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A NOTE ON GAGLIARDO–NIRENBERG TYPE INEQUALITIES ON ANALYTIC SETS

To my advisor Pierre Milman, on the occasion of his coming birthday

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Presented by Pierre Milman, FRSC

Abstract. Given an analytic set X and $x \in X$, we show that X admits (in a relatively compact neighbourhood of x) a modified Gagliardo–Nirenberg inequality, depending on a certain exponent $s \geqslant 1$ ($s = 1$ in case of a manifold). The infimum of the set of all such s characterizes, in a sense, the type of singularity at x.

Résumé. Etant donné un ensemble analytique X et $x \in X$, nous montrons que X admet (dans un voisinage relativement compact de x) une inégalité de Gagliardo–Nirenberg modifiée, en fonction d'un certain exposant $s \geqslant 1$ ($s = 1$ dans le cas d'une variété). La borne inférieure de l'ensemble de tous ces s caractérise, en un sens, le type de singularité en x.

The classical Gagliardo–Nirenberg inequality is valid for functions defined on a compact domain in $\mathbb{R}^n$ having sufficiently "good" boundary. As was shown in [5], in the case when the boundary of the domain has outward pointing cusps, a Gagliardo–Nirenberg inequality holds but with a certain exponent $s \geqslant 1$ ($s = 1$ in the case of a boundary). In the present note we show the validity of Gagliardo–Nirenberg inequalities (in quotient norms) for sufficiently "good" functions defined on a real or complex analytic set, also with a certain exponent $s \geqslant 1$ ($s = 1$ in case of a manifold), and we give an example of an algebraic set with an isolated singularity which does not admit a Gagliardo–Nirenberg inequality with any exponent smaller than a certain value $s > 1$.

1. Introduction: Gagliardo–Nirenberg inequalities in the smooth case. Consider first the case of a real analytic set. Let N be a relatively compact open subset of $\mathbb{R}^n$. Given $f \in C^\infty(\bar{N})$ and $m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, we define

$$|f|_m^N := \sum_{|\gamma| \leqslant m} |D^\gamma f|^N, \tag{1.1}$$

where $\gamma = (\gamma_1, \dots, \gamma_n) \subset \mathbb{Z}_+^n$, $D^\gamma = \frac{\partial^{\gamma_1}}{\partial^{\gamma_1} x_1} \cdots \frac{\partial^{\gamma_n}}{\partial^{\gamma_n} x_n}$, and $|\cdot|^N$ is the ordinary sup-norm on N.

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Let X be a real analytic set in $\mathbb{R}^n$ (assumed to be the closure in $\mathbb{R}^n$ of the set of its smooth points), $x \in X$. Given a relatively compact neighbourhood U of $x \in \mathbb{R}^n$ and a function $f \in C^\infty(\bar{U})$, we define its quotient-norm

$$\|f\|_{m,X}^U := \inf_{(h=f)|_{U\cap X}} |h|_m^U, \tag{1.2}$$

where the infimum is taken over all $h \in C^\infty(\bar{U})$ such that $h = f$ on $X \cap U$.

We will say that X admits a Gagliardo–Nirenberg inequality at $x \in X$ with exponent $s \geqslant 1$ if there exist a neighbourhood U of $x \in \mathbb{R}^n$ and constants $C_m > 0$ such that for all $f \in C^\infty(\bar{U})$ ($f \not\equiv 0$ on $X \cap U$), all m and all $1 \leqslant k < m/s$ we have

$$\frac{\|f\|_{k,X}^U}{\|f\|_{0,X}^U} \leqslant C_0 C_m^k \left(\frac{\|f\|_{m,X}^U}{\|f\|_{0,X}^U} \right)^{\frac{sk}{m}}. \tag{1.3}$$

If $s = 1$, and the quotient norms in (1.3) are replaced with the ordinary sup-norms on a relatively compact domain in $\mathbb{R}^n$ with smooth boundary, (1.3) becomes the classical Gagliardo–Nirenberg inequality, see [5].

