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MORITA EQUIVALENT SUBALGEBRAS OF IRRATIONAL ROTATION ALGEBRAS AND REAL QUADRATIC FIELDS

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ABSTRACT. We determine the isomorphic classes of Morita equivalent subalgebras of irrational rotation algebras. It is based on the solution of the quadratic Diophantine equations. We determine the irrational rotation algebras that have locally trivial inclusions. We compute the index of the locally trivial inclusions of irrational rotation algebras.

RÉSUMÉ. Nous déterminons les classes isomorphe de sous-algébres d'algébres de la rotation irrationnelle qui sont Morita-équivalente à l'algébre ambiante. Il est basé sur la solution des équations diophantienne du second degré. Nous déterminons les algébres de la rotation irrationnelle qui ont des inclusions localement triviaux. Nous calculons l'indices des inclusions localement triviaux d'algébres de la rotation irrationnelle.

1. **Introduction** Let θ be an irrational number. An irrational rotation algebra A_θ is the universal C*-algebra generated by two unitaries u, v, with the relation $uv = e^{2\pi i\theta}vu$. It is simple and has a unique normalized trace. These algebras have been classified up to C*-isomorphism and Morita equivalence [7,8].

C*-index theory in [10] is a C*-algebraic version of index theory for subfactors by V. F. R. Jones [4]. Let $N \subseteq M$ be II_1-factors. If M is a hyperfinite factor and the Jones index $[M : N]$ is finite, then N is also a hyperfinite factor. Hence N is isomorphic to M as a von Neumann algebra. In C*-index theory, there exist many non-isomorphic subalgebras that are of finite index. We need to consider isomorphic classes of subalgebras. K. Kodaka studied endomorphisms of certain irrational rotation algebras [5]. Since A_θ is simple, the ranges of endomorphisms are isomorphic subalgebras of A_θ. In this paper, we extend his results and study those C*-subalgebras of A_θ that are Morita equivalent to A_θ.

Throughout the paper, we assume that a subalgebra has a common unit.

In Section 2 we determine the isomorphic classes of Morita equivalent subalgebras of irrational rotation algebras. For example, Morita equivalent subalgebras of $A_{\frac{5+\sqrt{5}}{10}}$ are isomorphic to $A_{\frac{5+\sqrt{5}}{10}}$ or $A_{\frac{5+\sqrt{5}}{2}}$. Morita equivalent subalgebras of $A_{\frac{3+\sqrt{3}}{6}}$ are isomorphic to $A_{\frac{3+\sqrt{3}}{6}}$, $A_{\frac{3+\sqrt{3}}{3}}$, $A_{\frac{3+\sqrt{3}}{2}}$, or $A_{\sqrt{3}}$. It is based on the solution of the quadratic Diophantine equations. The isomorphic classes of Morita

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equivalent subalgebras of irrational rotation algebras are related to arithmetic properties of real quadratic fields. We show that a part of the decomposition of prime ideals in real quadratic fields is connected with the isomorphic classes of Morita equivalent subalgebras of irrational rotation algebras. We expect that there exists a connection with the real multiplication program by Y. Manin [6].

In Section 3 we determine the irrational rotation algebras which have locally trivial inclusions, where an inclusion $B \subseteq A$ of C*-algebras is called a locally trivial inclusion if there exist a projection q in A and an isomorphism φ of qAq onto $(1-q)A(1-q)$ such that $B = \{x + \varphi(x) \in A \,;\, x \in qAq\}$. A is simple, then qAq and A are Morita equivalent. Hence a locally trivial inclusion is a construction of a Morita equivalent subalgebra.

In Section 4 we show that the index of a locally trivial inclusion of an irrational rotation algebra is 4, which is equal to the minimal index (due to F. Hiai [3]) in the case of a subfactor.

2. Morita equivalent subalgebras Let τ_θ denote a unique normalized trace of A_θ, and let $M_k(A_\theta)$ denote the algebra of all $k \times k$ matrices over A_θ. The notation $A \cong B$ means that A is isomorphic to B as a C*-algebra. We denote by $\tau_{k,\theta}$ the unique normalized trace on $M_k(A_\theta)$. We refer the reader to K. R. Davidson [1] for the basic facts on C*-algebras.

First we consider the condition that $k \in \mathbb{N}$ and $\eta \in \mathbb{R} - \mathbb{Q}$ such that $M_k(A_\eta)$ is isomorphic to a subalgebra of A_θ.

