C. R. Math. Rep. Acad. Sci. Canada Vol. **31** (1) 2009, pp. 16–19

ON AF EMBEDDABILITY OF CONTINUOUS FIELDS

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ABSTRACT. Let A be a separable and exact C*-algebra which is a continuous field of C*-algebras over a connected, locally connected, compact metrizable space. If at least one of the fibers of A is AF embeddable, then so is A. As an application we show that if G is a central extension of an amenable and residually finite discrete group by $\mathbb{Z}^n$, then the C*-algebra of G is AF embeddable.

RÉSUMÉ. Soit A une C*-algèbre séparable et exacte qui est un champ continu de C*-algèbres sur un espace connexe, localement connexe, compact et metrizable. Si au moins l'une des fibres de A est embeddable dans une AF algèbre donc la C*-algèbre A est aussi. Comme application, nous montrons que si G est une extension centrale d'un groupe discret amenable et résiduellement fini par le groupe $\mathbb{Z}^n$, alors la C*-algèbre de G est embeddable dans une AF algèbre.

1. **Introduction** Ozawa has shown that the cone over any separable exact C*-algebra embeds in an AF algebra [8], (see also [10] for a different proof). He then applied a technique of Spielberg [11] to prove that AF embeddability of exact separable C*-algebras is a homotopy invariant. In this note we show that the same techniques combined with a result of Blanchard [1] yield AF-embeddings for certain continuous fields.

THEOREM 1.1. *Let A be a separable exact C*-algebra. Suppose that A is a continuous field of C*-algebras over a connected, locally connected, compact metrizable space. If the fiber $A(x)$ of A is AF embeddable for some $x \in X$, then A is AF embeddable.*

This allows us to derive the following embedding theorem for group C*-algebras. The question of whether the C*-algebra of a discrete amenable group is embeddable in an AF algebra is widely open even for elementary amenable groups.

THEOREM 1.2. *Let $1 \to \mathbb{Z}^n \to G \to H \to 1$ be a central extension of second countable locally compact amenable groups. If $C^*(H)$ is AF embeddable, then so is $C^*(G)$.*

Received by the editors on December 18, 2008.
The author was partially supported by NSF grant #DMS-0801173
AMS Subject Classification: 46L05.
Keywords: continuous fields of C*-algebras, AF algebras, amenable groups.

Proof. By [9, Theorem 1.2], (see also [3, Lemma 6.3]), C*(G) is a nuclear continuous field of C*-algebras over the spectrum $\mathbb{T}^n$ of C*($\mathbb{Z}^n$). Moreover, the fiber over the trivial character of $\mathbb{Z}^n$ is isomorphic to C*(H), which is AF embeddable by hypothesis. The conclusion follows now from Theorem 1.1. $\square$

Corollary 1.3. *Let $1 \to \mathbb{Z}^n \to G \to H \to 1$ be a central extension of countable discrete groups. If H is amenable and residually finite, then $C^*(G)$ is AF embeddable.*

Proof. C*(H) is AF embeddable by [2, Corollary 4.6]. The result follows from Theorem 1.2. $\square$

Let us note that the group G in Corollary 1.3 is not in general residually finite. Indeed, Groves exhibits in [4] examples of finitely generated, non residually finite groups G which contain $\mathbb{Z}$ as a central subgroup and such that $G/\mathbb{Z}$ is metabelian and hence residually finite by a classical result of P. Hall [5].

2. Extensions of C*-algebras The following result is implicitly contained in Spielberg's paper [11].

Proposition 2.1. *Let $0 \to I \to A \to B \to 0$ be an essential semisplit extension of separable C^*-algebras whose class vanishes in* $\mathrm{Ext}^{-1}(B, I) \cong \mathrm{KK}_1(B, I)$. *Suppose that both I and B are AF embeddable. Then A is AF embeddable.*

