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TOWARDS AN OPTIMAL RESULT ON UNIQUE CONTINUATION FOR SOLUTIONS OF SCHRÖDINGER INEQUALITIES

Dedicated to our advisor Pierre Milman, on the occasion of the first anniversary of our Working Seminar

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ABSTRACT. We establish the property of unique continuation (also known as quasianalyticity property for C^∞ functions) for functions u satisfying a differential inequality $|\Delta u| \leq |V|\,|u|$ with potentials V from a wide class of functions (including locally $L^{\frac{d}{2},\infty}(\mathbb{R}_d)$ spaces) for which the self-adjoint Schrödinger operator is well defined.

Motivating question: Is it true that for potentials V, for which the self-adjoint Schrödinger operator is well defined, its eigenfunctions satisfy the unique continuation property?

RÉSUMÉ. On montre la propriété de l'extension unique (également connue comme quasianaliticité des fonctions C^∞) des fonctions u qui satisfout l'inégalité differentiel $|\Delta u| \leq |V|\,|u|$ avec des potentiels V d'une grande classe de fonctions (y compris des espaces $L^{\frac{d}{2},\infty}(\mathbb{R}_d)$) pour lesquelles l'opérateur auto-adjoint de Schrödinger est bien défini.

Pour motiver la question: est-ce que les fonctions propres de tous les potentiels V pour qui l'opérateur auto-adjoint de Schrödinger est bien défini satisfait la propriété d'extension unique?

1. Introduction. Let Ω be a connected open subset of $\mathbb{R}^d$ ($d \geq 3$),

$$\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2},$$

$X_p := L^p_{\mathrm{loc}}(\Omega)$ the space of measurable functions f with $|f|^p$ locally integrable on Ω, $D' = D'(\Omega)$ the space of distributions over $\mathcal{C}_0^\infty(\Omega)$, $H^{m,p}_{\mathrm{loc}}(\Omega)$ the Sobolev space (of functions in X_p such that all their weak derivatives up to order m are in X_p, see [SW]) and finally

$$\mathcal{L}^{2,1}_{\mathrm{loc}} := \{f \in X_1 : \Delta f \in D' \cap X_1\}.$$

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Let $V \in L^1_{\rm loc}(\Omega)$ and $Y_V \subset \mathcal{L}^{2,1}_{\rm loc}(\Omega)$ be a class of functions depending on V (see Definition 1 below). We say that the differential inequality

$$(1.1) \qquad |\Delta u(x)| \leq |V(x)|\,|u(x)| \quad \text{(almost everywhere in } \Omega)$$

has the property of *weak* unique continuation (WUC) in Y_V provided that whenever u in Y_V satisfies (1.1) and vanishes in a non-empty open subset of Ω, it follows that u is identically equal to 0 in Ω. We also say that the differential inequality (1.1) has the property of *strong* unique continuation (SUC) in Y_V if whenever u in Y_V satisfies (1.1) and vanishes to an infinite order at $x_0 \in \Omega$, *i.e.*,

$$\lim_{\rho\to 0} \rho^{-N} \int_{B(x_0,\rho)} |u(x)|^2\,dx = 0 \quad \text{for any } N > 0,$$

it follows that u is identically equal to 0 on Ω.

An important problem related to WUC arising in quantum mechanics, is the question of absence of positive eigenvalues for a self-adjoint Schrödinger operator (see [Si] and Section 2 below). T. Kato proved [K1] that if potential V is bounded then the respective Schrödinger operator H does not have positive eigenvalues. In this article we extend Kato's result (see Theorem 3).

The precise classes Y_V that we consider are as follows.

Definition 1.

$Y_V^{\rm weak} := \mathcal{L}^{2,1}_{\rm loc} \cap \{f : |V|^{1/2} f \in X_2\}$

$Y_V^{\rm str} := Y_V^{\rm weak} \cap H^{1,p}_{\rm loc}(\Omega)$ with $p = \frac{2d}{d+2}$.