PROPOSITION 1. *Suppose that x is a regular point of X. Then X admits a Gagliardo–Nirenberg inequality at x with exponent $s = 1$.*

PROOF. Let $x = 0 \in X$. There exists a relatively compact neighbourhood U of $0 \in \mathbb{R}^n$ such that $X \cap U$ is an analytic manifold. Without loss of generality, $X \cap U$ is a coordinate chart. There exists an analytic diffeomorphism $\psi \colon U' \mapsto U$, where $U' \subset \mathbb{R}^n$ is relatively compact, such that $\psi^{-1}(X \cap U) = U' \cap \mathbb{R}^p$, $p \leqslant n$. Let $U_0' \subset U'$ be an open subset such that $U_0' \cap \mathbb{R}^p$ has smooth boundary and $U_0 := \psi(U_0') \ni 0$. Then the norms

$$\|\cdot\|_{m,X}^{U_0} \quad \text{and} \quad \|\psi^* \cdot\|_{m,\psi^{-1}(X\cap U_0)}^{U_0'}$$

are equivalent, and

$$\|\psi^* f\|_{m,\psi^{-1}(X\cap U_0)}^{U_0'} = |\psi^* f|_{m,\psi^{-1}(X\cap U_0)}^{U_0'},$$

since $\psi^* f := f \circ \psi$ can be extended from $\psi^{-1}(X \cap U)$ to U' as being constant with respect to the new variables. Since $\psi^{-1}(X \cap U)$ has smooth boundary and is relatively compact, it admits a classical Gagliardo–Nirenberg inequality with exponent $s = 1$. That is, there exist $C_m > 0$ such that

$$\frac{|\psi^* f|_{k,\psi^{-1}(X\cap U)}}{|\psi^* f|_{0,\psi^{-1}(X\cap U)}} \leqslant C_0 C_m^k \left(\frac{|\psi^* f|_{m,\psi^{-1}(X\cap U)}}{|\psi^* f|_{0,\psi^{-1}(X\cap U)}} \right)^{\frac{k}{m}}$$

for all m and all $1 \leqslant k < m$. This implies the required result. $\square$

2. **Gagliardo–Nirenberg inequalities near singular points.** Let us turn to the case when $x \in X$ can be a singular point.

Let M be a smooth manifold and N be a relatively compact open subset of M. There exist finitely many coordinate charts S_i on M with coordinate maps $\eta_i \colon S_i \mapsto \mathbb{R}^l$ ($l = \dim(M)$) which cover N. Furthermore, we can find finitely many relatively compact open subsets $T_j \subset M$ such that $T_j \Subset S_i$ for certain $i = i(j)$ (in what follows, let us fix some choice of i), and $N \subset \bigcup_j T_j$. Given $f \in C^\infty(\bar{N})$, define

$$|f|_m^N = \max_j |\eta_i^* f|_m^{\eta_i^{-1}(T_j \cap N)} < \infty, \tag{2.1}$$

where $\eta_i^{-1}(T_j \cap N)$ is, evidently, relatively compact for each j and $i = i(j)$, and the norms in the right-hand side of (2.1) are defined by (1.1).

The norm defined by (2.1) is determined by our choice of S_i, η_i, T_j and $i = i(j)$. Nevertheless, it is easy to see that any two of such norms are equivalent. The following estimate follows straightforwardly from [4, pp. 774–775] and [1, p. 2].

Proposition 2 ([1], [4]). *Let $X \subset \mathbb{R}^n$ be an analytic set with $0 \in X$, and let $U \subset \mathbb{R}^n$ be a relatively compact neighbourhood of $0 \in \mathbb{R}^n$. Let M be an analytic manifold and let $\varphi \colon M \mapsto \mathbb{R}^n$ be a proper analytic map such that $\varphi(M) = X$. Then there exists a constant $c \in \mathbb{N}$ such that for all $f \in C^\infty(\bar{U})$ we have*

$$\|f\|_{k,X}^U \leqslant C|\varphi^* f|_{ck}^{\psi^{-1}(X \cap U)}$$

for some fixed $C > 0$ for all $k \geqslant 1$.

