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ON CONTROLLABILITY OF PARTIALLY PRESCRIBED PAIRS OF MATRICES

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ABSTRACT. Let F be an infinite field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let

$$(C_1, C_2) = \left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right),$$

where the blocks $C_{i,j}$ are of type $p_i \times p_j$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. We analyse the possibility of the pair (C_1, C_2) being completely controllable, when

(i) $C_{1,2}$, $C_{1,3}$, and $C_{2,1}$ are fixed and the other blocks vary;
(ii) $C_{1,1}$, $C_{1,2}$, and $C_{2,1}$ are fixed and the other blocks vary.

We still describe the possible characteristic polynomials of a partitioned matrix of the form $C = [C_{i,j}] \in F^{n \times n}$, where the blocks $C_{i,j}$ are of type $p_i \times p_j$, $i, j \in \{1, 2, 3\}$, when one of the conditions (i) or (ii) occurs.

RÉSUMÉ. Soit F un corps infini et soient n, p_1, p_2, p_3 des entiers positifs tels que $n = p_1 + p_2 + p_3$. Soit

$$(C_1, C_2) = \left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right),$$

où les blocs $C_{i,j}$ sont de type $p_i \times p_j$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. Nous établions conditions pour lesquelles (C_1, C_2) est controllable, quand

(i) $C_{1,2}, C_{1,3}$, et $C_{2,1}$ sont connus et les autres blocs varient;
(ii) $C_{1,1}, C_{1,2}$, et $C_{2,1}$ sont connus et les autres blocs varient.

Soit $C = [C_{i,j}] \in F^{n \times n}$, où les blocs $C_{i,j}$ sont de type $p_i \times p_j, i, j \in \{1, 2, 3\}$. Nous étudions le polynôme caractéristique de la matrice C, quand une des conditions (i) ou (ii) est satisfait.

1. **Motivation** Control theory is an important branch of mathematics that has many applications in technology, engineering, sociology, and so on. Initially, during the decades between 1940 and 1960, control theory emerged associated with systems described by linear differential equations with constant coefficients and characterized by a control input. The development of this theory was mainly associated with electronics and mechanics. During this period, control theory was unable to deal with nonlinear systems and stochastic effects.

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Later, the development of the missile and aerospace industries, with particular relevance to trajectory optimization, gave rise to a deep transformation in control theory. Indeed, the motion and performance of aerospace, in general, can be influenced by several available control inputs. On the other hand, in the early sixties, the emergence of digital computers had a great impact on the nature of control problems. This fact led to a transition from analytical solutions to algorithmic solutions, which gave rise to a considerable amount of subsequent research. Indeed, during the last decades many authors have studied different types of control problems.

An important problem in control theory, is the following.

Problem 1. Given a system

$$\dot{\chi}(t) = A\chi(t) + B\zeta(t), \tag{1.1}$$

where $\chi(t) \in \mathbb{R}^{p\times 1}$ denotes the state of a certain physical system to be controllable by the input $\zeta(t) \in \mathbb{R}^{q\times 1}$, and $A \in \mathbb{R}^{p\times p}, B \in \mathbb{R}^{p\times q}$, how to select the input $\zeta(t)$ in such way that $\chi(t)$ is driven to a certain desirable state?

This problem is known as the Pole Assignment Problem. The aim of this problem is to establish conditions under which the system of the form (1.1) is completely controllable, *i.e.,* the pair (A, B) is completely controllable. In the next result we recall the concept of controllability.

Definition 1.1. Let F be a field and let $A \in F^{p\times p}, B \in F^{p\times q}$. The pair (A, B) is said to be *completely controllable* if all the invariant factors of the matrix pencil $[xI_p - A - B]$ are constant.

From now on, consider F a field and let $A \in F^{p\times p}, B \in F^{p\times q}$. The characterization of (1.1) being completely controllable when some entries of $[A\ B]$ are prescribed (*i.e.,* are fixed) and the others are unknown has been considered by many authors. The general idea is to study the possibility of "completing" the matrix $[A\ B]$, when some of its entries are prescribed, in such a way that the pair (A, B) is completely controllable. In this context "to complete" means to attribute values to the remaining entries. In particular, when several entries of $[A\ B]$ are prescribed as zero, the problem is completely solved, see [6], [7], [8], [10]. When the prescribed entries are not necessarily equal to zero, only some partial solutions are known.