Note that, even though the dependence on V of Y_V does not appear explicitly in other papers, [ABG], [ChSa], [JK], [St], [Sa], [W], it is implicit (see Section 3 for details).

In the case that $d = 2$ and V is a locally bounded function Carleman proved in 1939 [C] that (1.1) has the SUC property in Y_V when $Y_V = H^{2,2}_{\rm loc}(\Omega)$. Following Carleman, unique continuation was studied by many authors, *e.g.* [ABG], [ChSa], [JK], [K1], [St], [Sa], [W]. In this article we also follow Carleman's scheme and show that, for potentials V from a wide class of functions, differential inequality (1.1) has SUC and WUC in the respective classes $Y_V^{\rm str}$ and $Y_V^{\rm weak}$.

Traditionally proofs of unique continuation rely on the Carleman type estimate on the norms of the appropriate operators from L^p to L^q, for some p and q (*e.g.* [ABG, inequality (18)], [JK, Theorem 2.1], [St, Theorem 1]). Our method is based on an $L^2 \to L^2$ estimate of Proposition 1 (main inequality). Consequently, we prove WUC for (1.1) in $Y_V^{\rm weak}$ (previously it was derived only for $Y_V = H^{2,q}_{\rm loc}(\Omega)$ with $q > 1$).

The proof of our main inequality (Proposition 1) is based on Stein's interpolation theorem [St1] and relies on Lemma 1—a generalized variant of inequalities considered in [Sa], [St2] (*cf.* Lemma 1 in [Sa], Lemma 5 in [St]).

In the second section we formulate our results. In the third section we compare the latter with similar results in [Sa], [JK] and [St], and in the last section we sketch the main ideas of our approach (see complete details in [KS]).

2. **Main results.** Let $V \in X_1$ be a potential and let H_0 be defined as the dual of $-\Delta|_{C^\infty(\Omega)}$ on $L^2(\Omega)$ with $D(H_0) = H^{2,2}(\Omega) \subset L^2(\Omega)$. Suppose that V is such that the so-called Schrödinger operator

$$H = H_0 \dotplus V \tag{2.1}$$

is well defined in the sense of the form-sum (see [K2, Chapter VI]). Then, H is a self-adjoint extension of the algebraic sum $-\Delta + V$.

Below $\|A\|_{2\to2}$ denotes the operator norm of $A\colon L^2 \to L^2$.

Remark 1. The form-sum $\dotplus$ operation has many remarkable properties (see [KPS]). An easy to derive (see [KS]) sufficient condition for the existence of the form-sum is as follows: there exist $\beta' < 1$ and $\lambda_0 \in \mathbb{R}$ such that for all $\lambda > \lambda_0$

$$\big\||V|^{\frac{1}{2}}(H_0+\lambda)^{-1}|V|^{\frac{1}{2}}\big\|_{2\to2} \le \beta'. \tag{2.2}$$

Also, it is easy to show (see [KS]) that if (2.2) takes place, then

$$D(H) \subset Y_V^{\mathrm{weak}}.$$

To formulate our main results we need the following notations and definitions.

Notation. Here $\mathbf{1}_{x_0,\rho,K}$ denotes the characteristic function of the intersection of a compact set $K \subset \Omega$ with the open ball $B(x_0,\rho)$ having center x_0 and radius ρ. Set

$$(-\Delta)^{-\frac{z}{2}} f(x) = c_z \int_\Omega (-\Delta)^{-\frac{z}{2}}(x,y) f(y)\,dy,$$

where

$$(-\Delta)^{-\frac{z}{2}}(x,y) := |x-y|^{z-d} \quad \text{and} \quad c_z := \Gamma\Big(\frac{d-z}{2}\Big)\Big(\pi^{d/2}2^z\Gamma\Big(\frac{z}{2}\Big)\Big)^{-1}$$

(see [SW]).

