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AF EMBEDDING OF CROSSED PRODUCTS OF CERTAIN GRAPH C*-ALGEBRAS BY QUASI-FREE ACTIONS

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Abstract. We introduce the notion of quasi-free action of a locally compact abelian group on a graph C*-algebra of a row-finite directed graph, with respect to a labeling of the edges of the graph by elements of the dual group, which we shall call a labeling map. A sufficient condition for AF embedding is given: if the row-finite directed graph is constructed by possibly attaching 1-loops to a row-finite directed graph each weakly connected component of which is a rooted (possibly infinite) directed tree, and the labeling map is almost proper, then the crossed product can be embedded into an AF algebra.

Résumé. On introduit la notion d'action quasi-libre d'un groupe localement compact abélien sur la C*-algèbre d'un graphe dirigé dont les rangs sont finis, par rapport à un choix d'étiquettes pour les bords du graphe par éléments du groupe dual, qu'on appellera une application d'étiquette. Une condition suffissante pour que la C*-algèbre soit enfoncée dans une C-algèbre AF (c'est-à-dire, limite de C*-algèbres de dimesion finie), est donnée, dans laquelle interviennent et le graphe lui-même et l'application d'étiquette.

1. **Introduction** In the last twenty years important progress has been made in the classification of amenable C*-algebras. Much is now known about many special classes of C*-algebras, for example, AH algebras, purely infinite C*-algebras, the crossed products associated with certain C*-algebra dynamical systems, quasi-diagonal C*-algebras and so on. It is well known that the existence of an AF embedding implies quasi-diagonality. Since M. Pimsner and D. Voiculescu's AF embedding result for irrational rotation C*-algebras [11], much effort has been made to embed more general crossed product C*-algebras into AF algebras (for example see [2], [9]–[12]), which sometimes also implies some K-theory information. While the C*-algebra one starts with must be embeddable if the group acting is discrete, T. Katsura [7] successfully embedded into AF algebras certain crossed products of the Cuntz algebras $\mathcal{O}_n$, which are purely infinite simple C*-algebras, by quasi-free continuous actions of a locally

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compact abelian group. Meanwhile, the quasi-free actions on $\mathcal{O}_n$ have been studied for many years, especially by A. Kishimoto.

In this paper, we introduce the notion of quasi-free action of a locally compact abelian group G, with dual group Γ, on the graph C*-algebra $C^*(E)$ of a row-finite directed graph E, corresponding to what will be called a labeling map ω from E^* to Γ. Keeping the same embedding strategy as in [7], we will prove that the crossed product of $C^*(E)$ by G can be embedded into an AF algebra for certain special E and ω, thus generalizing the main result Theorem 3.8 in [7] from $\mathcal{O}_n$ to a much bigger class of C*-algebras $C^*(E)$, which contains both simple C*-algebras and non-simple C*-algebras, and also contains both purely infinite C*-algebras and finite C*-algebras.

2. Quasi-free actions and almost proper maps A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable (possibly infinite) sets E^0 of *vertices*, E^1 of *edges*, and maps $r, s\colon E^1 \to E^0$ identifying the range and source of each edge. The graph is called *row-finite* if each vertex emits at most finitely many edges. We write E^n for the set of *paths* $\mu = e_1 e_2 \cdots e_n$ with length $|\mu| = n$, which are sequences of edges e_i such that $r(e_i) = s(e_{i+1})$ for $1 \le i < n$. Then the maps r, s extend naturally to $E^* = \bigcup_{n \ge 0} E^n$ and s extends naturally to the set of infinite paths $\mu = e_1 e_2 \cdots$. In particular, we have $r(v) = s(v) = v$ for $v \in E^0$. A path μ is called *closed* if $s(\mu) = r(\mu)$. A path μ with $|\mu| \ge 1$ is called a *loop* if $s(\mu) = r(\mu)$ and it has distinct vertices except for $s(\mu) = r(\mu)$, and a loop μ is called a 1-*loop* if $|\mu| = 1$. A vertex $v \in E^0$ which emits no edges is called a *sink*. The relation $\le_E$ on E^0 is defined by $v \le_E w$ if there is a path $\mu \in E^*$ with $s(\mu) = w$ and $r(\mu) = v$.

For a directed graph $E = (E^0, E^1, r, s)$, the *weakly connected* relation $\sim$ in E^0 is defined as follows: for $v, w \in E^0$, $v \sim w$ if and only if $v = w$ or there are $e_1, e_2, \ldots, e_n$ in E^1, and $v_1, v_2, \ldots, v_n$ in E^0 such that $v_0 = v$, $v_n = w$, and $\{v_{i-1}, v_i\} = \{r(e_i), s(e_i)\}$ $(i = 1, 2, \ldots, n)$. Clearly, $\sim$ is an equivalence relation in E^0. A directed graph $F = (F^0, F^1, r_F, s_F)$ is called a *weakly connected component* of E if F^0 is an equivalence class of $\sim$ in E^0,

$$F^1 = \{e \in E^1 \mid r(e) \in F^0\} = \{e \in E^1 \mid s(e) \in F^0\},$$

and $r_F = r|_{F^1}$, $s_F = s|_{F^1}$. A directed graph $E = (E^0, E^1, r, s)$ is called a *rooted directed tree* if there is a $v_0 \in E^0$ with the property that there exists a unique path in E^* from v_0 to every other vertex in E^0, but no path with length larger than 0 and from v_0 to v_0.