Proposition 3. *Given an analytic set X, it admits a Gagliardo–Nirenberg inequality at any of its points x (with a certain exponent $s = s(x) \geqslant 1$).*

Proof. Without loss of generality, $0 \in X$, and we show that X admits a Gagliardo–Nirenberg inequality at 0 with certain exponent $s \geqslant 1$. Let $U = \{x \in \mathbb{R}^n : b_\varepsilon(x) < 0\}$, where $b_\varepsilon = \sum_{k=1}^n x_k^2 - \varepsilon$, be an open ball of sufficiently small radius $\varepsilon > 0$ centered at 0. There exists an analytic manifold M with $\dim M = \dim X$ and a proper analytic function $\varphi \colon M \mapsto \mathbb{R}^n$ such that $\varphi(M) = X$ (see [2]). Define $N = \varphi^{-1}(X \cap U)$. Then N is a relatively compact subset of M.

First, we show that the sets T_j in the definition of the norm (2.1) can be chosen in such a way that $\eta_i^{-1}(T_j \cap N)$ is a relatively compact and semianalytic subset of $\mathbb{R}^l$ ($l = \dim M = \dim X$). Let ι be an (analytic) embedding of M in $\mathbb{R}^q$ for some q. Define

$$T_j := \iota^{-1}\big(\iota(M) \cap B(x_j, \varepsilon_j)\big),$$

a relatively compact subset of M, where $B(x_j, \varepsilon_j) \subset \mathbb{R}^q$ is the open ball of radius $\varepsilon_j > 0$ centered at $x_j \in \iota(M)$, and $\varepsilon_j > 0$ and x_j are chosen in such a way that the remaining conditions on T_j in (2.1) are satisfied. We have

$$\eta_i^{-1}(T_j) = \{x \in \mathbb{R}^l : (\eta_i^* \iota^* b_{\varepsilon_j})(x) < 0\},$$

where $\eta_i^* \iota^* b_{\varepsilon_j}$ is an analytic function, and by definition, $\eta_i^{-1}(T_j)$ is semianalytic. Now since η_i is a diffeomorphism, $\eta_i^{-1}(T_j \cap N) = \eta_i^{-1}(T_j) \cap \eta_i^{-1}(N)$, where

$$\eta_i^{-1}(N) = \{x \in \mathbb{R}^l : (\eta_i^* \varphi^* b'_c)(x) < 0\}, \quad b'_c(y) = \sum_{k=1}^n y_k^2 - c,$$

and $\eta_i^* \varphi^* b'_c$ is an analytic function. Thus, $\eta_i^{-1}(N)$ is also semianalytic, and $\eta_i^{-1}(T_j) \cap \eta_i^{-1}(N)$ is semianalytic as well.

Secondly, we show that there exist $\hat{s} \geqslant 1$ and $\hat{C}_m > 0$ such that, given any $f \in C^\infty(\bar{N})$, we have for all m and all $1 \leqslant k < m/\hat{s}$

$$\frac{|\varphi^* f|_k^N}{|\varphi^* f|_0^N} \leqslant \hat{C}_0 \hat{C}_m^k \left(\frac{|\varphi^* f|_m^N}{|\varphi^* f|_0^N} \right). \tag{2.2}$$

Indeed, since $\eta_i^{-1}(T_j \cap N)$ are semianalytic and relatively compact, we have, according to [5], that there exist $\hat{s}_j \geqslant 1$ and $\hat{C}_{mj} > 0$ such that for all m and all $1 \leqslant k < m/\hat{s}_j$

$$\frac{|\eta_i^* \varphi^* f|_k^{\eta_i^{-1}(T_j \cap N)}}{|\eta_i^* \varphi^* f|_0^{\eta_i^{-1}(T_j \cap N)}} \leqslant \hat{C}_{0j} \hat{C}_{mj}^k \left(\frac{|\eta_i^* \varphi^* f|_m^{\eta_i^{-1}(T_j \cap N)}}{|\eta_i^* \varphi^* f|_0^{\eta_i^{-1}(T_j \cap N)}} \right).$$