With $W \in X_1$, $x_0 \in \Omega$, $\rho > 0$, we introduce the following notation:

$$\theta_K(W,x_0,\rho) := \big\|\mathbf{1}_{x_0,\rho,K}|W|^{\frac{1}{2}}(-\Delta)^{-1}|W|^{\frac{1}{2}}\mathbf{1}_{x_0,\rho,K}\big\|_{2\to2}.$$

Definition 2. We say that W belongs to $\mathcal{F}_{\beta,\mathrm{loc}}$ if

$$\sup_K \overline{\lim_{\rho\to 0}} \sup_{x_0\in K} \theta_K(W,x_0,\rho) \le \beta,$$

where K is a compact subset of Ω.

Next we introduce a class of potentials V for which we prove WUC and SUC. Let $x_0 \in \Omega \subset \mathbb{R}^d$ ($d \ge 3$), $\rho > 0$ and $W \in X_{\frac{d-1}{2}}$. Define

$$\tau_K(W,x_0,\rho) := \left\| \mathbf{1}_{x_0,\rho,K} |W|^{\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{2}} |W|^{\frac{d-1}{4}} \mathbf{1}_{x_0,\rho,K} \right\|_{2\to 2}.$$

Definition 3. We say that a potential W is in $\mathcal{F}^d_{\beta,\mathrm{loc}}$ if

$$\sup_K \overline{\lim_{\rho\to 0}} \sup_{x_0\in K} \tau(W,x_0,\rho) \le \beta.$$

Note that for $d = 3$ the class $\mathcal{F}_{\beta,\mathrm{loc}}$ coincides with $\mathcal{F}^d_{\beta,\mathrm{loc}}$ and for $d \ge 4$ the class $\mathcal{F}^d_{\beta,\mathrm{loc}}$ is a proper subclass of $\mathcal{F}_{\beta,\mathrm{loc}}$ (which follows from $\theta \le \tau$ proved in [KS]).

Our main result states that (1.1) has the WUC and SUC properties with potentials V from $\mathcal{F}^d_{\beta,\mathrm{loc}}$. The difference between the cases of WUC and SUC is in the classes Y_V within which we look for solutions to (1.1).

Theorem 1. *There exists a constant $\beta < 1$ such that if $V \in \mathcal{F}^d_{\beta,\mathrm{loc}}$, then* (1.1) *satisfies WUC in Y_V^{weak}.*

Theorem 2. *There exists a constant $\beta < 1$ such that if $V \in \mathcal{F}^d_{\beta,\mathrm{loc}}$, then* (1.1) *satisfies SUC in Y_V^{str}.*

Finally, we state our result concerning the eigenvalue problem.

Theorem 3. *Assume that H is defined by* (2.1)*, and that $V \in \mathcal{F}^d_{\beta,\mathrm{loc}}$ has a compact support and also that $\beta < 1$ is 'sufficiently' small. Then the only solution to eigenvalue problem $Hu = \lambda u$, $\lambda > 0$, $u \in D(H)$ is $u \equiv 0$.*

In reference to our motivating question we wonder whether Theorems 1 and 3 remain valid replacing $\mathcal{F}^d_{\beta,\mathrm{loc}}$ by $\mathcal{F}_{\beta,\mathrm{loc}}$.

3. Comparison with classical results.

In this section we compare our results with the classical results that followed Carleman's foundational paper on unique continuation [C] by identifying subclasses of $\mathcal{F}^d_{\beta,\mathrm{loc}}$ (see [Sa], [JK], [St]).