Let E be a row-finite directed graph, and let A be a C*-algebra. A *Cuntz–Krieger E-family* in A consists of a set $\{p_v : v \in E^0\}$ of mutually orthogonal projections in A and a set $\{s_e : e \in E^1\}$ of partial isometries in A such that $s_e^* s_e = p_{r(e)}$ for $e \in E^1$ and $p_v = \sum_{\{e : s(e) = v\}} s_e s_e^*$ whenever v is not a sink. Clearly, if each s_e $(e \in E^1)$ is not zero, then the product $s_\mu = s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}$, where $\mu_i \in E^*(1 \le i \le n)$ and $\mu = \mu_1 \mu_2 \cdots \mu_n$, is non-zero precisely when μ is a path in E^*. Since the range projections $s_e s_e^* (e \in E^1)$ are mutually orthogonal, we have $s_e^* s_f = 0$ unless $e = f$ for any $e, f \in E^1$. For convenience, since vertices are paths of length 0, we write $s_v = p_v$ for $v \in E^0$.

Let E be a row-finite directed graph. As was shown in [1], there is a C*-algebra $C^*(E)$ (called the *graph* C*-*algebra of* E) which is generated by a Cuntz–Krieger E-family $\{s_e, p_v\}$ in $C^*(E)$ of non-zero elements such that, for any Cuntz–Krieger E-family $\{S_e, P_v\}$ in $B(\mathcal{K})$ for a Hilbert space $\mathcal{K}$, there is a representation $\pi = \pi_{S,P}$ of A on $\mathcal{K}$ such that $\pi(s_e) = S_e$, $\pi(p_v) = P_v$, for all $e \in E^1$, $v \in E^0$. With the convention that $p_v = s_v = s_v s_v^*$ for $v \in E^0$, the C*-algebra $C^*(E)$ is generated as a Banach space by the subset

$$\{s_\mu s_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu) \in E^0\}.$$

Let G be a (always assumed to be second countable) locally compact abelian group with dual group Γ (also assumed to be a second countable locally compact abelian group).

Definition 2.1. A map $\omega \colon E^* = \bigcup_{n \geq 0} E^n \to \Gamma$ is called a *labeling map*, if $\omega(\mu) = \omega(e_1) + \omega(e_2) + \cdots + \omega(e_n)$ for $\mu = e_1 e_2 \cdots e_n \in E^n$ and $\omega(\mu) = 1_\Gamma$ for $\mu \in E^0$, where 1_Γ is the unit of Γ, A map ω will be called *almost proper* if $\omega|_{E^* \setminus E^0}$ is proper with respect to the discrete topology on E^*, *i.e.,* for any compact subset A of Γ, $\omega^{-1}(A) \setminus E^0$ is a finite set.

It is clear that a labeling map ω is determined by $\omega|_{E^1}$, and so is really just a labeling of the edges of the directed graph E by elements of the group Γ. For convenience, we denote $\omega(\mu)$ by ω_μ. Clearly, the image $\omega(E^*)$ of ω is a countable set. It is easy to see that if $E^* \setminus E^0$ is an infinite set and ω is almost proper, then Γ is not a compact set, which is equivalent to G not being discrete. If E has a closed path γ and w is almost proper, then $w_\gamma \neq 1_\Gamma$.

Let $\omega \colon E^* \to \Gamma$ be a labeling map. For any $t \in G$, set $\widetilde{s_e} = (t, \omega_e) s_e$, $\widetilde{p_v} = p_v$. It is easy to see that $(\widetilde{s_e}, \widetilde{p_v})$ is a Cuntz–Krieger E-family in $C^*(E)$. We have then an endomorphism $\alpha_t^\omega \colon C^*(E) \to C^*(E)$ with $\alpha_t^\omega(s_e) = \widetilde{s_e}$ and $\alpha_t^\omega(p_v) = \widetilde{p_v}$. Since $\alpha_{-t}^\omega = (\alpha_t^\omega)^{-1}$, α_t^ω is an automorphism of $C^*(E)$, and moreover $(C^*(E), G, \alpha^\omega)$ is a C*-dynamical system. It is easy to see that $\alpha_t^\omega(s_\mu s_\nu^*) = (t, \omega_\mu - \omega_\nu) s_\mu s_\nu^*$ for any $\mu, \nu \in E^*$. We shall call the action α^ω of G on $C^*(E)$ the *quasi-free action* corresponding to ω.