LEMMA 2.1. *If $M_k(A_\eta)$ is isomorphic to a subalgebra B of A_θ with a common unit, then there exists a natural number n such that $M_k(A_\eta) \cong A_{n\theta}$.*

PROOF. Let $\varphi\colon M_k(A_\eta) \to B$ be an isomorphism. Since the trace is unique and a subalgebra has a common unit, $\tau_\theta(\varphi(x)) = \tau_{k,\eta}(x)$ for any $x \in M_k(A_\eta)$. By [8, Proposition 1.3], there exist an integer l and a projection q_1 in $M_k(A_\eta)$ such that $\tau_{k,\eta}(q_1) = \eta + l$. Since $\varphi(q_1)$ is a projection in A_θ, $\eta + l \in (\mathbb{Z} + \mathbb{Z}\theta) \cap [0,1]$. There exist integers m_0, m_1 such that $\eta + l = m_0\theta + m_1$. Hence $A_\eta \cong A_{|m_0|\theta}$. Define $n := |m_0|$. Let q_2 and q_3 be projections in $M_k(A_\eta)$ such that $\tau_{k,\eta}(q_2) = \frac{\eta+l}{k} = \frac{m_0\theta+m_1}{k}$ and $\tau_{k,\eta}(q_3) = \frac{1-\eta-l}{k} = \frac{1-m_0\theta-m_1}{k}$. Since $\varphi(q_2)$ and $\varphi(q_3)$ are projections in A_θ, we have $\frac{m_0\theta+m_1}{k}, \frac{1-m_0\theta-m_1}{k} \in (\mathbb{Z}+\mathbb{Z}\theta) \cap [0,1]$. Hence $\frac{m_1}{k}$ and $\frac{1-m_1}{k}$ are integers. This implies $k = 1$. Therefore $M_k(A_\eta) \cong A_{n\theta}$. □

We shall consider Morita equivalent subalgebras of A_θ. Let $\mathrm{GL}(2,\mathbb{Z})$ denote the group of 2×2 matrices with entries in $\mathbb{Z}$ and with determinant ± 1, and let $\mathrm{GL}(2,\mathbb{Z})$ act on the set of irrational numbers by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\theta = \frac{a\theta+b}{c\theta+d}$.

PROPOSITION 2.2. *A C*-algebra B is isomorphic to a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ if and only if there exist $n \in \mathbb{N}$ and $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $n\theta = g\theta$ and $B \cong A_{n\theta}$.*

PROOF. We assume that B is isomorphic to a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ. Since B is Morita equivalent to

A_θ and has a unit, there exist $k \in \mathbb{N}$ and $g_0 \in \mathrm{GL}(2,\mathbb{Z})$ such that $B \cong M_k(A_{g_0\theta})$ by [9, Corollary 2.6]. Hence, there exists a natural number n such that $B \cong A_{n\theta}$ by Lemma 2.1. Therefore $k = 1$ and $A_{n\theta} \cong A_{g_0\theta}$. There exists an integer l such that $n\theta = g_0\theta + l$ or $n\theta = -g_0\theta + l$ by [8, Theorem 2]. Let $g_1, g_2 \in \mathrm{GL}(2,\mathbb{Z})$ be $g_1 = \left(\begin{smallmatrix} 1 & l \\ 0 & 1 \end{smallmatrix}\right)$, $g_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Define $g := g_1 g_0$ or $g := g_1 g_2 g_0$. Then $g \in \mathrm{GL}(2,\mathbb{Z})$ and $n\theta = g\theta$. Consequently, there exist $n \in \mathbb{N}$ and $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $n\theta = g\theta$ and $B \cong A_{n\theta}$.

Conversely, assume that there exist $n \in \mathbb{N}$ and $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $n\theta = g\theta$ and $B \cong A_{n\theta}$. We consider the subalgebra generated by u^n and v. Since $u^n v = e^{2\pi i n\theta} v u^n$, it is isomorphic to $A_{n\theta}$. Since $n\theta = g\theta$, it is Morita equivalent to A_θ by [8, Theorem 4]. Consequently, B is isomorphic to a subalgebra of A_θ and is Morita equivalent to A_θ. □

We study the isomorphic classes of subalgebras of A_θ. Note that if $A_{n_0\theta}$ is isomorphic to $A_{n_1\theta}$ where n_0 and n_1 are natural numbers, then $n_0 = n_1$ by the results of [7, 8]. We consider the case where θ is not a quadratic irrational number.