Proof. For the convenience of the reader we spell out the whole argument. By assumption there is an injective $*$-homomorphism $\varphi\colon I \to J$ where J is an AF algebra. After replacing J by the hereditary C*-subalgebra of J generated by $\varphi(I)$ we may assume that φ is approximately unital and so it extends to an injective $*$-homomorphism $\widetilde{\varphi}\colon M(I) \to M(J)$ between the corresponding multiplier algebras. Since I is essential in A, we have that $A \subset M(I)$, and we can identify A with its image $\widetilde{\varphi}(A) \subset M(J)$. Therefore since $A \cap J = I$, we obtain just as in [11, 1.11] a commutative diagram with exact rows and injective vertical maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & J & \longrightarrow & A+J & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

The extension $0 \to J \to A + J \to B \to 0$ is obviously semisplit and essential (since $A + J \subset M(J)$) and moreover its class in $\mathrm{Ext}^{-1}(B, J)$ vanishes since it corresponds to the image of zero under the group morphism $\varphi_*\colon \mathrm{Ext}^{-1}(B, I) \to \mathrm{Ext}^{-1}(B, J)$. We take the tensor product of the extension by the C*-algebra $\mathcal{K}$ of all compact operators on a separable Hilbert space $\mathcal{H}$. Thus we have reduced the proof to the case of an extension $0 \to I \to A \to B \to 0$ as in the statement of the theorem with the additional property that $I \cong I \otimes \mathcal{K}$ is an AF algebra. Let us denote by $\sigma\colon B \to M(I)$ the Busby map corresponding to this extension. If $a \mapsto \dot{a}$ denotes the quotient map $M(I) \to M(I)/I$, then we can identify A with

$$E(\sigma) = \{a \in M(I) : \dot{a} \in \sigma(B)\}.$$

By the assumption on B there is an injective $*$-homomorphism $\lambda\colon B \to D$ where D is a separable AF algebra. Fix an injective representation $\gamma\colon D \to M(\mathcal{K}) \subset M(I \otimes \mathcal{K}) \cong M(I)$ of infinite multiplicity. Let us set $\psi = \gamma \circ \lambda\colon B \to M(I)$ and observe that there is an embedding $E(\sigma) \hookrightarrow E(\sigma \oplus \dot{\psi})$ given by $a \mapsto a \oplus \psi(\dot{a})$. By Kasparov's absorption theorem [7], $E(\sigma \oplus \dot{\psi})$ in isomorphic $E(\dot{\psi})$. On the other hand, $E(\dot{\psi}) \subset E(\dot{\gamma})$ and $E(\dot{\gamma})$ is an AF algebra since it is a split extension of AF algebras $0 \to I \to E(\dot{\gamma}) \to D \to 0$. □

Lemma 2.2. *If X is a connected, locally connected, compact metrizable space, then for any point $x \in X$, $C_0(X \setminus \{x\})$ is embeddable in $C_0[0,1) \otimes \mathcal{O}_2$.*

Proof. By the Hahn–Mazurkiewicz theorem [6, Theorem. 3-30], there is a continuous surjection $h\colon [0,1] \to X$ and hence an injective $*$-homomorphism $C(X) \to C[0,1]$. If $t \in [0,1]$ is such that $h(t) = x$, then

$$C_0(X \setminus \{x\}) \hookrightarrow C_0[0,t) \oplus C_0(t,1].$$

The latter C*-algebra embeds in $C_0[0,1) \otimes \mathcal{O}_2 \otimes M_2(\mathbb{C}) \cong C_0[0,1) \otimes \mathcal{O}_2$. □

Proof of Theorem 1.1. By the main result of [1], there is a $C(X)$-linear injective $*$-homomorphism $A \hookrightarrow C(X) \otimes \mathcal{O}_2$. Therefore A embeds in the C*-algebra

$$E = \{f \in C(X) \otimes \mathcal{O}_2 : f(x) \in A(x)\}.$$

The evaluation map at x gives a split extension

$$0 \to C_0(X \setminus \{x\}) \otimes \mathcal{O}_2 \to E \to A(x) \to 0.$$

We have that $A(x)$ is AF embeddable by hypothesis and that $C_0(X \setminus \{x\}) \otimes \mathcal{O}_2$ is embeddable in $C_0[0,1) \otimes \mathcal{O}_2$ by Lemma 2.2. By the main result of [8], $C_0[0,1) \otimes \mathcal{O}_2$ is AF embeddable. We conclude that E and hence A is AF embeddable by applying Proposition 2.1.

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