With $C^*(E)$ viewed as a C*-algebra on a Hilbert space $\mathcal{H}$, by the regular representation and the Fourier transformation from $L^2(G, \mathcal{H})$ to $L^2(\Gamma, \mathcal{H})$, we have

$$\begin{aligned} C^*(E) \times_{\alpha^\omega} G &= \overline{\operatorname{span}}\{s_\mu f s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) \in E^0, f \in C_0(\Gamma)\} \\ &\subseteq \mathbf{B}(L^2(\Gamma, \mathcal{H})), \end{aligned}$$

where by "span" we mean the linear span, and f acts on $L^2(\Gamma, \mathcal{H})$ by pointwise multiplication. For $f \in L^\infty(\Gamma)$, set $\sigma_{\gamma_0}(f)(\gamma) = f(\gamma + \gamma_0)$ (for $\gamma, \gamma_0 \in \Gamma$). Then $(s_\mu \eta)(\sigma) = s_\mu(\eta(\sigma - \omega_\mu))($ for $\eta \in L^2(\Gamma, \mathcal{H}))$, and $f s_\mu = s_\mu \sigma_{\omega_\mu}(f)($on $L^2(\Gamma, \mathcal{H}))$.

Therefore for $f \in L^\infty(\Gamma)$, f commutes with $s_\mu s_\mu^*$ and p_v for any $\mu \in E^*$ and $v \in E^0$.

For any subset S of Γ, we denote by χ_S the characteristic function on Γ of S. Let $\{U_i\}_{i\in\mathbf{I}}$ be a countable open base of Γ such that for any $\gamma \in \omega(E^*) \subseteq \Gamma$, $j \in \mathbf{I}$, $\overline{U_j}$ is compact and $U_j - \gamma \in \{U_i : i \in \mathbf{I}\}$, and let Λ be the directed set consisting of all finite and not empty subsets of $\mathbf{I}$ with the inclusion order. Let $D_0(\Gamma)$ be the C*-subalgebra of $L^\infty(\Gamma)$ generated by all the characteristic functions χ_{U_i} (for all $i \in \mathbf{I}$), and for any $\lambda \in \Lambda$, let $D_\lambda(\Gamma)$ be the C*-subalgebra of $L^\infty(\Gamma)$ generated by all the characteristic functions χ_{U_i} (for all $i \in \lambda$). Let $\mathcal{F}(E)$ be the C*-subalgebra of $\mathbf{B}(L^2(\Gamma, \mathcal{H}))$ generated by the set $\{s_\mu f s_\nu^* : \mu, \nu \in E^*, f \in D_0(\Gamma)\}$, and moreover let $\mathcal{F}_\lambda(E)$ be the C*-subalgebra of $\mathbf{B}(L^2(\Gamma, \mathcal{H}))$ generated by the set $\{s_\mu f s_\nu^* : \mu, \nu \in E^*, f \in D_\lambda(\Gamma)\}$, which is equal to the set

$$\{s_\mu f s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) \in E^0, f \in D_\lambda(\Gamma)\}.$$

Then we easily have the following relations.

- $C_0(\Gamma) \subseteq D_0(\Gamma)$ and $\mathrm{C}^*(E) \times_{\alpha^\omega} G \subseteq \mathcal{F}(E)$. If moreover Γ is discrete (equivalently G is compact), $\mathbf{I} = \Gamma$, and $U_i = \{i\}$ for any $i \in \mathbf{I} = \Gamma$, then $\mathrm{C}^*(E) \times_{\alpha^\omega} G = \mathcal{F}(E)$.
- $D_0(\Gamma)$ is the inductive limit of $D_\lambda(\Gamma)$, and $D_0(\Gamma)$ is invariant under the actions of σ_γ ($\gamma \in \omega(E^*)$).
- $\mathcal{F}(E)$ is the inductive limit of $\mathcal{F}_\lambda(E)$ with the coherent family of morphisms $\phi_{\lambda_1,\lambda_2} : \mathcal{F}_{\lambda_1}(E) \to \mathcal{F}_{\lambda_2}(E)$, where $\phi_{\lambda_1,\lambda_2}$ is the inclusion map for $\lambda_1 \subseteq \lambda_2$.