In a previous paper [3] we studied the existence of a completely controllable pair of the form

$$\left(\begin{bmatrix} C_{1,1} & \cdots & C_{1,k-1} \\ \vdots & & \vdots \\ C_{k-1,1} & \cdots & C_{k-1,k-1} \end{bmatrix}, \begin{bmatrix} C_{1,k} \\ \vdots \\ C_{k-1,k} \end{bmatrix} \right), \tag{1.2}$$

where $C_{i,j} \in F^{p\times p}, i \in \{1,\dots,k-1\}$, $j \in \{1,\dots,k\}$, when $k-1$ of its blocks are fixed and the others vary. When the blocks are not necessarily of the same size and k of its blocks are prescribed, the problem becomes more difficult, and is not yet solved. To give some insight into this question, we start by studying case $k = 3$. Indeed, we characterize the possibility of (1.2) being completely controllable when $k = 3$, F is infinite, and one of the following conditions is satisfied:

(i) $C_{1,2}$, $C_{1,3}$ and $C_{2,1}$ are fixed and the remaining blocks vary;
(ii) $C_{1,1}$, $C_{1,2}$ and $C_{2,1}$ are fixed and the remaining blocks vary.

Next, we study another question concerning matrix completion problems. This type of problem consists in studying the possibility to "complete" a matrix when some of its entries are prescribed (*i.e.,* are fixed), such that the resulting matrix satisfies certain properties. In other words, given a matrix and a part of the given matrix (such as a submatrix, some entries, so on), the aim of these problems is to describe conditions under which we can fill the unknown entries, such that the resultant matrix satisfies the required properties. In particular, an important problem that motivates our work is the following.

Problem 2. [9] Let F be a field and let n, p, q be positive integers such that $n = p + q$. Let $f(x) \in F[x]$ be a monic polynomial of degree n. Let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \tag{1.3}$$

be a partitioned matrix, where $A_{1,1} \in F^{p\times p}$, $A_{2,2} \in F^{q\times q}$. Suppose that some of the blocks $A_{i,j}, i, j \in \{1,2\}$ are known. Under which conditions does there exist a matrix of the form (1.3) with characteristic polynomial $f(x)$?

Notice this problem can be split into essentially seven distinct problems, according to the location of the prescribed blocks:

(P_1) $A_{1,1}$ prescribed;

(P_2) $A_{1,2}$ prescribed;

(P_3) $A_{1,1}$ and $A_{1,2}$ prescribed;

(P_4) $A_{1,1}$ and $A_{2,2}$ prescribed;

(P_5) $A_{1,2}$ and $A_{2,1}$ prescribed;

(P_6) $A_{1,1}, A_{1,2}$ and $A_{2,2}$ prescribed;

(P_7) $A_{1,1}, A_{1,2}$ and $A_{2,1}$ prescribed.

Many authors have studied this list of problems. Some of them are completely solved; however for some prescription of blocks only few partial solutions are known. In particular, concerning problem (P_7) we do not know any reference with nontrivial results.

A natural question that arises is the following.

PROBLEM 3. Let F be an arbitrary field. Let $n, k, p_1, \ldots, p_k$ be positive integers such that $n = p_1 + \cdots + p_k$. Let

$$C = \begin{bmatrix} C_{1,1} & \cdots & C_{1,k} \\ \vdots & & \vdots \\ C_{k,1} & \cdots & C_{k,k} \end{bmatrix} \in F^{n \times n}, \tag{1.4}$$

where the blocks $C_{i,j} \in F^{p_i \times p_j}, i, j \in \{1, \ldots, k\}$. Suppose that some of the blocks $C_{i,j}$ are prescribed. Under which conditions does there exist a matrix of the form (1.4) with prescribed eigenvalues or characteristic polynomial?