Theorem 2.3. *Let θ be an irrational number. Assume that θ is not a quadratic number. If B is a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ, then B is isomorphic to A_θ.*

Proof. On the contrary, assume that A_θ had a non-isomorphic Morita equivalent subalgebra. Then there exist $n \in \mathbb{N}$ and $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $n\theta = g\theta$ and $n \neq 1$ by Proposition 2.2. There exist integers a, b, c, d such that $ad - bc = \pm 1$ and $n\theta = \frac{a\theta+b}{c\theta+d}$. Hence we have $nc\theta^2 + (dn-a)\theta - b = 0$. Since θ is not a quadratic irrational number and n is a natural number, $c = 0$, $dn - a = 0, b = 0$. By $ad - bc = \pm 1$, $ad = \pm 1$. Hence $n = \pm 1$. This is a contradiction. □

We consider the case where θ is a quadratic irrational number. We may assume that θ satisfies $k\theta^2 + l\theta + m = 0$ with a natural number k and integers l,m such that $\gcd(k, l, m) = 1$. The equation is uniquely determined. Let $D = l^2 - 4km$ be the discriminant of θ.

Lemma 2.4. *Let θ be a quadratic irrational number with $k\theta^2 + l\theta + m = 0$ as above. If B is a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ, then there exists a divisor n of k such that B is isomorphic to $A_{n\theta}$.*

Proof. By Proposition 2.2, there exist $n \in \mathbb{N}$ and $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $n\theta = g\theta$ and $B \cong A_{n\theta}$. There exist integers a, b, c, d such that $ad - bc = \pm 1$ and $n\theta = \frac{a\theta+b}{c\theta+d}$. Hence we have $nc\theta^2 + (dn-a)\theta - b = 0$. In the case $\theta = \frac{-l+\sqrt{D}}{2k}$, we have

$$\frac{nc(l^2 - 2l\sqrt{D} + D)}{4k^2} + \frac{(dn-a)(-l+\sqrt{D})}{2k} - b = 0.$$

Since $\sqrt{D}$ is an irrational number,

$$a = \frac{n}{k}(kd - lc), \quad b = \frac{nc(l^2 + D) - 2kdnl + 2kal}{4k^2} = -\frac{n}{k}mc.$$

By $ad - bc = \pm 1$, $kd^2 - ldc + mc^2 = \pm\frac{k}{n}$. Consequently, $\frac{k}{n} \in \mathbb{Z}$. The case $\theta = \frac{-l-\sqrt{D}}{2k}$ is proved in the same way. $\square$

THEOREM 2.5. *Let θ be a quadratic irrational number with $k\theta^2 + l\theta + m = 0$ as above. Assume that n is a divisor of k and $k = \alpha n$ for a natural number α. Then there exists a subalgebra B of A_θ with a common unit such that B is isomorphic to $A_{n\theta}$ and is Morita equivalent to A_θ if and only if either $nx^2 - lxy + \alpha my^2 = 1$ or $nx^2 - lxy + \alpha my^2 = -1$ has integer solutions for x and y.*

PROOF. We assume that there exists a subalgebra B of A_θ with a common unit such that B is isomorphic to $A_{n\theta}$ and is Morita equivalent to A_θ. Since A_θ and $A_{n\theta}$ are Morita equivalent, there exists a $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $n\theta = g\theta$. Let a, b, c, d be integers such that $ad - bc = \pm 1$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. In the case $\theta = \frac{-l+\sqrt{D}}{2k}$, by the proof of Lemma 2.4 we have $a = nd - \frac{lc}{\alpha} \in \mathbb{Z}$ and $b = -\frac{mc}{\alpha} \in \mathbb{Z}$. There exists an integer t such that $c = t\alpha$. If not, it contradicts $\gcd(k, l, m) = 1$. We have $ad - bc = nd^2 - ldt + \alpha mt^2$. Since $ad - bc = \pm 1$, $(x, y) = (d, t)$ is a solution of $nx^2 - lxy + \alpha my^2 = 1$ or $nx^2 - lxy + \alpha my^2 = -1$. The case $\theta = \frac{-l-\sqrt{D}}{2k}$ is proved in the same way.

Conversely, assume that either $nx^2 - lxy + \alpha my^2 = 1$ or $nx^2 - lxy + \alpha my^2 = -1$ has integer solutions for x and y. Let (d, t) be a solution of this equation. Define $a := nd - lt$, $b := -mt$, $c := \alpha t$, $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $ad - bc = \pm 1$. Therefore $g \in \mathrm{GL}(2,\mathbb{Z})$. Elementary calculations show $g\theta = n\theta$. By Proposition 2.2, there exists a subalgebra B of A_θ with a common unit such that B is isomorphic to $A_{n\theta}$ and is Morita equivalent to A_θ. $\square$

There exists an algorithm for solving the quadratic Diophantine equations of the theorem above, see for example [2]. Hence we can determine the isomorphic classes of Morita equivalent subalgebras of irrational rotation algebras. We shall show some examples.