Since $D_\lambda(\Gamma)$ is of finite dimension and abelian, with the $\lambda \in \Lambda$ fixed from now on, there are mutually orthogonal minimal projections $p_1, p_2, \ldots, p_M$ in $D_\lambda(\Gamma)$ such that $D_\lambda(\Gamma)$ consists of all their linear combinations. Let p be the unit of $D_\lambda(\Gamma)$. Then p is the characteristic function of $U = \bigcup_{i\in\lambda} U_i$, and is the sum of all p_i. It is easy to see that

$$\mathcal{F}_\lambda(E) = \overline{\text{alg-span}}\{s_\mu p_i s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) \in E^0, i = 1, 2, \ldots, M\},$$

where for a set X in a topological algebra A, $\overline{\text{alg-span}}\, X$ is the closed subalgebra of A generated by X, *i.e.,* the smallest closed subalgebra of A that contains X. For the sake of convenience, we let $1 = \chi_\Gamma$, which is the identity operator on $L^2(\Gamma, \mathcal{H})$, and let $p_0 = 1 - p$.

Let $\mathcal{A}_\lambda(E))$ be the (not closed) algebra generated by

$$\{1\}, \quad D_\lambda(\Gamma), \quad \text{and} \quad \{s_\mu f s_\mu^* : \mu \in E^*, f \in \{1\} \cup D_\lambda(\Gamma)\}.$$

Then $\mathcal{A}_\lambda(E))$ is a $*$-subalgebra of $M(\mathcal{F}(E)))$. For any $v \in E^0$ and $k \geq 1$, we define a map ρ_{vk} on $\mathcal{A}_\lambda(E)$ by

$$\rho_{vk}(x) = \sum_{\substack{s(\mu)=v \\ |\mu|=k}} s_\mu x s_\mu^* \quad (\text{for } x \in \mathcal{A}_\lambda(E)).$$

We note that if v is a sink, then $\rho_{vk} = 0$. Since E is row-finite, the right-hand side of the equation above is a finite sum. Then it is not difficult to see that ρ_{vk} is a $*$-endomorphism on $\mathcal{A}_\lambda(E)$.

3. AF-embedding Let $T = (T^0, T^1, r, s)$ be a row-finite directed graph, each weakly connected component of which is a rooted (possibly infinite) directed tree, and let $E = (E^0, E^1, r, s)$ be the row-finite directed graph constructed from T by attaching n_v $(0 \le n_v < +\infty)$ 1-loops to each vertex v in T. It is clear that the directed graph with one vertex and n 1-loops, whose C*-algebra is the Cuntz algebra, is a special one of these E. In this section we always assume E to be of this form and the labeling map $\omega\colon E^* \to \Gamma$ to be almost proper. It is easy to see that the C*-algebras $C^*(E)$ of the graphs under consideration can be simple or non-simple, and also can be purely infinite or finite.

For any $v \in E^0$, $k \ge 1$, $0 \le l \le k$, let

$$E_v^{kl} = \big\{\mu = e_1 e_2 \cdots e_k \in E^k : s(\mu) = v \text{ and } e_1, e_2, \ldots, e_l \text{ are } 1\text{-loops, but } e_{l+1} \text{ is not a 1-loop}\big\},$$

$$\rho_{vk}^l(x) = \sum_{\mu \in E_v^{kl}} s_\mu x s_\mu^* \quad (\text{for } x \in \mathcal{A}_\lambda(E)).$$

As with ρ_{vk} above, ρ_{vk}^l is still a homomorphism. Let $E_v^k = \{\mu \in E^k : s(\mu) = v\}$. Then $E_v^k = \bigcup_{l=0}^k E_v^{kl}$, and therefore $\rho_{vk} = \sum_{l=0}^k \rho_{vk}^l$.

Set $F = \omega^{-1}(\overline{U - U}) \backslash E^0$ with $U = \bigcup_{i \in \lambda} U_i$. Since $\overline{U - U}$ is compact, and ω is almost proper, F is finite, and we denote it by $\{\mu_1, \mu_2, \ldots, \mu_N\}$ if it is not empty. Then it is easy to check that $p s_\mu p = 0$ for any $\mu \in E^* \backslash (E^0 \cup F)$. Set $W = \{s(\mu_1), s(\mu_2), \ldots, s(\mu_N)\}$; then $(W, \le_T)$ is a partially ordered set. Let V be the subset of W consisting of all the maximal elements in $(W, \le_T)$, and let

$$m = \max_{1 \le i \le N} |\mu_i| + \max\big\{|\mu| : \mu \text{ is a path in } T \text{ from a vertex in } V \text{ to a vertex in } W\big\}.$$

For $v \in V$, $j \ge 0$, let $V(v, j)$ be all the vertices to which there is a (unique) path μ in T from v with $|\mu| = j$, and let

$$E_F^* = \bigcup_{v \in V} \bigcup_{j=0}^{m-1} \bigcup_{u \in V(v,j)} \{\mu : s(\mu) = u, 1 \le |\mu| \le m - j\}.$$

