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A REMARK ON ORTHOGONALITY OF ELEMENTS OF A C*-ALGEBRA

GEORGE A. ELLIOTT, FRSC

Abstract. It is shown that any two non-zero hereditary sub-C*-algebras of a C*-algebra that has no minimal projections have approximately orthogonal elements of norm one. (The question of exact orthogonality is left open.)

Résumé. On démontre que, dans une C*-algèbre sans projecteur minimal, deux sous-C*-algèbres héréditaires qui ne sont pas égales à zéro possèdent des éléments de norme un qui sont approximativement orthogonals.

1. It would seem to be a reasonable question whether, in a C*-algebra, any two non-zero hereditary sub-C*-algebras (see [5]) must contain orthogonal elements—provided that at least one of them is not one-dimensional. If the C*-algebra is of type I, or if the hereditary sub-C*-algebras each have a unit, then it is not difficult to see that the answer to this question is affirmative.

The purpose of the present note is to show that the answer to the corresponding approximate version of the question is affirmative in general.

Interestingly, perhaps, the answer is based on the remarkable homogeneity result of Kishimoto, Ozawa, and Sakai for an arbitrary separable C*-algebra ([2]).

Theorem. *Let A be a C*-algebra, and let B_1 and B_2 be non-zero hereditary sub-C*-algebras of A. Suppose that B_1 is not one-dimensional. It follows that, for each $\epsilon > 0$, there exist elements $b_1 \in B_1$ and $b_2 \in B_2$ of norm one such that $\|b_1 b_2\| \leq \epsilon$.*

Proof. It is sufficient to consider the case that the C*-algebra A is separable. (One may replace A by the sub-C*-algebra generated by chosen non-zero elements of B_1 and B_2, with the chosen element of B_1 not just a scalar multiple of a projection, and replace B_1 and B_2 by their intersections with this sub-C*-algebra.)

It is sufficient to consider the case that the closed two-sided ideal of A generated by B_1 is equal to A. Indeed, if the intersection of this ideal with B_2 is equal to 0, then $B_1 B_2 = 0$ and the conclusion is clear. If the intersection is non-zero, then we may replace A by the ideal and B_2 by the intersection.

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Consider first the case that the C*-algebra A is of type I. In this case A has an essential closed two-sided ideal with Hausdorff primitive spectrum, and the intersections of this ideal with B_1 and B_2 necessarily still have the properties assumed for B_1 and B_2 (both non-zero, and the second not one-dimensional), and so equal to A. If the primitive spectrum of A has at least two points, then by the Dauns–Hofmann theorem there exist non-zero central multipliers (equivalently, bounded continuous functions on the primitive spectrum of A) f_1, f_2 and g_1, g_2 such that $f_1 + f_2 = 1$ and $f_1 g_1 = f_2 g_2 = 0$. Choose $0 \neq a_2 \in B_2$, and note that either $f_1 a_2 \neq 0$ or $f_2 a_2 \neq 0$; by symmetry we may suppose the latter. Set $f_2 a_2 = b_2$, and choose $0 \neq b_1 \in g_2 B_1$. Then $0 \neq b_i \in B_i$, $i = 1, 2$, and $b_1 b_2 = 0$, as desired.

If the primitive spectrum of A consists of only one point, then as we are still in the case that A is of type I, A must be the C*-algebra of compact operators on some Hilbert space. Then both B_1 and B_2 contain non-zero projections and as B_1 is not one-dimensional, *i.e.,* not just scalar multiples of a single projection, it must contain a two-dimensional projection. Then let p_1 be a two-dimensional projection in B_1 and let p_2 be a one-dimensional projection in B_2. Since the product $p_1 p_2 p_1$ has rank at most one, its kernel projection in $p_1 B_1 p_1$ is a non-zero projection in B_1 orthogonal to p_2, as desired.

Now consider the case that the C*-algebra A has no non-zero ideal of type I. Since A is generated as a closed two-sided ideal by the hereditary sub-C*-algebra B_1, also B_1 is not of type I. By a well-known result of Glimm and Dixmier ([5]) there exist two inequivalent irreducible representations of B_1 with the same kernel. With f_1 and f_2 pure states of B_1 corresponding to these representations, and with f_1 and f_2 denoting also the unique extensions of these states to (pure) states of A, by Theorem 1.1 of [2] there exists an automorphism β of B_1 such that

$$f_1 \beta = f_2.$$

A further reduction is now needed. If B_2 and B_1 are not orthogonal, then there exist $0 \neq a_2 \in B_2$ and $0 \neq a_1 \in B_1$ with $a_2 a_1 \neq 0$. With u the partially isometric part of $a_2 a_1$ in the bidual of A, if B_2 and B_1 are replaced by the hereditary sub-C*-algebras corresponding to the (open) range and support projections of u, respectively, then the map

$$\begin{aligned} \operatorname{Ad} u \colon B_1 &\to B_2 \\ b &\mapsto ubu^* \end{aligned}$$

is an isomorphism of B_1 onto B_2. The combined map, $(\operatorname{Ad} u)\beta$, is then an isomorphism of B_1 onto B_2; denote this by α.