Let $\hat{s} = \max_j \hat{s}_j$, $\hat{C}_m = \max_j \hat{C}_{mj}$. Taking into account that

$$\max_j |\eta_i^* \varphi^* f|_0^{\eta_i^{-1}(T_j \cap N)} = |\varphi^* f|_0^N,$$

we obtain, by definition of the norm (2.1), the required inequality. According to Proposition 2 there exists $c \in \mathbb{N}$ such that for all $f \in C^\infty(\bar{U})$ we have

$$\|f\|_{k,X}^U \leqslant C |\varphi^* f|_{ck}^N \tag{2.3}$$

for some fixed $C > 0$ for all $k \geqslant 1$. Along with that, it follows from the definition of the norm (2.1) and the fact that the sets $\eta_i^{-1}(T_j \cap N)$ are relatively compact, that there exists $B > 0$ such that

$$|\varphi^* h|_m^N \leqslant B |h|_m^U \tag{2.4}$$

for any $h \in C^\infty(\bar{U})$. Since, given any $h \in C^\infty(\bar{U})$ such that $h|_X = f|_X$, we have $|\varphi^* h|_m^N = |\varphi^* f|_m^N$, we can take the infimum in (2.4) over all such h to get

$$|\varphi^* f|_m^N \leqslant B \|f\|_{m,X}^U. \tag{2.5}$$

Next, as follows from the estimates (2.2), (2.3) and (2.5),

$$\frac{\|f\|^U_{k,X}}{\|f\|^U_{0,X}} \leqslant C\hat{C}_0\hat{C}_m^{ck}B^{\frac{\hat{s}ck}{m}}\left(\frac{\|f\|^U_{m,X}}{\|f\|^U_{0,X}}\right)^{\frac{\hat{s}ck}{m}}.$$

for all m and all $1 \leqslant k \leqslant \frac{m}{\hat{s}c}$. To complete our proof we put $C_0 := C\hat{C}_0$, $C_m := \hat{C}_m^c B^{\frac{\hat{s}c}{m}}$ $(m \geqslant 1)$ and $s := \hat{s}c$. □

Let us note that the proof of Proposition 3 actually provides an estimate for s.

3. Examples

Example 1. Let $X = \{(x, y) \in \mathbb{R}^2 : y^q = x^p\}$, where q is even and $\frac{p}{q} > 1$.

Let us show that X does not admit a Gagliardo–Nirenberg inequality at 0 with any exponent smaller than $s = \frac{p}{q}$. We employ, with slight modification, the family of functions that was used in [5] for the proof of an analogous statement for a compact domain in $\mathbb{R}^n$.

(1) Suppose first that p is odd. Consider on $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ the family of functions $f_k(x, y) = y\varphi(1 - kx)$, where

$$\varphi(x) = \begin{cases} e^{-1/x} & x > 0, \\ 0 & x \leqslant 0. \end{cases}$$

Note that for each k

$$(3.1)\quad e^{-1} = \left|\frac{\partial f_k}{\partial y}(0,0)\right| \leqslant \inf_{g_k|_{X\cap U}=f_k|_{X\cap U}} \sup_{(x,y)\in U}\left\{\left|\frac{\partial g_k}{\partial x}(x,y)\right| + \left|\frac{\partial g_k}{\partial y}(x,y)\right|\right\}$$
$$\leqslant \|f_k\|^U_{1,X}.$$

The last inequality follows straightforwardly from the definition of the quotient norm. The first inequality holds due to the following two facts:

(a) given any $g_k \in C^\infty(U)$ such that $g_k|_{X\cap U} = f_k|_{X\cap U}$ we have that $\nabla g_k(0,0) = \nabla f_k(0,0)$, since for every k

$$\frac{\partial g_k}{\partial x}(0,0) = \lim_{x\to 0+}\frac{f_k(x, x^{\frac{p}{q}}) - f_k(0,0)}{x}$$
$$= \lim_{x\to 0+}\frac{\frac{\partial f_k}{\partial x}(0,0)x + \frac{\partial f_k}{\partial y}(0,0)x^{\frac{p}{q}} - o(|x|)}{x} = \frac{\partial f_k}{\partial x}(0,0),$$

and, similarly,

$$\frac{\partial g_k}{\partial y}(0,0) = \lim_{x\to 0+}\frac{f_k(x, x^{\frac{p}{q}}) - f_k(x, -x^{\frac{p}{q}})}{2x^{\frac{p}{q}}} = \frac{\partial f_k}{\partial y}(0,0),$$

(b) for any such g we have

$$\Big|\frac{\partial f_k}{\partial y}(0,0)\Big| = \Big|\frac{\partial g_k}{\partial y}(0,0)\Big| \leqslant \Big|\frac{\partial g_k}{\partial x}(0,0)\Big| + \Big|\frac{\partial g_k}{\partial y}(0,0)\Big|$$
$$\leqslant \sup_{(x,y)\in U}\Big\{\Big|\frac{\partial g_k}{\partial x}(x,y)\Big| + \Big|\frac{\partial g_k}{\partial y}(x,y)\Big|\Big\},$$

so

$$\Big|\frac{\partial f_k(0,0)}{\partial y}\Big| \leqslant \inf_{g_k|_{X\cap U}=f_k|_{X\cap U}} \sup_{(x,y)\in U}\Big\{\Big|\frac{\partial g_k}{\partial x}(x,y)\Big| + \Big|\frac{\partial g_k}{\partial y}(x,y)\Big|\Big\}.$$

Thus, $e^{-1} \leqslant \|f_k\|^U_{1,X}$ for all k. Also, observe that we always have

$$\|f_k\|^U_{0,X} = \sup_{(x,y)\in X\cap U} |f_k(x,y)|,$$

and, as a result, for all k

$$\|f_k\|^U_{0,X} = \sup_{(x,y)\in X\cap U} |f_k(x,y)| = \sup_{(x,y)\in X\cap U} |y\varphi(1-kx)| = \sup_{0\leqslant x<1/k} |x^{\frac{p}{q}}\varphi(1-kx)|,$$

since for any $x \geqslant 1/k$ we have $\varphi(1-kx)=0$. Furthermore,

$$\sup_{0\leqslant x<1/k} |x^{\frac{p}{q}}\varphi(1-kx)| \leqslant \sup_{0\leqslant x<1/k} |x^{\frac{p}{q}}| \sup_{0\leqslant x<1/k} |\varphi(1-kx)| \leqslant k^{-\frac{p}{q}}e^{-1}.$$

To estimate $\|f_k\|^U_{m,X}$, we take the extension of $f_k|_{X\cap U}$ from $X\cap U$ to U, namely, f_k itself. Then

$$\|f_k\|^U_{m,X} \leqslant |f_k|^U_m \leqslant C_m k^m$$

for a certain $C_m > 0$, as follows from the definition of f_k. The inequalities obtained above imply that for all k and m,

$$e^{-1} \leqslant C_0 e^{-1} C (C_1)^m k^{-\frac{p}{q}(1-\frac{s}{m})} k^{m\frac{s}{m}},$$

so fixing m and taking $k\to\infty$, we obtain that

$$-\frac{p}{q}\Big(1-\frac{s}{m}\Big) + s \geqslant 0.$$

Thus, $s \geqslant \frac{p}{q}$; otherwise taking m sufficiently large and then letting $k\to\infty$, we arrive at a contradiction $e^{-1} \leqslant 0$.