Obviously the prescription of the characteristic polynomial is more general since it covers the situation of the eigenvalues of the matrix being outside of the field F.

When all the blocks are of the same size, we showed in [4] that it is always possible to prescribe $2k-3$ blocks of the matrix and the eigenvalues in F, except if either all the principal blocks are prescribed or all the blocks of one row or column are prescribed.

Still considering all the blocks of the same size, we showed in [3] that it is possible to prescribe $k-1$ blocks of the matrix and the characteristic polynomial, except if all the nonprincipal blocks of a row or column are prescribed equal to 0 and the characteristic polynomial has no divisor of degree p.

When the blocks are not necessarily of the same size, the description of the eigenvalues of a matrix of the form (1.4) when some of its blocks are prescribed and the others are unknown, becomes more difficult. In [5] we established conditions under which is possible to prescribe the eigenvalues of (1.4) when a diagonal of blocks is prescribed.

In this paper we study the particular case when $k = 3$, F is infinite, and one of the following conditions is satisfied:

(i) $C_{1,2}$, $C_{1,3}$ and $C_{2,1}$ are fixed and the remaining blocks vary;
(ii) $C_{1,1}$, $C_{1,2}$ and $C_{2,1}$ are fixed and the remaining blocks vary.

2. Main Results

From now on, suppose that $k = 3$, and let F be a field. We denote by $F^{p \times q}$ the set of all matrices of type $p \times q$ with entries in F.

In the following result we identify conditions under which the pair of the form (1.2) is completely controllable, when $C_{1,2}$, $C_{1,3}$, and $C_{2,1}$ are fixed and the other blocks vary.

THEOREM 2.1. [2] *Let F be an infinite field. Let n, p_1, p_2, and p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$, and $C_{2,1} \in F^{p_2 \times p_1}$. Then there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, and $C_{2,3} \in F^{p_2 \times p_3}$ such* *that the pair*

$$(C_1, C_2) = \left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right) \tag{2.1}$$

is completely controllable, except if the following condition (E) *holds:*

$$C_{1,2} = 0 \quad \text{and} \quad C_{1,3} = 0. \tag{E}$$

Sketch of the proof. We start by showing that if the condition (E) occurs, then there exists no completely controllable pair of the form (2.1) with prescribed form.

From now on, let us assume that condition (E) is not satisfied. Let $r = \operatorname{rank} C_{1,3}$. Without loss of generality, we may assume that

$$C_{1,3} = \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix}.$$

Now we can take $C'_{1,2} \in F^{p_1 \times p_2}$ with an appropriate form, such that the pair $(C_{1,1}, \begin{bmatrix} C'_{1,2} & C_{1,3} \end{bmatrix})$ is completely controllable. Then we choose $C'_{2,1} \in F^{p_2 \times p_1}$ with an appropriate form, such that the number of nonconstant invariant factors of the matrix pencil $\begin{bmatrix} xI_{p_1} - C_{1,1} & -C'_{2,1} \end{bmatrix}^\top$ is less than or equal to 1. Then, according to the main result of [13], there exist $C_{2,2} \in F^{p_2 \times p_2}$ and $C_{2,3} \in F^{p_2 \times p_3}$ such that the pair

$$(C'_1, C'_2) = \left(\begin{bmatrix} C_{1,1} & C'_{1,2} \\ C'_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right)$$

is completely controllable. Finally, we can prove that $\begin{bmatrix} C'_1 & C'_2 \end{bmatrix}$ is block-similar to $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ where $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ has the prescribed form. Consequently, the matrix pencils $\begin{bmatrix} xI_{p_1} - C'_1 & -C'_2 \end{bmatrix}$ and $\begin{bmatrix} xI_{p_1} - C_1 & -C_2 \end{bmatrix}$ have the same invariant factors, so the invariant factors of $\begin{bmatrix} xI_{p_1} - C_1 & -C_2 \end{bmatrix}$ are constant. Therefore, (C_1, C_2) is completely controllable and has the prescribed form. □

In the following result, for the same prescription of blocks when condition (E) holds, we describe the characteristic polynomial of the matrix of the form

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}, \tag{2.2}$$

over arbitrary fields.

