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CORDES CHARACTERIZATION FOR PSEUDODIFFERENTIAL OPERATORS WITH SYMBOLS VALUED IN A NONCOMMUTATIVE C*-ALGEBRA

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Abstract. Given a separable unital C*-algebra $\mathcal{A}$ with norm $\|\cdot\|$, let E denote the Banach-space completion of the $\mathcal{A}$-valued Schwartz space on $\mathbb{R}^n$ with norm $\|f\|_2 = \|\langle f, f\rangle\|^{1/2}$, $\langle f, g\rangle = \int f(x)^* g(x)\, dx$. The assignment of the pseudodifferential operator $B = b(x, D)$ with $\mathcal{A}$-valued symbol $b(x, \xi)$ to each smooth function with bounded derivatives $b \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$ defines an injective mapping O from $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$ to the set $\mathcal{H}$ of all operators with smooth orbit under the canonical action of the Heisenberg group on the algebra of all adjointable operators on the Hilbert C*-module E. It is known that O is surjective if $\mathcal{A}$ is commutative. In this paper, we show that if O is surjective for $\mathcal{A}$, then it is also surjective for $M_k(\mathcal{A})$.

Résumé. Étant donné une C*-algèbre A, séparable et avec unité, soit E l'espace de Banach obtenu par complétation de l'espace de Schwartz sur $\mathbb{R}^n$ avec valeurs dans A par la norme induite par le produit interne à valeurs dans A: $\langle f, g\rangle = \int f(x)^* g(x)\, dx$. L'association de l'opérateur pseudo-différentiel $B = b(x, D)$, ayant symbol $b(x, \xi)$ à valeurs dans A, à chaque fonction smooth b, à derivées bornées, define une application injective O de l'ensemble de tous ces symbols dans l'ensemble de tous les opérateurs ayant orbite lisse par l'action du group de Heisenberg sur l'algèbre de tous les opérateurs adjointables sur le C*-module de Hilbert E. Il est bien connu que, si A est commutatif, alors O est surjective. Dans cet article nous montrons que, si O est surjective pour une algèbre quelconque A, alors elle est surjective aussi pour l'algèbre des matrices k par k à coefficients dans A.

1. **Introduction** Let $\mathcal{A}$ be a separable C*-algebra with norm $\|\cdot\|$ and unit $\mathbf{1}$, and let $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ denote the set of all $\mathcal{A}$-valued smooth (Schwartz) functions on $\mathbb{R}^n$ which, together with all their derivatives, are bounded by arbitrary negative powers of $|x|$, $x \in \mathbb{R}^n$. We equip it with the $\mathcal{A}$-valued inner-product

$$\langle f, g\rangle = \int f(x)^* g(x)\, dx,$$

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which induces the norm $\|f\|_2 = \|\langle f, f\rangle\|^{1/2}$, and denote by E its Banach-space completion with this norm. The inner product $\langle \cdot, \cdot \rangle$ turns E into a Hilbert C*-module [5]. The set of all (bounded) adjointable operators on E is denoted by $\mathcal{B}^*(E)$.

Let $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$ denote the set of all smooth bounded functions from $\mathbb{R}^{2n}$ to $\mathcal{A}$ whose derivatives of arbitrary order are also bounded. For each b in $\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$, a linear mapping from $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ to itself is defined by the formula

$$(Bu)(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} b(x,\xi)\hat{u}(\xi)\, d\xi, \tag{1.1}$$

where $\hat{u}$ denotes the Fourier transform,

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-iy\cdot\xi} u(y)\, dy.$$

The operator $B := b(x, D)$ extends to an element of $\mathcal{B}^*(E)$ whose norm satisfies the following estimate. There exists $K > 0$ depending only on n such that

$$\|B\| \leq K \sup\{\|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)\|; (x,\xi) \in \mathbb{R}^{2n} \text{ and } \alpha, \beta \leq (1, \dots, 1)\}. \tag{1.2}$$

This generalization of the Calderón–Vaillancourt Theorem [1] was proved by Merklen [8], [9]; see also [4], [10], [11].