(2) In the case when p is even, X is symmetric with respect to y-axis, and we consider the family of functions $f_k(x,y) = y\varphi(1-kx)\varphi(1+kx)$. We have

$$\|f_k\|^U_{0,X} = \sup_{(x,y)\in X\cap U} |f_k(x,y)| = \sup_{(x,y)\in X\cap U} |y\varphi(1-kx)\varphi(1+kx)|$$
$$= \sup_{-1/k\leqslant x<1/k} |x^{\frac{p}{q}}\varphi(1-kx)\varphi(1+kx)|$$
$$\leqslant \sup_{-1/k\leqslant x<1/k} x^{\frac{p}{q}} \sup_{-1/k\leqslant x<1/k} |\varphi(1-kx)\varphi(1+kx)|$$
$$\leqslant k^{-\frac{p}{q}}e^{-1/2}e^{-1/2} = k^{-\frac{p}{q}}e^{-1}.$$

The proof of the inequality $e^{-2} \leqslant \|f_k\|^U_{1,X}$, $k \in \mathbb{N}$, is analogous to the proof of (3.1). Similarly, we can show that $\|f_k\|^U_{m,X} \leqslant |f_k|^U_m \leqslant C'_m k^m$ for a certain C'_m for all $k \in \mathbb{N}$. This gives us the required result.

In the complex analytic case (changing our definition of norm (1.1)), we have

$$\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n, \quad |\gamma| = \sum_{j=1}^{n} |\gamma_j|, \quad D^\gamma = \frac{\partial^{\gamma_1}}{\partial^{\gamma_1} x_1} \cdots \frac{\partial^{\gamma_n}}{\partial^{\gamma_n} x_n},$$

$$\frac{\partial^{\gamma_j}}{\partial^{\gamma_j} x_j} := \frac{\partial^{\gamma_j}}{\partial^{\gamma_j} \operatorname{Re}(x_j)} \text{ if } \gamma_j > 0, \quad \frac{\partial^{\gamma_j}}{\partial^{\gamma_j} x_j} := \frac{\partial^{|\gamma_j|}}{\partial^{|\gamma_j|} \operatorname{Im}(x_j)} \text{ if } \gamma_j < 0,$$

$$|f|^N_m := \sum_{|\gamma| \leqslant m} |D^\gamma f|^N,$$

which gives us complex analogs of the norms (1.2) and (2.1). Our set $X \subset \mathbb{C}^n$ can be viewed as a real analytic set embedded in $\mathbb{R}^{2n}$. The algebras $\mathbb{C}^\infty(U)$ and $\mathbb{C}^\infty(\bar{U})$, where U is a relatively compact domain, remain the same regardless of whether we consider U as a subdomain of $\mathbb{C}^n$ or $\mathbb{R}^{2n}$, so Propositions 1 and 3 remain valid.

Example 2. Let $X = \{(x, y) \in \mathbb{C}^2 : y^q = x^p\}$, where $\frac{p}{q} > 1$. Let us show that X does not admit a Gagliardo–Nirenberg inequality at 0 with any exponent smaller than $s = \frac{p}{q}$. We consider the family of functions

$$f_k(x, y) = \begin{cases} y e^{-\frac{1}{1-k^2 x\bar{x}}} & |x| \leqslant \frac{1}{k}, \\ 0 & |x| > \frac{1}{k}. \end{cases}$$

Let $U = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 \leqslant 1\}$. First, let q be even. We have $\operatorname{Re}(y)^q = \operatorname{Re}(x)^p$ for any $(x, y) \in X$. Then the same argument as in Example 1 gives us that

$$e^{-1} = \Big| \frac{\partial f_k}{\partial \operatorname{Re}(y)}(0, 0) \Big| \leqslant \|f_k\|^U_{1,X}.$$

Similarly,

$$\|f_k\|^U_{0,X} \leqslant k^{-\frac{p}{q}} e^{-1}, \quad \|f_k\|^U_{m,X} \leqslant |f_k|^U_m \leqslant C''_m k^m$$

for certain $C''_m > 0$ for all k. The argument used in Example 1 gives us the required inequality $s \geqslant \frac{p}{q}$.