Kato-type class (*cf.* [MS]): let

$$\mathcal{K}^d_{\beta,\mathrm{loc}} := \{W \in X_{\frac{d-1}{2}} : \sup_K \overline{\lim_{\rho\to 0}} \sup_{x_0\in K} e_K(W,x_0,\rho) \le \beta\},$$

where

$$e_K(W,x_0,\rho) := \|\mathbf{1}_{x_0,\rho,K}(-\Delta)^{-\frac{d-1}{2}}W^{\frac{d-1}{2}}\mathbf{1}_{x_0,\rho,K}\|_\infty.$$

(In the case when $d=3$, this class was considered by Sawyer [Sa].) The inclusion

$$\mathcal{K}^d_{\beta,\mathrm{loc}} \subsetneq \mathcal{F}^d_{\beta,\mathrm{loc}} \quad \text{for all } d \geq 3$$

is an obvious consequence of (an easy to prove) inequality $\tau(V,x_0,\rho) \leq e(V,x_0,\rho)$ for $V \in \mathcal{K}^d_{\beta,\mathrm{loc}}$, see [KS]. The latter inclusion is strict. For example, for $d=3$ potential

$$V_\beta(x) := \frac{\beta(\frac{d-2}{2})^2}{|x|^2},$$

is in $\mathcal{F}^d_{\beta,\mathrm{loc}} \backslash \mathcal{K}^d_{\beta,\mathrm{loc}}$. Indeed, one can verify directly that $V_\beta \in \mathcal{F}^d_{\beta,\mathrm{loc}}$ by means of the well known Hardy's inequality. However, $e(V,0,\rho)=\infty$ for all $\rho>0$, and hence $V_\beta \notin \mathcal{K}^d_{\beta,\mathrm{loc}}$.

Note that the class Y_V in [Sa] includes our Y_V^{weak} with V from a narrower class of potentials (see details in [KS]).

Next we examine classes $L^{d/2}_{\mathrm{loc}}(\Omega)$ and the weak type $d/2$ Lorentz space $L^{d/2,\infty}_{\mathrm{loc}}(\Omega)$ (for the definition of the latter see [SW]). Below, $\|\cdot\|_{p,\infty}$ denotes the weak type p Lorentz norm.

It is easy to show that $L^{d/2}_{\mathrm{loc}}(\Omega) \subsetneq \bigcap_{\beta>0} \mathcal{F}^d_{\beta,\mathrm{loc}}$ and

$$L^{\frac{d}{2},\infty}_{\mathrm{loc}}(\Omega) \subsetneq \bigcup_{\beta>0} \mathcal{F}^d_{\beta,\mathrm{loc}} \tag{3.1}$$

(see [KS]). The latter inclusion can be obtained by showing that $\tau(V,x_0,\rho)$ is bounded from above by $\|\mathbf{1}_{x_0,\rho}V\|^{d-1}_{\frac{d}{2},\infty}$ whenever V is in $L^{\frac{d}{2},\infty}_{\mathrm{loc}}(\Omega)$. To see that inclusion (3.1) is strict, we introduce the following family of potentials:

$$V(x) := \frac{C\big(\mathbf{1}_{1+\delta}(x) - \mathbf{1}_{1-\delta}(x)\big)}{\big(|x|-1\big)^{\frac{2}{d-1}}\big(-\ln\big||x|-1\big|\big)^b}, \quad \text{where } b > \frac{2}{d-1} \quad \text{and} \quad 0<\delta<1.$$

A straightforward computation shows that V belongs to $\mathcal{K}^d_{\beta,\mathrm{loc}}$ and thus to $\mathcal{F}^d_{\beta,\mathrm{loc}}$ for some β. Moreover, a direct calculation reveals that

$$V \in L^{\frac{d-1}{2}}(\Omega) \setminus L^{\frac{d-1}{2}+\varepsilon}(\Omega) \quad \text{for any } \varepsilon > 0,$$

and therefore $V \notin L^{\frac{d-1}{2},\infty}(\Omega)$.