We note that if $F = \varnothing$, then $E_F^* = \varnothing$. It is clear that for any $\mu_i \in F$, there are $v \in V$ and $j \in \mathbb{N} \cup \{0\}$ with $0 \le j \le m - \max_{1 \le i \le N} |\mu_i|$ such that $s(\mu_i) \in V(v, j)$, and so $\mu_i \in E_F^*$. Since each weakly connected component of T is a rooted directed tree, for any $u, v \in V$ with $u \ne v$, there is no path in E from one vertex in $V(u, i)$ to another vertex in $V(v, j)$ for any $i, j \ge 0$, and also there is no path in E from one vertex in $V(v, k)$ to another vertex in $V(v, l)$ for any $k > l$. Now we let

$$q = \prod_{\mu \in E_F^*} (1 - s_\mu p s_\mu^*) p,$$

where p is the unit of $D_\lambda(\Gamma)$ for the fixed $\lambda \in \Lambda$ as mentioned above. We note that if $F = \varnothing$, then $q = p$. Since $1 - \rho_{vk}(p) = \prod_{s(\mu)=v,|\mu|=k}(1 - s_\mu p s_\mu^*)$,

$$q = \prod_{v\in V}\prod_{j=0}^{m-1}\prod_{u\in V(v,j)}\prod_{k=1}^{m-j}(1-\rho_{uk}(p))p$$
$$= \prod_{v\in V}\Big(\prod_{k=1}^{m}(1-\rho_{vk}(p))\prod_{u\in V(v,1)}\prod_{k=1}^{m-1}(1-\rho_{uk}(p))\cdots\prod_{u\in V(v,m-1)}(1-\rho_{u1}(p))\Big)p,$$

which is a projection dominated by p in $\mathcal{A}_\lambda(E) \subseteq M(\mathcal{F}(E)) \subseteq \mathbf{B}(L^2(\Gamma, \mathcal{H}))$.

Lemma 3.1. *With the notations as above, we have the following.*

(i) *q and $\rho_{uk}(q)$ commute with f and p_v for any $u, v \in E^0$, $k \geq 1$ and $f \in D_0(\Gamma)$.*
(ii) *For any $\mu, \nu \in E^*$, $(s_\mu q s_\mu^*)(s_\nu q s_\nu^*) = \delta_{\mu,\nu} s_\mu q s_\mu^*$. If moreover $\mu \notin E^0$, $q s_\mu q = 0$.*

Proof. (i) is clear from the direct computation, and (ii) is also clear if $\mu, \nu \in E^0$. Since $(s_\mu q s_\mu^*)^2 = s_\mu q s_\mu^*$ by (i), it is enough to prove that for any $\mu \in E^*\backslash E^0$, $q s_\mu q = 0$, which is equivalent to $q s_\mu q s_\mu^* q = 0$. If $\mu \notin F$, then $q s_\mu q s_\mu^* q = q(p s_\mu p) q s_\mu^* q = 0$. If $\mu \in F$, by the discussion above $\mu \in E_F^*$. Therefore $0 \leq q s_\mu q s_\mu^* q \leq q(s_\mu p s_\mu^*)q = 0$. □

Lemma 3.2. *With the notations as above, let $v \in E^0$. Then there is a finite subset $E^*(v)$ of E^* such that $p_v p = \sum_{\mu\in E^*(v)} s_\mu q s_\mu^* p$. Moreover if $v \notin \bigcup_{u\in V, 0\leq j\leq m-1} V(u,j)$, $E^*(v) = \{v\}$ and if $v \in \bigcup_{u\in V,0\leq j\leq m-1} V(u,j)$, $E^*(v)$ is a subset of finite set $E_F^* \bigcup(\bigcup_{u\in V,0\leq j\leq m-1} V(u,j))$.*

Proof. By definition, $q = \prod_{u\in V}\prod_{j=0}^{m-1}\prod_{w\in V(u,j)}\prod_{k=1}^{m-j}(1-\rho_{wk}(p))p$.

If $v \notin \bigcup_{u\in V,0\leq j\leq m-1} V(u,j)$, it is clear by direct computation that $p_v q p_v p = p_v p$.

If now $v \in V$, then since $\rho_{wk}(p)p_v = p_v\rho_{wk}(p) = \delta_{w,v}\rho_{wk}(p)$,

$$p_v q p_v p = p_v \prod_{k=1}^{m}(1-\rho_{vk}(p))p_v p = \prod_{k=1}^{m}(p_v - \rho_{vk}(p))p. \tag{3.1}$$

Let $1 \leq k \leq m$, since there is no path from v to any vertex in $V(u,j)$ for any $0 \leq j \leq m-1$ and $u \in V$ with $u \neq v$,

$$\begin{aligned}
\rho_{vk}(q)p &= \rho_{vk}\Big(\prod_{j=0}^{m-1}\prod_{u\in V(v,j)}\prod_{s=1}^{m-j}(1-\rho_{us}(p))p\Big)p \\
&= \prod_{j=0}^{m-1}\prod_{u\in V(v,j)}\prod_{s=1}^{m-j}(p_v-\rho_{vk}(\rho_{us}(p)))\rho_{vk}(p)p \\
&\qquad\qquad\qquad\qquad\text{(since } \rho_{vk} \text{ is a homomorphism)} \\
&= \prod_{j=0}^{m-1}\prod_{s=1}^{m-j}\prod_{u\in V(v,j)}(p_v-\rho_{vk}(\rho_{us}(p)))\rho_{vk}(p)p\ .
\end{aligned}$$