The estimate (1.2) implies that the mapping

$$\mathbb{R}^{2n} \ni (z,\zeta) \longmapsto B_{z,\zeta} = T_{-z} M_{-\zeta} B M_\zeta T_z \in \mathcal{B}^*(E) \tag{1.3}$$

is smooth (*i.e.*, C^∞ with respect to the norm topology), where T_z and M_ζ are defined by $T_z u(x) = u(x-z)$ and $M_\zeta u(x) = e^{i\zeta\cdot x} u(x)$, $u \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$. That follows just as in the scalar case [3, Chapter 8].

Definition 1.1. We call *Heisenberg smooth* an operator $B \in \mathcal{B}^*(E)$ for which the mapping (1.3) is smooth, and denote by $\mathcal{H}$ the set of all such operators.

The elements of $\mathcal{H}$ are the smooth vectors for the canonical action of the Heisenberg group on $\mathcal{B}^*(E)$.

We therefore have a mapping

$$\begin{array}{rccc} O_{\mathcal{A}}\colon & \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n}) & \longrightarrow & \mathcal{H} \\ & b & \longmapsto & b(x,D). \end{array} \tag{1.4}$$

It is a standard result that in the scalar case ($\mathcal{A} = \mathbb{C}$), $O_{\mathcal{A}}$ is injective. For general $\mathcal{A}$, injectiveness follows from the scalar case by a duality argument. Cordes [2] proved that $O_{\mathcal{A}}$ is surjective in the scalar case. We have shown [7] that this also happens if $\mathcal{A}$ is unital and commutative.

In this paper we show that if $O_{\mathcal{A}}$ is surjective, then $O_{M_k(\mathcal{A})}$ is also surjective. We show that by first noticing that the Hilbert C*-module E_k for the matrix case is a Banach-space direct sum of k^2 copies of E. Then it follows that a bounded operator on E_k (regarded only as a Banach space) is smooth under the action of the Heisenberg group if and only if it is a matrix whose entries are operators on E that are also smooth under the Heisenberg group. When we impose that such a matrix be an adjointable $M_k(\mathcal{A})$-module homomorphism, then we get precisely the pseudodifferential operators of the form (1.1).

Given a skew-symmetric $n \times n$ matrix J and $F \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, let us denote by L_F the pseudodifferential operator $a(x, D) \in \mathcal{B}^*(E)$ with symbol $a(x, \xi) = F(x - J\xi)$. Let us further denote by R_F the pseudodifferential operator with symbol $b(x, \xi) = F(x + J\xi)$ defined similarly as in (1.1), except that $b(x, \xi)$ multiplies $\hat{u}(\xi)$ on the right. At the end of Chapter 4 in [10], Rieffel made a conjecture that may be rephrased as follows: any $B \in \mathcal{H}$ that commutes with every R_G, $G \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$, is of the form $B = L_F$ for some $F \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^n)$.

Using Cordes' characterization of the Heisenberg-smooth operators in the scalar case, we have shown [6] that Rieffel's conjecture is true when $\mathcal{A} = \mathbb{C}$. The second author [8, Theorem 3.5] proved further that Rieffel's conjecture is true for any separable C^*-algebra $\mathcal{A}$ for which the operator $O_{\mathcal{A}}$ is a bijection.

The assumption of separability of $\mathcal{A}$ is needed to justify several results about vector-valued integration [9, Apêndice].

2. Adjointable operators

Let us denote by E_k the Hilbert C*-module obtained using the procedure described in the first paragraph of this paper with $\mathcal{A}$ replaced by $M_k(\mathcal{A})$, the C*-algebra of k-by-k matrices with entries in $\mathcal{A}$.