Finally, we compare our results stated in Section 2 with a similar result in [St] (and in [JK]). The result in [St] can be formulated as follows. Let $d \geq 3$ and $V \in L^{d/2,\infty}_{\mathrm{loc}}(\Omega)$. There exists a constant $\beta \in (0,1)$ such that if

$$\sup_{x_0\in\Omega} \limsup_{\rho\to 0} \|\mathbf{1}_{x_0,\rho} V\|_{\frac{d}{2},\infty} \leq \beta,$$

then (1.1) has SUC in $H^{2,\bar{p}}_{\mathrm{loc}}(\Omega)$ with $\bar{p} = \frac{2d}{d+2}$. (See [W] for a discussion on the size of β.)

As a consequence of (3.1) the latter result of Stein follows from Theorem 2: indeed, let $L^{p,q}$ be the (p,q) Lorentz space from [SW]. By the Sobolev embedding theorem for Lorentz spaces, $H^{2,\bar{p}}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{\bar{q},\bar{p}}_{\mathrm{loc}}(\Omega)$ with $\bar{q} = \frac{2d}{d-2}$, see [A]. Hence, by the Hölder inequality in the Lorentz spaces, $|V|^{\frac{1}{2}}u \in L^2_{\mathrm{loc}}(\Omega)$ whenever $u \in L^{\bar{q},\bar{p}}_{\mathrm{loc}}(\Omega)$ and $V \in L^{d/2,\infty}_{\mathrm{loc}}$. Also, $H^{2,\bar{p}}_{\mathrm{loc}}(\Omega) \hookrightarrow H^{1,\bar{p}}_{\mathrm{loc}}(\Omega)$, and therefore $H^{2,\bar{p}}_{\mathrm{loc}}(\Omega) \subset Y^{\mathrm{str}}_V$.

4. **Key ideas and a sketch of the proofs.** In this section we sketch the proof of Theorem 1 (complete details can be found in [KS]). First, we introduce some notations. Let $C^\infty_0(\Omega)$ denote the space of C^∞ functions on Ω with compact support. In what follows, we omit index K and write simply $\mathbf{1}_\rho := \mathbf{1}_{\rho,0,K}$. Let $\mathbf{1}_{\rho\setminus a} := \mathbf{1}_\rho - \mathbf{1}_a$ where $0 < a < \rho$, $\mathbf{1}^c_\rho := 1 - \mathbf{1}_\rho$ and $\varphi_t(x) := |x|^{-t}$.

Let $T^{N-1}_{x,0}(f)$ denote the Taylor polynomial of degree $N-1$ of a function f with respect to variable $x \in \mathbb{R}^d$ at $x = 0$. We define an integral operator

$$[(-\Delta)]_N^{-\frac{z}{2}} f(x) := \int_\Omega [(-\Delta)]_N^{-\frac{z}{2}}(x,y) f(y)\, dy, \quad 0 \leq \mathrm{Re}(z) \leq d-1,$$

where

$$[(-\Delta)]_N^{-\frac{z}{2}}(x,y) := c_z(|x-y|^{z-d} - T^{N-1}_{x,0}|x-y|^{z-d}).$$

We also introduce

$$[(-\Delta)]^{-z}_{N,t} := \varphi_t (-\Delta)^{-z}_N \varphi_t^{-1}.$$

Our proofs of Theorems 1–3 follow Carleman's method with our 'main inequality' being the key input.

Proposition 1 (Main inequality). *If $\tau(V,0,\rho) < \infty$, then there exists a constant $C = C(\rho,\alpha,d) > 0$ such that*

$$\left\|\mathbf{1}_{\rho\setminus a}|V|^{\frac{1}{2}}\varphi_{N^\alpha_d}[(-\Delta)]^{-1}_N \varphi^{-1}_{N^\alpha_d}|V|^{\frac{1}{2}}\mathbf{1}_{\rho\setminus a}\right\|_{2\to 2} \leq C\tau(V,0,\rho)^{\frac{1}{d-1}},$$

where $N^\alpha_d := N + (\frac{d}{2} - \alpha)\frac{d-3}{d-1}$, $0 < \alpha < 1/2$ and N is a positive integer.

Below we sketch the proof of Theorem 1.