Since $\mu \notin F(\subseteq E_F^*)$ for any $\mu \in E^*$ such that $s(\mu) \in V(v,j)$ (for $v \in V$, $0 \leq j \leq m-1$) and $|\mu| > m-j$, we have

$$\begin{aligned}
\rho_{vk}(q)p &= \prod_{j=0}^{m-1}\prod_{s=1}^{m}\prod_{u\in V(v,j)}(p_v-\rho_{vk}(\rho_{us}(p)))\rho_{vk}(p)p \\
&\qquad\text{(since } \rho_{vk}(\rho_{us}(p))p=0 \text{ for } s>m-j \text{ and } u\in V(v,j)) \\
&= \prod_{s=1}^{m}\prod_{j=0}^{m-1}\prod_{u\in V(v,j)}(p_v-\rho_{vk}(\rho_{us}(p)))\rho_{vk}(p)p \\
&= \prod_{s=1}^{m-k}\prod_{j=0}^{m-1}\prod_{u\in V(v,j)}(p_v-\rho_{vk}(\rho_{us}(p)))\rho_{vk}(p)p \\
&\qquad\text{(since } \rho_{vk}(\rho_{us}(p))p=0 \text{ for } s>m-k \text{ and } u\in V(v,j)) \\
&= \prod_{s=1}^{m-k}\prod_{j=0}^{m-1}\prod_{u\in V(v,j)}\Big(p_v-\sum_{l=0}^{k}\rho_{vk}^{l}(\rho_{us}(p))\Big)\rho_{vk}(p)p \\
&= \prod_{s=1}^{m-k}\Big(p_v-\sum_{j=0}^{m-1}\sum_{u\in V(v,j)}\sum_{l=0}^{k}\rho_{vk}^{l}(\rho_{us}(p))\Big)\rho_{vk}(p)p \\
&= \prod_{s=1}^{m-k}\Big(p_v-\sum_{l=0}^{k}\sum_{j=0}^{m-1}\sum_{u\in V(v,j)}\rho_{vk}^{l}(\rho_{us}(p))\Big)\rho_{vk}(p)p.
\end{aligned}$$

Since

$$\sum_{j=0}^{m-1} \sum_{u\in V(v,j)} \rho_{vk}^k(\rho_{us}(p)) = \rho_{vk}^k(\rho_{vs}(p)) = \sum_{l=k}^{k+s} \rho_{v(k+s)}^l(p),$$

for $1 \le s \le m-k$, and

$$\sum_{j=0}^{m-1} \sum_{u\in V(v,j)} \rho_{vk}^l(\rho_{us}(p)) = \rho_{v(k+s)}^l(p), \quad \text{for } 0 \le l \le k-1,$$

we have

$$\rho_{vk}(q)p = \prod_{s=1}^{m-k} (p_v - \sum_{l=0}^{k+s} \rho_{v(k+s)}^l(p))\rho_{vk}(p)p = \prod_{s=1}^{m-k} (p_v - \rho_{v(k+s)}(p))\rho_{vk}(p)p.$$

$$\rho_{vk}(q)p = \prod_{s=k+1}^{m} (p_v - \rho_{vs}(p))\rho_{vk}(p)p \ . \tag{3.2}$$

Therefore

$$p_v p = p_v q p_v p + \sum_{k=1}^{m} \rho_{vk}(q)p$$

by (3.1), (3.2) and direct computation.

If $v \in V(u,i)$ for some $u \in V$, $1 \le i \le m-1$, the discussion is similar except for replacing m with $m-i$ and replacing $V(v,j)$ $(0 \le j \le m-1)$ with $\widetilde{V(v,j)}$ $(0 \le j \le m-i-1)$, where $\widetilde{V(v,j)}$ consists of all vertices to which there is a unique path μ in T from v with $|\mu| = j$. This completes the proof. □

Recall that $p_0 = 1-p$, and $p_1, p_2, \ldots, p_M$ are the mutually orthogonal minimal projections in D_λ with $\sum_{i=1}^M p_i = p$, which are defined in Section 2. We arbitrarily chose a vertex $\tilde{v} \in E^0$, and let $\widetilde{E_F^*} = E_F^* \cup \{\tilde{v}\}$. Let $\mathbf{J}$ be the finite set of all maps from $\widetilde{E_F^*}$ to the set $\{0, 1, 2, \ldots, M\}$, and $\mathbf{K}_1 = \{(v,\tau) : v \in E^0, \tau \in \mathbf{J}\}$. For any $(v,\tau) \in \mathbf{K}_1$, let

$$q_{(v,\tau)} = q \prod_{\mu \in \widetilde{E_F^*}} \sigma_{w_\mu}(p_{\tau(\mu)})p_v \in \mathbf{B}(L^2(\Gamma, \mathcal{H})).$$

Clearly, $q_{(v,\tau)}$ is a projection with $q_{(v,\tau)} \le q \le p$. Let

$$\mathbf{K} = \{(v,\tau) \in \mathbf{K}_1 : q_{(v,\tau)} \ne 0\},$$

which is clearly a countable (possibly infinite) set.