Using that the norm $\|((a_{ij}))_{1\le i,j\le k}\|_\infty := \max\{\|a_{i,j}\|\ ;\ 1 \le i, j \le k\}$ is equivalent to the C*-norm $\|\cdot\|$ of $M_k(\mathcal{A})$ ($\|\cdot\|_\infty \le \|\cdot\| \le k^2\|\cdot\|_\infty$), one easily proves that a given function $f = ((f_{ij}))_{1\le i,j\le k} \colon \mathbb{R}^n \to M_k(\mathcal{A})$ belongs to $\mathcal{S}^{M_k(\mathcal{A})}(\mathbb{R}^n)$ if and only if each f_{ij} belongs to $\mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$.

PROPOSITION 2.1. *For each (l, m), $1 \le l, m \le k$, the maps*

$$P_{lm} \colon \mathcal{S}^{M_k(\mathcal{A})}(\mathbb{R}^n) \ni ((f_{ij}))_{1\le i,j\le k} \longmapsto f_{lm} \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$$

and

$$I_{lm} \colon \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n) \ni f \longmapsto ((\delta_{il}\delta_{jm} f))_{1\le i,j\le k} \in \mathcal{S}^{M_k(\mathcal{A})}(\mathbb{R}^n)$$

($\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \ne q$) extend continuously to

$$P_{lm} \colon E_k \longrightarrow E \quad \text{and} \quad I_{lm} \colon E \longrightarrow E_k.$$

Moreover, $\|P_{lm}\| = 1$ and I_{lm} is an isometry.

PROOF. For each $f \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$ and each (i, j), we have:

$$\Big(\int I_{lm}(f)(x)^* I_{lm}(f)(x)\, dx\Big)_{ij} = \delta_{im}\delta_{jm} \int f(x)^* f(x)\, dx;$$

and then,

$$\|I_{lm}(f)\|_2^2 = \|((\delta_{im}\delta_{jm}\mathbf{1}))_{1\le i,j\le k}\| \cdot \|f\|_2^2 = \|f\|_2^2.$$

This shows that I_{lm} is an isometry.

Given $f = ((f_{ij}))_{1\le i,j\le k} \in \mathcal{S}^{M_k(\mathcal{A})}(\mathbb{R}^n)$, we have:

$$\|P_{ml}(f)\|_2^2 = \Big\|\int f_{ml}(x)^* f_{ml}(x)\,dx\Big\| \le \Big\|\sum_{j=1}^k \int f_{jl}(x)^* f_{jl}(x)\,dx\Big\|$$

(we have used that $\|a\| \le \|a+b\|$ if a and b are two positive elements of any C*-algebra, and that the integral of a positive valued function is also positive). The right-hand side of the previous inequality equals

$$\Big\|\Big(\int f(x)^* f(x)\,dx\Big)_{ll}\Big\| \le \Big\|\int f(x)^* f(x)\,dx\Big\|_\infty \le \Big\|\int f(x)^* f(x)\,dx\Big\|.$$

This shows that $\|P_{ml}\| \le 1$. The equality holds because, for any $g \in \mathcal{S}^{\mathcal{A}}(\mathbb{R}^n)$, $P_{ml}(I_{ml}(g)) = g$ and $\|g\|_2 = \|I_{ml}(g)\|_2$. □

PROPOSITION 2.2. *The map*

$$E_k \ni f \longmapsto ((P_{ij}(f)))_{1\le i,j\le k} \in \bigoplus_{\substack{1\le i\\ j\le k}} E \tag{2.1}$$

is a Banach space isomorphism. The right action of $M_k(\mathcal{A})$ on E_k is then given by matrix multiplication, while the $M_k(\mathcal{A})$-valued inner-product on E_k is given by

$$\big\langle ((f_{ij}))_{1\le i,j\le k}, ((g_{ij}))_{1\le i,j\le k}\big\rangle = \Big(\Big(\sum_{l=1}^k \langle f_{li}, g_{lj}\rangle\Big)\Big)_{1\le i,j\le k}. \tag{2.2}$$