Proposition 2.2. [2] *Let F be an arbitrary field. Let n, p_1, p_2, and p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $f \in F[x]$ be a monic polynomial of degree n. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$, and $C_{2,1} \in F^{p_2 \times p_1}$.*

If the exceptional condition (E) *is satisfied, then there exist* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, *and* $C_{3,3} \in F^{p_3 \times p_3}$ *such that the matrix of the form* (2.2) *has characteristic polynomial* f *if and only if* f *has a divisor of degree* p_1.

Sketch of the proof. The necessary condition follows from [11], [14]. For the sufficient condition, we construct directly a matrix with prescribed characteristic polynomial and prescribed form. □

As a simple consequence of Theorem 2.1 and [15], we can still describe the characteristic polynomial of the matrix of the form (2.2) when condition (E) is not satisfied.

Corollary 2.3. [2] *Let* F *be an infinite field. Let* n, p_1, p_2, *and* p_3 *be positive integers such that* $n = p_1 + p_2 + p_3$. *Let* $f \in F[x]$ *be a monic polynomial of degree* n. *Let* $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$, *and* $C_{2,1} \in F^{p_2 \times p_1}$. *If* (E) *is not satisfied, then there exist* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, *and* $C_{3,3} \in F^{p_3 \times p_3}$ *such that the matrix of the form* (2.2) *has characteristic polynomial* f.

In the following result we show that there always exists a matrix of the form (2.2) with prescribed blocks and arbitrary prescribed eigenvalues.

Proposition 2.4. [2] *Let* F *be an arbitrary field. Let* $c_1, \ldots, c_n \in F$. *Let* $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$, *and* $C_{2,1} \in F^{p_2 \times p_1}$. *Then there exist* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, *and* $C_{3,3} \in F^{p_3 \times p_3}$ *such that the matrix of the form* (2.2) *has eigenvalues* $c_1, \ldots, c_n$.

Sketch of the proof. If either condition (E) is satisfied or $C_{2,1} = 0$, we construct directly a matrix with prescribed eigenvalues and prescribed form.

If (E) is not satisfied and $C_{2,1} \neq 0$, we use the main result of [12] to conclude the proof. □

Remark 1. When F is infinite, the proof of Proposition 2.4 can be immediately obtained from Proposition 2.2 and Corollary 2.3.

In the following result we analyse the possibility of the pair of the form (2.1) being completely controllable when $C_{1,1}$, $C_{1,2}$, and $C_{2,1}$ are prescribed and the other blocks vary.

Theorem 2.5. [1] *Let* F *be an infinite field and let* n, p_1, p_2, p_3 *be positive integers such that* $n = p_1 + p_2 + p_3$. *Let* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, *and* $C_{2,1} \in F^{p_2 \times p_1}$. *Then there exist* $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,2} \in F^{p_2 \times p_2}$, *and* $C_{2,3} \in F^{p_2 \times p_3}$ *such that the pair of the form* (2.1) *is completely controllable if and only if* $2p_1 + p_2 \leq n$.

Sketch of the proof. The necessary condition follows from [11], [14]. Conversely, let us assume that $2p_1 + p_2 \leq n$.

Case 1. Suppose that $C_{2,1} \neq 0$. Let $r = \operatorname{rank} C_{2,1}$. Without loss of generality, we may assume that

$$C_{2,1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}.$$

Let $C_{1,3} = \begin{bmatrix} I_{p_1} & 0 \end{bmatrix} \in F^{p_1 \times p_3}, C_{2,2} = C(x^{p_2})$ (where $C(x^{p_2})$ denotes the companion matrix of the polynomial x^{p_2}), $C_{2,3} = 0 \in F^{p_2 \times p_3}$. Then the pair of the form (2.1) is completely controllable.