LEMMA 3.3.

(i) $\{q_{(v,\tau)}\}_{(v,\tau)\in\mathbf{K}}$ *are mutually orthogonal projections in* $\mathcal{F}(E)$.
(ii) *For any fixed* $v \in E^0$,

$$p_v q = \sum_{\tau\in\mathbf{J}} q_{(v,\tau)} = \sum_{\tau:(v,\tau)\in\mathbf{K}} q_{(v,\tau)}.$$

PROOF. (i) It is clear that $\{q_{(v,\tau)}\}_{(v,\tau)\in\mathbf{K}}$ are mutually orthogonal. Since $D_0(\Gamma)$ is invariant under the action of σ_γ (for all $\gamma \in \omega(E^*)$),

$$\prod_{\mu\in\widetilde{E_F^*}} \sigma_{w_\mu}(p_{\tau(\mu)})p_v p \in \mathcal{F}(E).$$

Therefore, $q_{(v,\tau)} = q\prod_{\mu\in\widetilde{E_F^*}} \sigma_{w_\mu}(p_{\tau(\mu)})p_v p \in \mathcal{F}(E)$ for $q \in \mathcal{A}_\lambda(E) \subseteq M(\mathcal{F}(E))$.

(ii)

$$\begin{aligned} p_v q &= p_v q \prod_{\mu\in\widetilde{E_F^*}} \sigma_{w_\mu}(p_0 + p_1 + \dots p_M) \\ &= p_v q \sum_{\tau\in\mathbf{J}} \prod_{\mu\in\widetilde{E_F^*}} \sigma_{w_\mu}(p_{\tau(\mu)}) = \sum_{\tau\in\mathbf{J}} q_{(v,\tau)}. \end{aligned}$$

□

LEMMA 3.4. *The* C*-*algebra* $\mathcal{F}_\lambda$ *generated by* $\{s_\mu q_{(v,\tau)} s_\nu^*\}_{\mu,\nu\in E^*,(v,\tau)\in\mathbf{K}}$ *is isomorphic to* $\bigoplus_{(v,\tau)\in\mathbf{K}} \mathcal{K}_{(v,\tau)}$, *and hence is an AF subalgebra of* $\mathcal{F}(E)$, *where* $\mathcal{K}_{(v,\tau)}$ *is a compact operator algebra over a finite dimensional or a separable infinite dimensional Hilbert space.*

PROOF. First, since $q_{(v,\tau)} \in \mathcal{F}(E)$, $s_\mu q_{(v,\tau)} s_\nu^* = (s_\mu p) q_{(v,\tau)} (s_\nu p)^* \in \mathcal{F}(E)$. Let (v_1,τ_1), $(v_2,\tau_2) \in \mathbf{K}$, by Lemma 3.1, for any $\mu_1,\mu_2,\nu_1,\nu_2 \in E^*$,

$$\begin{aligned} (s_{\mu_1} q_{(v_1,\tau_1)} s_{\nu_1}^*)(s_{\mu_2} q_{(v_2,\tau_2)} s_{\nu_2}^*) &= s_{\mu_1} q_{(v_1,\tau_1)} p_{v_1} (q s_{\nu_1}^* s_{\mu_2} q) q_{(v_2,\tau_2)} s_{\nu_2}^* \\ &= \delta_{\nu_1,\mu_2} \delta_{v_1,r(\nu_1)} \delta_{(v_1,\tau_1),(v_2,\tau_2)} s_{\mu_1} q_{(v_1,\tau_1)} s_{\nu_2}^*. \end{aligned}$$

Let $(v,\tau) \in \mathbf{K}$. If $r(\mu)$ or $r(\nu)$ is not v, then $s_\mu q_{(v,\tau)} s_\nu^* = 0$. Let $E^{*,v} = \{\mu \in E^* : r(\mu) = v\}$, by the computation above, $\{s_\mu q_{(v,\tau)} s_\nu^*\}_{\mu,\nu\in E^{*,v}}$ is a matrix unit, and generates the same C*-algebra, denoted by $\mathcal{F}_{(v,\tau)}$, as $\{s_\mu q_{(v,\tau)} s_\nu^*\}_{\mu,\nu\in E^*}$. Moreover $\mathcal{F}_{(v,\tau)}$ is isomorphic to a compact operator algebra $\mathcal{K}_{(v,\tau)}$ over a finite dimensional or separable infinite dimensional Hilbert space. This completes the proof. □