PROOF. Using that $P_{lm}I_{lm}$ equals the identity on E for every (l,m), that $P_{lm}I_{pq} = 0$ if $l \ne p$ or $m \ne q$, and that $\sum_{lm} I_{lm}P_{lm}$ equals the identity on E_k, it follows that

$$\bigoplus_{i,j=1}^k E \ni ((f_{ij}))_{1\le i,j\le k} \longmapsto \sum_{l,m=1}^k I_{lm}(f_{lm}) \in E_k$$

is an inverse for the map defined in (2.1). The statements about the action of $M_k(\mathcal{A})$ and about the inner-product follow by density, since they hold on $\mathcal{S}^{M_k(\mathcal{A})}(\mathbb{R}^n)$. □

Let $\mathcal{L}(E_k)$ denote the algebra of bounded operators on the Banach space E_k. In order to describe which elements of $\mathcal{L}(E_k)$ belong to $\mathcal{B}^*(E_k)$ (*i.e.,* which of them are adjointable $M_k(\mathcal{A})$-module homomorphisms), it is convenient to define an isomorphism

$$\bigoplus_{i,j=1}^{k} E \simeq \bigoplus_{p=1}^{k^2} E, \tag{2.3}$$

using the bijection $\phi\colon \{1,\dots,k\}\times\{1,\dots,k\} \to \{1,2,\dots,k^2\}$ defined by listing the pairs (l,m) column after column,

$$\phi(1,1)=1,\dots,\phi(k,1)=k, \phi(1,2)=k+1,\dots,\phi(k,2)=2k,$$
$$\dots,\phi(1,k)=k^2-(k-1),\dots,\phi(k,k)=k^2.$$

The composition of the two isomorphisms defined in (2.1) and (2.3) induces the isomorphism

$$E_k \simeq \bigoplus_{p=1}^{k^2} E, \tag{2.4}$$

which, in turn, induces

$$\mathcal{L}(E_k) \ni T \mapsto ((P_p T I_q))_{1\le p,q\le k^2} \in M_{k^2}(\mathcal{L}(E_k)). \tag{2.5}$$

Here, abusing notation, we have written P_p and I_q, where we really meant $P_{\phi^{-1}(p)}$ and $I_{\phi^{-1}(q)}$.

Since its proof is purely algebraic, the following theorem could be stated for general rings and modules.

Theorem 2.3. *Using the isomorphism* (2.5) *as an identification, a given* $T = ((T_{pq}))_{1\le p,q\le k^2} \in \mathcal{L}(E_k)$ *is a (right)* $M_k(\mathcal{A})$*-module homomorphism if and only if*

$$T = \begin{bmatrix} \tilde{T} & 0 & \cdots & 0 \\ 0 & \tilde{T} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{T} \end{bmatrix}, \tag{2.6}$$

where $\tilde{T}$ *is a* k*-by-*k *matrix of bounded (right)* $\mathcal{A}$*-module homomorphisms and* 0 *denotes the* k*-by-*k *zero block.*

Proof. Given $T = ((T_{pq}))_{1\le p,q\le k^2} \in \mathcal{L}(E_k)$, each $T_{pq} = P_p T I_q$ is obviously bounded. If T is an $M_k(\mathcal{A})$-module homomorphism, then T_{pq} is an $\mathcal{A}$-module homomorphism since, for every $a \in \mathcal{A}$ and $f \in E$, we have

$$I_q(fa) = I_q(f) \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}.$$

Given an integer l, $1 \le l \le k^2$, consider the integers l_1 and l_2 defined by $0 \le l_1 \le k-1$, $1 \le l_2 \le k$, and $l = kl_1 + l_2$. The product of two matrices can then be expressed by

$$\begin{bmatrix} a_1 & a_{1+k} & \cdots & a_{1+(k-1)k} \\ a_2 & a_{2+k} & \cdots & a_{2+(k-1)k} \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_{k+k} & \cdots & a_{k+(k-1)k} \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_{1+k} & \cdots & b_{1+(k-1)k} \\ b_2 & b_{2+k} & \cdots & b_{2+(k-1)k} \\ \vdots & \vdots & \ddots & \vdots \\ b_k & b_{k+k} & \cdots & b_{k+(k-1)k} \end{bmatrix}$$
$$= \begin{bmatrix} c_1 & c_{1+k} & \cdots & c_{1+(k-1)k} \\ c_2 & c_{2+k} & \cdots & c_{2+(k-1)k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & c_{k+k} & \cdots & c_{k+(k-1)k} \end{bmatrix},$$