Case 2. Suppose that $C_{2,1} = 0$. Since F is infinite we can choose $C_{2,2} \in F^{p_2 \times p_2}$ with an appropriate form, such that $\sigma(C_{1,1}) \cap \sigma(C_{2,2}) = \varnothing$ where $\sigma(C_{i,i})$ denotes the spectrum of $C_{i,i}$. For an appropriate choice of $C'_{1,2} \in F^{p_1 \times p_2}, C_{1,3} \in F^{p_1 \times p_3}$ it is not hard to prove that the pair

$$(C'_1, C'_2) = \left(\begin{bmatrix} C_{1,1} & C'_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right)$$

is completely controllable. Bearing in mind that $\sigma(C_{1,1}) \cap \sigma(C_{2,2}) = \varnothing$, we can prove that $\begin{bmatrix} C'_1 & C'_2 \end{bmatrix}$ is block-similar to $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ where $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ has the prescribed form. Since (C'_1, C'_2) is completely controllable, it follows that (C_1, C_2) is also completely controllable. Clearly, the pair (C_1, C_2) has the prescribed form.

Remark 2. If $C_{2,1} \neq 0$, Theorem 2.5 is still valid for arbitrary fields [1].

When $C_{2,1} \neq 0$, from Remark 2 and [15] we can deduce the following result where we describe the characteristic polynomial of the matrix of the form (2.2) for the same prescription of blocks for arbitrary fields.

Corollary 2.6. [1] *Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $f \in F[x]$ be a monic polynomial of degree n. Let $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, and $C_{2,1} \in F^{p_2 \times p_1}$. If $C_{2,1} \neq 0$ and $2p_1 + p_2 \leq n$, there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, and $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form* (2.2) *has characteristic polynomial f.*

In the following result we can still describe the characteristic polynomial of the matrix of the form (2.2), for arbitrary prescription of $C_{2,1}$, over infinite fields. The proof of this result follows easily from Theorem 2.5.

Corollary 2.7. [1] *Let F be an infinite field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $f \in F[x]$ be a monic polynomial of degree n. Let $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, and $C_{2,1} \in F^{p_2 \times p_1}$. If $2p_1 + p_2 \leq n$, there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, and $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form* (2.2) *has characteristic polynomial f.*

Remark 3. The condition $2p_1 + p_2 \leq n$ is sufficient for the existence of a matrix of the form (2.2) with prescribed form and prescribed characteristic polynomial, however is not necessary.

Clearly, we still obtain the following result which describes the possible list of eigenvalues of the matrix of the form (2.2) for the same prescription of blocks.

Corollary 2.8. *Let F be an infinite field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $c_1, \ldots, c_n \in F$. Let $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, and $C_{2,1} \in F^{p_2 \times p_1}$. If $2p_1 + p_2 \leq n$, there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, and $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (2.2) has eigenvalues $c_1, \ldots, c_n$.*

Remark 4. If $C_{2,1} \neq 0$, Corollary 2.8 is still valid for arbitrary fields.

Notice Propositions 2.2, and 2.4 and Corollaries 2.3, 2.6, 2.7, and 2.8 are extensions of the problem proposed by G. N. Oliveira in [9]. Clearly, Propositions 2.2 and 2.4 and Corollary 2.6 are extensions of Oliveira's problem, while Corollaries 2.3, 2.7, and 2.8 are extensions of Oliveira's problem when F is infinite.

3. Concluding Remarks

In this paper we identify conditions under which the pair of the form (2.1) is completely controllable when some of its entries are prescribed. However, the more general problem of describing conditions under which the pair of the form (1.2) is completely controllable, is not yet solved.

We still solve another question on matrix completion problems, namely, we describe the possible characteristic polynomials, or the possible list of eigenvalues of a matrix of the form (2.2) when some of its blocks are fixed and the others are unknown. The general problem of describing the possible characteristic polynomials of a matrix of the form (1.4), when $k > 3$, and some of its entries are prescribed is still open. Notice that when the prescribed positions correspond to "large" submatrices, there are necessary and sufficient interlacing inequalities involving the invariant factors [11], [14]. The proof of the sufficiency of these inequalities can be very difficult.

To conclude, we believe that our work is an advance on this type of problem, since our contribution provides some insight for the case $k = 3$. Furthermore, we still believe that the techniques used can be extended to the more general situation $k > 3$. These results can be a starting point for subsequent research on this field.

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