PROOF. Let $u \in Y_V^{\text{weak}}$. We may assume without loss of generality that there exist $a > 0$ and $\rho \in (a, 1)$ such that $u \equiv 0$ on $B(0, a)$ and $\bar{B}(0, 3\rho) \subset \Omega$. Note that if V is a potential from $\mathcal{F}^d_{\beta,\text{loc}}$, and $V_1 := |V| + 1$, then for a fixed $x_0 \in \Omega$

$$\tau(V_1, x_0, \rho) \leq \tau(V, x_0, \rho) + \varepsilon(\rho),$$

where $\varepsilon(\rho) \to 0$ as $\rho \to 0$. Also, if function u satisfies (1.1), then u satisfies (1.1) with V replaced by V_1. It is easy to see that there is a function $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B(0, 2\rho)$, $\eta \equiv 0$ on $\Omega \setminus B(0, 3\rho)$, $|\nabla\eta| \leq \frac{c}{\rho}$, $|\Delta\eta| \leq \frac{c}{\rho^2}$. Denote $u_\eta := u\eta$ and let $E_\eta(u) := 2\nabla\eta\nabla u + u\Delta\eta$. Then

$$\Delta u_\eta = \eta\Delta u + E_\eta(u),$$

and

$$\operatorname{supp} \eta\Delta u \subset \bar{B}(0, 3\rho) \setminus B(0, a), \quad \operatorname{supp} E_\eta(u) \subset \bar{B}(0, 3\rho) \setminus B(0, 2\rho),$$

and therefore, $\mathbf{1}_\rho^c \eta\Delta u = \mathbf{1}_{3\rho\setminus\rho}\Delta u$, $\mathbf{1}_\rho^c E_\eta(u) = \mathbf{1}_{3\rho\setminus 2\rho} E_\eta(u)$. Note that $\Delta u_\eta \in X_1$ since both u and the error term $E_\eta(u)$, belong to L^1_{loc}. Therefore,

$$u_\eta = (-\Delta)^{-1}(-\Delta u_\eta).$$

Recall that u_η is identically equal to 0 on $B(0, a)$. By subtracting the $N-1$-th degree Taylor polynomial of u_η at 0 and interchanging the signs of differentiation and integration, it follows that $u_\eta = [(-\Delta)]_N^{-1}(-\Delta u_\eta)$. Obviously,

$$\Delta u_\eta = (\mathbf{1}_{\rho\setminus a} + \mathbf{1}_\rho^c)\Delta u_\eta.$$

Using these two identities we conclude

$$\begin{aligned} I := \mathbf{1}_{\rho\setminus a} V_1^{\frac{1}{2}} \varphi_{N_d^\alpha} u &= \mathbf{1}_{\rho\setminus a} V_1^{\frac{1}{2}} [(-\Delta)]^{-1}_{N, N_d^\alpha} V_1^{\frac{1}{2}} \mathbf{1}_{\rho\setminus a} \varphi_{N_d^\alpha} \frac{-\Delta u}{V_1^{\frac{1}{2}}} \\ &\quad + \mathbf{1}_{\rho\setminus a} V_1^{\frac{1}{2}} [(-\Delta)]^{-1}_{N, N_d^\alpha} V_1^{\frac{1}{2}} \mathbf{1}_\rho^c \varphi_{N_d^\alpha} \frac{-\eta\Delta u}{V_1^{\frac{1}{2}}} \\ &\quad + \mathbf{1}_{\rho\setminus a} V_1^{\frac{1}{2}} [(-\Delta)]^{-1}_{N, N_d^\alpha} \mathbf{1}_{3\rho\setminus 2\rho} \varphi_{N_d^\alpha} \big(-E_\eta(u)\big) \\ &= I_1 + I_1^c + I_2, \end{aligned}$$

where I_1, I_1^c and I_2 refer to the summands of I (we assume that $0 < \alpha < 1/2$ is fixed throughout the proof). Note that for $d \geq 4$ it is not known whether a