with

$$c_l = \sum_{j=1}^{k} a_{l_2+k(j-1)} b_{j+kl_1}.$$

This formula holds if the two matrices that we multiply belong to $M_k(\mathcal{A})$ or if the left one is an element of E_k regarded as a matrix by (2.1). With this notation, if a given $T = ((T_{pq}))_{1 \le p,q \le k^2} \in \mathcal{L}(E_k)$ is an $M_k(\mathcal{A})$-module homomorphism, then for every $(a_l)_{1 \le l \le k^2} \in E_k$ (we now refer to the isomorphism (2.4)) and for every

$$\begin{bmatrix} b_1 & b_{1+k} & \cdots & b_{1+(k-1)k} \\ b_2 & b_{2+k} & \cdots & b_{2+(k-1)k} \\ \vdots & \vdots & \ddots & \vdots \\ b_k & b_{k+k} & \cdots & b_{k+(k-1)k} \end{bmatrix} \in M_k(\mathcal{A}),$$

we have, for every $1 \le p \le k^2$,

$$\sum_{l=1}^{k^2} \sum_{j=1}^{k} T_{pl} a_{l_2+k(j-1)} b_{j+kl_1} = \sum_{j=1}^{k} \sum_{l=1}^{k^2} T_{p_2+k(j-1),l}\, a_l b_{kp_1+j}. \tag{2.7}$$

We now apply this equality, for each $a \in E$ and each (q,r), $1 \le q, r \le k^2$ to $a_l = \delta_{ql} a$ and $b_l = \delta_{rl} \mathbf{1}$. The only nonvanishing term in the left-hand sum will satisfy $l_2 + k(j-1) = q$ (hence $j - 1 = q_1$ and $l_2 = q_2$) and $kl_1 + j = r$ (hence $l_1 = r_1$ and $j = r_2$); hence $l = kr_1 + q_2$ and $j = q_1 + 1 = r_2$. The only nonvanishing term in the right-hand sum will satisfy $l = q$ and $kp_1 + j = r$ (hence $p_1 = r_1$ and $j = r_2$). Equation (2.7) then becomes

$$T_{p,kr_1+q_2} \delta_{q_1+1,r_2} = T_{p_2+k(r_2-1),q} \delta_{p_1,r_1}.$$

We now can see that if $q_1 + 1 \ne r_2$ and $p_1 = r_1$, then $T_{p_2+k(r_2-1),q} = 0$. This proves that for each (p,q), $T_{pq} = 0$ unless $p = p_2 + kq_1$. In other words, if $p_1 \ne q_1$, then $T_{p,q} = 0$. Therefore, all blocks outside the diagonal in (2.6) indeed vanish.

Finally, letting $q_1 + 1 = r_2$ and $p_1 = r_1$, we get $T_{p,kp_1+q_2} = T_{p_2+kq_1,q}$ or $T_{kp_1+p_2,kp_1+q_2} = T_{kq_1+p_2,kq_1+q_2}$, proving that all blocks along the diagonal in (2.6) are indeed equal.

We have proved that any module homomorphism on E_k is of the form (2.6). To see that the converse is also true, we only need to remark that under the description of E_k given by Proposition 2.2, the action of a T as in (2.6) on E_k is given by left multiplication by $\tilde{T}$. □

Theorem 2.4. *A given $T = ((T_{pq}))_{1\le p,q\le k^2} \in \mathcal{L}(E_k)$ belongs to $\mathcal{B}^*(E_k)$ if and only if it is of the form* (2.6) *with $\tilde{T} \in M_k(\mathcal{B}^*(E))$.*