Denote the support of f_1 in the bidual of B_1 by p. Then $\alpha(p)$ is the support of a pure state of A which is not equivalent to f_1 (as it is equivalent to f_2!), and so $\|p - \alpha(p)\| = 1$. Hence by [4, Corollary 5.3] (which is applicable by [4, Remark 5.2], as α is an isomorphism from B_1 to B_2),

$$\inf_{\substack{b \in B_1 \\ \|b\| = 1}} \|b\alpha(b)\| = 0.$$

Given $\epsilon > 0$, choose $b \in B_1$ of norm one such that $\|b\alpha(b)\| \leq \epsilon$. The elements $b_1 = b \in B_1$ and $b_2 = \alpha(b) \in B_2$ have the properties described in the conclusion of the theorem. □

2. Corollary. *Let A be a C^*-algebra, and let B_1 and B_2 be non-zero hereditary sub-C^*-algebras of A. Suppose that B_1 does not contain a one-dimensional hereditary sub-C^*-algebra. It follows that, for each finite subset S of A and each $\epsilon > 0$, there exist elements $b_1 \in B_1$ and $b_2 \in B_2$ of norm one such that*

$$\|b_1 a b_2\| < \epsilon, \ \ a \in S.$$

Proof. Let S be a finite subset of A, and let $\epsilon > 0$. Consider first the case that S has just one element, say a. If $aB_2 = 0$ the conclusion is trivial. If $aB_2 \neq 0$, then choose $b \in B_2$ such that $ab \neq 0$ and $\|ab\| = 1$, and consider the polar decomposition $ab = u|ab|$ of ab in the bidual of A. Then $|ab| = (b^*a^*ab)^{1/2} \in B_2$ and the map $x \mapsto uxu^*$ is an isomorphism of the hereditary sub-C*-algebra $(|ab|B_2|ab|)^-$ onto the hereditary sub-C*-algebra $(abB_2(ab)^*)^-$. Choose a non-zero open spectral projection e of $|ab|$ (in the bidual of A) on which $|ab|$ is close to being the identity (*i.e.,* such that $|ab|e$ is close to e), the degree of closeness to be specified. By Theorem 1 applied to the hereditary sub-C*-algebras B_1 and ueB_2eu^* of A, there exist elements $b_1 \in B_1$ and $b_2' \in eB_2e$ of norm one such that

$$\|b_1(ub_2'u^*)\| \leq \epsilon/2,$$

i.e., $\|b_1ub_2'\| \leq \epsilon/2$. Then, if $\||ab|e - e\| < \epsilon/2$, as we may suppose, so that also $\||ab|b_2' - b_2'\| < \epsilon/2$, setting $bb_2' = b_2$ we have

$$\begin{aligned}\|b_1ab_2\| &= \|b_1abb_2'\| = \|b_1u|ab|b_2'\| \\ &\leq \|b_1u(|ab|b_2' - b_2')\| + \|b_1ub_2'\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

as desired.

Now consider the case that S is any finite subset of A, and write $S = \{a\} \cup S'$ with $a \notin S'$. By the preceding case, there exist $b_1' \in B_1$ and $b_2' \in B_2$ of norm one such that

$$\|b_1'ab_2'\| < \epsilon.$$

Replacing b_1' by $(b_1'^*b_1')^{1/2}$ and b_2' by $(b_2'b_2'^*)^{1/2}$, we may suppose that b_1' and b_2' are positive. Since $\|b_1'\| = \|b_2'\| = 1$, there exist non-zero open spectral projections e_1 and e_2 of b_1' and b_2' (in the bidual of A) on which b_1' and b_2' are arbitrarily close to the identity (*i.e.,* $b_1'e_1$ close to e_1 and $b_2'e_2$ close to e_2 in norm), sufficiently close that

$$\|e_1ae_2\| < \epsilon.$$

Assuming, inductively, that the conclusion holds for the set S' with $e_1B_1e_1$ and $e_2B_2e_2$ in place of B_1 and B_2, we have elements $b_1 \in e_1B_1e_1$ and $b_2 \in e_2B_2e_2$ of norm one such that, for all $a' \in S'$,

$$\|b_1a'b_2\| < \epsilon.$$

Since $b_1e_1 = b_1$ and $e_2b_2 = b_2$, this inequality also holds with a in place of a', in other words, for all $a' \in S$, as desired. □

3. Using Corollary 2, Kucerovsky and Ng obtained the following very interesting result.

Theorem. ([3, Theorem 3.5]) *Let A and B be separable nuclear C*-algebras, and suppose that A has no one-dimensional hereditary sub-C*-algebras. The following two properties of an embedding ρ of A in the corona of B, the quotient multiplier algebra $M(B)/B$, are equivalent:*

(i) *The image of ρ has the corona factorization property, i.e., for every $0 \neq a \in A$ there exists $c \in M(B)/B$ such that $c\rho(a)c^* = 1$ (one might say, in analogy with the definition of Cuntz* [1]*, that the image of ρ is purely infinite simple with respect to the corona).*

(ii) *Every completely positive map from A to $M(B)/B$ factorizes as the map*

$$a \mapsto v^*\rho(a)v$$

for some $v \in M(B)/B$.

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Department of Mathematics,
University of Toronto,
Toronto, ON M5S 2E4
email: elliott@math.toronto.edu