priori I may not be in L^2 (but only $I \in L^s$, $s < d/(d-2)$). As a consequence we have to prove that I_1, I_1^c and I_2 are in L^2, so that $I \in L^2$ as well. We show in [KS] that $\|I_1^c\| \leq c_1 \varphi_{N_d^\alpha}(\rho)$, $\|I_2\| \leq c_2 \varphi_{N_d^\alpha}(\rho)$ and $\|I_1\|_2 \leq \delta \|I\|_2$, $\delta < 1$. Then we may conclude that

$$(1-\delta)\|I\|_2 \leq (c_1 + c_2)\varphi_{N_d^\alpha}(\rho)$$

and therefore

$$\Big\| \mathbf{1}_{\rho \backslash a} \frac{\varphi_{N_d^\alpha}}{\varphi_{N_d^\alpha}(\rho)} u \Big\|_2 \leq \frac{c_1 + c_2}{1 - \delta}.$$

Now letting $N \to \infty$, it follows that $u \equiv 0$ on $B(0, \rho)$. □

Remark 2. Our estimates of I_1 and I_1^c are derived easily using the 'main inequality' (*i.e.*, Proposition 1) but an estimation of I_2 requires additional arguments (see [KS]).

The following inequality is crucial for our proof of Proposition 1.

Lemma 1. *There exist constants $C > 0$ and $c > 0$ such that*

$$\big| |x-y|^{-1+i\gamma} - T_{x,0}^{N-1}(|x-y|^{-1+i\gamma}) \big| \leq C e^{c\gamma^2} \Big(\frac{|x|}{|y|} \Big)^N |x-y|^{-1} \tag{4.1}$$

for all x, $y \in \mathbb{R}^d$, all $\gamma \in \mathbb{R}$ and all positive integers N.

For a proof see [KS].

Proof of Proposition 1 (Sketch) The proof is based on Stein's interpolation theorem [SW]. Consider an operator-valued function

$$F(z) := \mathbf{1}_{\rho\backslash a} |V|^{\frac{d-1}{4}z} \varphi_{N+(\frac{d}{2}-\alpha)(1-z)} [(-\Delta)]_N^{-\frac{d-1}{2}z} \varphi^{-1}_{N+(\frac{d}{2}-\alpha)(1-z)} |V|^{\frac{d-1}{4}z} \mathbf{1}_{\rho\backslash a}.$$

F is continuous on the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$ and analytic in its interior. In order to use the interpolation theorem we need to estimate $\|F(z)\|_{2\to 2}$ on the boundary of the strip.

The estimate on $\mathrm{Re}(z) = 0$ is due to Jerison and Kenig [JK]: there exist constants $C_2 = C_2(\rho, \alpha, d)$ and $c_2 = c_2(\rho, \alpha, d) > 0$ such that

$$\|\mathbf{1}_{\rho\backslash a} (-\Delta)^{-i\gamma}_{N, N+\frac{d}{2}-\alpha} \mathbf{1}_{\rho\backslash a}\|_{2\mapsto 2} \leq C_2 e^{c_2|\gamma|},$$

for all $\gamma \in \mathbb{R}$ and all positive integers N, where $0 < \alpha < 1/2$.

The estimate on $\mathrm{Re}(z) = 1$ follows from the definition of class $\mathcal{F}^d_{\beta,\mathrm{loc}}$ and our key inequality (Lemma 1). Now the assumption of the interpolation theorem are fulfilled. The statement of the Proposition follows by letting $z = 2/(d-1)$. □

Remark 3. Inequality (4.1) for $\gamma = 0$ appears in [Sa, Lemma 1], and a variant of (4.1) with exponent $-1 + i\gamma$ replaced by $-\sigma + i\gamma$, $0 < \sigma < 1$, and γ^2 replaced by $|\gamma|$, appears in [St, Lemma 5].

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