Proof. Given T and S in $\mathcal{L}(E_k)$ of the form (2.6), with corresponding $\tilde{T} = ((\tilde{T}_{ij}))_{1\le i,j\le k}$ and $\tilde{S} = ((\tilde{S}_{ij}))_{1\le i,j\le k}$, and given $f = ((f_{ij}))_{1\le i,j\le k}$ and $g = ((g_{ij}))_{1\le i,j\le k}$ in $E_k \simeq \bigoplus_{1\le i,j\le k} E$, by (2.2) we have

$$\langle Tf, g\rangle_{ij} = \sum_{l,m=1}^{k} \langle \tilde{T}_{lm} f_{mi}, g_{lj}\rangle \quad \text{and} \quad \sum_{l,m=1}^{k} \langle f_{mi}, \tilde{S}_{lm} g_{lj}\rangle = \langle f, Sg\rangle_{ij}. \tag{2.8}$$

From this it follows that if each $\tilde{T}_{ij}$ is adjointable and if $\tilde{S}_{ij} = T^*_{ji}$ for all (i,j), then S is the adjoint of T.

Conversely, suppose that T is adjointable and that its adjoint is S. The equality of the two sums in (2.8) for particular choices of f and g will imply that each $\tilde{T}_{ij}$ is adjointable. □

3. Heisenberg-smooth adjointable operators

The mapping (1.3) may be defined for any B in $\mathcal{L}(E)$ or in $\mathcal{L}(E_k)$. It thus makes sense to talk about Heisenberg-smooth operators in $\mathcal{L}(E)$ or in $\mathcal{L}(E_k)$. Given $B = ((B_{pq}))_{1\le p,q\le k^2} \in \mathcal{L}(E_k)$, we have

$$B_{z,\zeta} = (((B_{pq})_{z,\zeta}))_{1\le p,q\le k^2} = ((P_p B_{z,\zeta} I_q))_{1\le p,q\le k^2}$$

(it is enough to check these equalities on the dense subset of Schwartz functions). We then get the following.

Proposition 3.1. *A given $B = ((B_{pq}))_{1\le p,q\le k^2} \in \mathcal{L}(E_k)$ is Heisenberg-smooth if and only if each $B_{pq} \in \mathcal{L}(E)$ is Heisenberg-smooth.*

This leads to our main result.

Theorem 3.2. *If the unital separable C^*-algebra $\mathcal{A}$ is such that the map $O_{\mathcal{A}}$ defined in* (1.4) *is a bijection, then the map $O_{M_k(\mathcal{A})}$ is also a bijection.*

PROOF. Given any Heisenberg-smooth operator $B \in \mathcal{B}^*(E_k)$, we must show that it is of the form $B = b(x,D)$ for some $b \in \mathcal{B}^{M_k(\mathcal{A})}(\mathbb{R}^{2n})$. Let $((B_{pq}))_{1\leq p,q\leq k^2}$ be the matrix that corresponds to B by the isomorphism (2.5). By Proposition 3.1 and by the assumption that $O_{\mathcal{A}}$ is surjective, each B_{pq} is a pseudodifferential operator of the type defined in (1.1). By Theorem 2.3, B is of the form (2.6). That is, there exist $b_{ij} \in \mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n})$, $1 \leq i,j \leq k$ such that with

$$\tilde{T} = \begin{bmatrix} b_{11}(x,D) & b_{12}(x,D) & \cdots & b_{1k}(x,D) \\ b_{21}(x,D) & b_{22}(x,D) & \cdots & b_{2k}(x,D) \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1}(x,D) & b_{k2}(x,D) & \cdots & b_{kk}(x,D) \end{bmatrix},$$

we have

$$B = \begin{bmatrix} \tilde{T} & 0 & \cdots & 0 \\ 0 & \tilde{T} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{T} \end{bmatrix}.$$

This implies that B and $b(x,D)$ are equal, if $b \in \mathcal{B}^{M_k(\mathcal{A})}(\mathbb{R}^{2n}) = M_k(\mathcal{B}^{\mathcal{A}}(\mathbb{R}^{2n}))$ is given by $b = ((b_{ij}))_{1\leq i,j\leq k}$. Indeed, the equality of the two operators can be easily verified on $\mathcal{S}^{M_k(\mathcal{A})}(\mathbb{R}^n)$. □

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