Let q be odd. The estimates for $\|f_k\|^U_{0,X}$ and $\|f_k\|^U_{m,X}$ remain the same. Let us show that

$$\Big| \frac{\partial g_k(0, 0)}{\partial \operatorname{Im}(y)} \Big| = e^{-1} \tag{3.2}$$

for any extension g_k of f_k from $X \cap U$ to U, so that $\|f_k\|^U_{1,X} \geqslant e^{-1}$. Then an argument identical to the one used in Example 1 will give us $s \geqslant \frac{p}{q}$. Let $x = re^{i\theta}$. Denote

$$y_1 = e^{\frac{2\pi i}{q}} r^{\frac{p}{q}} e^{i\frac{p}{q}\theta}, \quad y_2 = e^{-\frac{2\pi i}{q}} r^{\frac{p}{q}} e^{i\frac{p}{q}\theta}.$$

Then $(x, y_1) \in X$, $(x, y_2) \in X$ for any value of θ, so we can put $\theta = 0$. We have

$$\begin{aligned}\frac{\partial g_k(0,0)}{\partial \operatorname{Im}(y)} &= \lim_{r\to 0+} \frac{f_k(re^{i\theta}, r^{\frac{p}{q}} e^{\frac{2\pi i}{q}} e^{i\frac{p}{q}\theta}) - f_k(re^{i\theta}, r^{\frac{p}{q}} e^{-\frac{2\pi i}{q}} e^{i\frac{p}{q}\theta})}{r^{\frac{p}{q}} e^{i\frac{p}{q}\theta}(e^{\frac{2\pi i}{q}} - e^{-\frac{2\pi i}{q}})}\Big|_{\theta=0} \\ &= \lim_{r\to 0} \frac{\begin{array}{c} f_k\big(r, 0, r^{\frac{p}{q}}\cos(2\pi/q), r^{\frac{p}{q}}\sin(2\pi/q)\big) \\ - f_k\big(r, 0, r^{\frac{p}{q}}\cos(2\pi/q), -r^{\frac{p}{q}}\sin(2\pi/q)\big)\end{array}}{2r^{\frac{p}{q}}\sin(2\pi/q)} = \frac{1}{e},\end{aligned}$$

which gives us equality (3.2), as required.

4. Singularity exponent for Gagliardo–Nirenberg inequalities on an irreducible complex curve.

It follows from the inequality $\|f\|^U_{m,X}/\|f\|^U_{0,X} \geqslant 1$ that if (1.3) holds for some exponent s, then it also holds for any larger exponent. Given an analytic set X and $x \in X$, we define

$$s_* = \inf\big\{s \geqslant 1 : X \text{ admits a Gagliardo–Nirenberg} \\ \text{inequality at } x \text{ with exponent } s\big\}.$$

The following is an open problem. Suppose that we are given an analytic set $X \subset \mathbb{C}^2$ whose germ at 0 is defined to be the zero set of irreducible power series

$$f(x, y) = y^d + \sum_{k=0}^{d-1} a_k(x) y^k,$$

where $a_k \in \mathbb{C}\{x\}$, $a_k(0) = 0$. According to Puiseaux's Theorem (see [3]), there exists $\varphi \in \mathbb{C}\{z\}$ such that

$$f(z^d, y) = \prod_{j=1}^{d} \big(y - \varphi(e^{i\frac{2\pi j}{d}} z)\big), \quad z, y \in \mathbb{C}.$$

Question 1. Let $\varphi(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$, $b_k \neq 0$, and

$$l_* := \min\{l : \gcd(n_1, \ldots, n_l) = 1\}.$$

Is it true that $s_* = \frac{n_{l_*}}{d}$ at 0?

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