LEMMA 3.5. *The C^*-algebra $\mathcal{F}_\lambda(E)$ is a subalgebra of $\mathcal{F}_\lambda$.*

PROOF. Since

$$\mathcal{F}_\lambda(E) = \overline{\text{alg-span}}\{s_\mu p_i s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) \in E^0, i = 1, 2, \ldots, M\},$$

it is enough to prove for each $1 \leq i \leq M$, $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, $s_\mu p_i s_\nu^* \in \mathcal{F}_\lambda$. For any $v \in E^0$, $1 \leq i \leq M$,

$$\begin{aligned}
p_v p_i = p_v p p_i &= \sum_{\gamma \in E^*(v)} s_\gamma q s_\gamma^* p_i \quad \text{(by Lemma 3.2)} \\
&= \sum_{\gamma \in E^*(v)} s_\gamma (p_{r(\gamma)} q) s_\gamma^* p_i \\
&= \sum_{\gamma \in E^*(v)} s_\gamma \Big(\sum_{\tau \in \mathbf{J}} q_{(r(\gamma), \tau)} \Big) s_\gamma^* p_i \quad \text{(by Lemma 3.3(ii))} \\
&= \sum_{\gamma \in E^*(v)} \sum_{\tau \in \mathbf{J}} s_\gamma q_{(r(\gamma), \tau)} \sigma_{\omega_\gamma}(p_i) s_\gamma^* \\
&= \sum_{\gamma \in E^*(v)} \sum_{\tau \in \mathbf{J}} s_\gamma q \prod_{\mu \in E_F^*} \sigma_{w_\mu}(p_{\tau(\mu)}) \sigma_{\omega_\gamma}(p_i) p_{r(\gamma)} s_\gamma^* \\
&= \sum_{\gamma \in E^*(v) \cap E_F^*} \sum_{\tau \in \mathbf{J} : \tau(\gamma) = i} s_\gamma q_{(r(\gamma), \tau)} s_\gamma^* + \sum_{\gamma \in E^*(v) \setminus E_F^*} \sum_{\tau \in \mathbf{J} : \tau(\tilde{v}) = i} s_\gamma q_{(r(\gamma), \tau)} s_\gamma^* \\
&\qquad \text{(since } E^*(v) \setminus E_F^* \subseteq E^0, \text{ and } \omega_v = 1_\Gamma \text{ for any } v \in E^0).
\end{aligned}$$

Let $r(\mu) = r(\nu) = v$. Then

$$\begin{aligned}
s_\mu p_i s_\nu^* = &\sum_{\gamma \in E^*(v) \cap E_F^*} \sum_{\tau \in \mathbf{J} : \tau(\gamma) = i} s_{\mu\gamma} q_{(r(\gamma), \tau)} s_{\nu\gamma}^* \\
&+ \sum_{\gamma \in E^*(v) \setminus E_F^*} \sum_{\tau \in \mathbf{J} : \tau(\tilde{v}) = i} s_{\mu\gamma} q_{(r(\gamma), \tau)} s_{\nu\gamma}^*,
\end{aligned}$$

which is in $\mathcal{F}_\lambda$. This completes the proof. □

THEOREM 3.6. *Let T be a row-finite directed graph, each weakly connected component of which is a rooted (possibly infinite) directed tree. Let E be a row-finite directed graph constructed by attaching n_v ($0 \leq n_v < +\infty$) 1-loops to each vertex v in T. Let G be a locally compact abelian group with dual group Γ. Let $\omega \colon E^* \to \Gamma$ be an almost proper labeling map, and let $(\mathrm{C}^*(E), G, \alpha^\omega)$ be the C^*-dynamical system with α^ω being the quasi-free action corresponding to ω. Then the crossed product $\mathrm{C}^*(E) \times_{\alpha^\omega} G$ can be embedded into an AF algebra. If moreover G is compact, then the crossed product $\mathrm{C}^*(E) \times_{\alpha^\omega} G$ itself is an AF algebra.*

Proof. Recall that $\mathcal{F}(E)$ is the inductive limit of $\mathcal{F}_\lambda(E)$. By Lemma 3.4 and Lemma 3.5, $\mathcal{F}_\lambda(E)$ is contained in the AF subalgebra $\mathcal{F}_\lambda$ of $\mathcal{F}(E)$. Therefore $\mathcal{F}(E)$ is an AF algebra by the local characterization of AF algebras. If moreover G is compact, and so Γ is discrete, we let $\mathbf{I} = \Gamma$, and $U_i = \{i\}$ for any $i \in \mathbf{I} = \Gamma$, then $\mathrm{C}^*(E) \times_{\alpha^\omega} G = \mathcal{F}(E)$. This completes the proof. □

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