)\overline{g(x)}dm(x),$$

for every $\varphi \in L^1(\mathcal{S}) \cap L^2(\mathcal{S})$. If $g \in L^1(K) \cap L^2(K)$, then $\widehat{g} \in L^\infty(\mathcal{S}) \cap L^2(\mathcal{S})$ defines a pseudomeasure $\Psi^{-1}(\widehat{g})$ which belongs to $L^2(K)$. In fact, for $\varphi \in L^1(\mathcal{S})$ we have

$$\int_{\mathcal{S}} \varphi(\alpha)\overline{\widehat{g}(\alpha)}d\pi(\alpha) = \int_K \check{\varphi}(x)\overline{g(x)}dm(x).$$

In particular, the convolution $\sigma * g = \Psi^{-1}(\Psi(\sigma)\widehat{g})$ of $\sigma \in P(K)$ and $g \in L^1(K) \cap L^2(K)$ is well defined as a convolution of pseudomeasures.

Theorem 3.5. *Let K be a hypergroup. Then the following statements are equivalent:*

(i) $T' \in M(L^2(K))$.
(ii) $T'f * g = T'(f * g) = f * T'g$ *for every* $f, g \in L^1(K) \cap L^2(K)$.

(iii) *There exists a unique* $\eta \in L^\infty(\mathcal{S})$ *such that* $T'(g) = \mathcal{P}^{-1}(\eta\mathcal{P}(g))$ *for every* $g \in L^2(K)$.

(iv) *There exists a unique pseudomeasure* $\sigma \in P(K)$ *such that* $T'(g) = \sigma * g$ *for every* $g \in L^1(K) \cap L^2(K)$. *Moreover,* $\|\eta\|_\infty = \|\sigma\|_P = \|T'\|$.

Proof. Lemma 3.2 implies (i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (iii) is due to Proposition 3.4. To establish (iii) $\Rightarrow$ (iv), let $\eta \in L^\infty(\mathcal{S})$ and denote by $\Phi\colon L^2(K) \to L^2(K)$ the map $\Phi(g) = \mathcal{P}^{-1}(\eta\mathcal{P}(g))$. Let $\sigma := \Psi^{-1}(\eta) \in P(K)$. We know that for $g \in L^1(K) \cap L^2(K)$, $\sigma * g$ exists as a pseudomeasure and $\Psi(\sigma * g) = \Psi(\sigma)\widehat{g} = \eta\mathcal{P}(g)$. Moreover, $\eta\mathcal{P}(g) \in L^2(\mathcal{S}) \cap L^\infty(\mathcal{S})$ and from the above considerations we have that

$$\Psi^{-1}(\eta\mathcal{P}(g)) = \sigma * g \in L^2(K),$$

and

$$\sigma * g = \Psi^{-1}(\eta\mathcal{P}(g)) = \mathcal{P}^{-1}(\eta\mathcal{P}(g)) = \Phi(g).$$

Other parts and the implication (iv) $\Rightarrow$ (i) are obvious. □

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