EXAMPLE 2.6. Let θ be an algebraic integer of a real quadratic field. If B is a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ, then B is isomorphic to A_θ.

EXAMPLE 2.7. Let $\theta = \frac{5+\sqrt{5}}{10}$. If B is a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ, then B is isomorphic to A_θ or $A_{5\theta}$.

EXAMPLE 2.8. Let $\theta = \frac{3+\sqrt{3}}{6}$. If B is a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ, then B is isomorphic to A_θ, $A_{2\theta}$, $A_{3\theta}$, or $A_{6\theta}$.

EXAMPLE 2.9. Let $\theta = \frac{-5+\sqrt{65}}{10}$. If B is a subalgebra of A_θ with a common unit and is Morita equivalent to A_θ, then B is isomorphic to A_θ.

We shall show that the isomorphic classes of Morita equivalent subalgebras of irrational rotation algebras are related to arithmetic properties of real quadratic fields.

COROLLARY 2.10. *Let θ be a quadratic irrational number with $p\theta^2 + l\theta + m = 0$. Assume that p is a prime number and l, m are integers such that $\gcd(p, l, m) = 1$. We denote by $\mathbb{K} := \mathbb{Q}(\theta)$ an algebraic field generated by θ. The ring of integers of $\mathbb{K}$ is denoted by $\mathcal{O}_\mathbb{K}$. If A_θ has a non-isomorphic Morita equivalent subalgebra, then $p\mathcal{O}_\mathbb{K}$ is not a prime ideal in $\mathcal{O}_\mathbb{K}$, that is, p splits completely or is ramified in $\mathbb{K}$.*

PROOF. Since A_θ has a non-isomorphic Morita equivalent subalgebra, $A_{p\theta}$ is isomorphic to a Morita equivalent subalgebra of A_θ by Lemma 2.4. By Theorem 2.5, either $px^2 - lxy + my^2 = 1$ or $px^2 - lxy + my^2 = -1$ has integer solutions for x and y. We denote one of them by (d, t). We have

$$\Big(pd - \frac{l}{2}t + \frac{\sqrt{D}}{2}t\Big)\Big(pd - \frac{l}{2}t - \frac{\sqrt{D}}{2}t\Big) = \pm p.$$

It is easy to see that

$$pd - \frac{l}{2}t + \frac{\sqrt{D}}{2}t, pd - \frac{l}{2}t - \frac{\sqrt{D}}{2}t \in \mathcal{O}_\mathbb{K},$$
$$pd - \frac{l}{2}t + \frac{\sqrt{D}}{2}t, pd - \frac{l}{2}t - \frac{\sqrt{D}}{2}t \notin p\mathcal{O}_\mathbb{K}.$$

Hence $p\mathcal{O}_\mathbb{K}$ is not a prime ideal in $\mathcal{O}_\mathbb{K}$, that is, $p\mathcal{O}_\mathbb{K}$ splits completely or is ramified. □

3. Locally trivial inclusions In Section 2, we considered a subalgebra generated by u^n, v, and the isomorphic classes of the subalgebra. In this section, we shall show that there exist other Morita equivalent subalgebras of certain irrational rotation algebras.

Let $B \subseteq A$ be C*-algebras. An inclusion $B \subseteq A$ is called a locally trivial inclusion if there exist a projection q in A and an isomorphism φ of qAq onto $(1-q)A(1-q)$ such that $B = \{x + \varphi(x) \in A\ ;\ x \in qAq\}$. It is easy to see that B is isomorphic to qAq. Hence if A is simple, then B is Morita equivalent to A.

K. Kodaka [5] determined the irrational rotation algebras which have a locally trivial inclusion $B \subseteq A_\theta$ with $B \cong A_\theta$. He showed that there exists a projection q in A_θ such that $A_\theta \cong qA_\theta q \cong (1-q)A_\theta(1-q)$ if and only if the discriminant of θ is 5. We determine the irrational rotation algebras which have a locally trivial inclusion $B \subseteq A_\theta$. We do not assume that B is isomorphic to A_θ.

Let c, d be integers with $\gcd(c,d)=1$. We also assume $c\neq 0$. Let $V_\theta(d,c\,;k)$ be the standard module defined in [9] where k is a natural number. It is an $M_k(A_{\frac{a\theta+b}{c\theta+d}})$-$A_\theta$-equivalence bimodule constructed in [9] for any integers a, b such that $ad-bc=\pm1$. Since $V_\theta(d,c\,;\,k)$ is a finitely generated projective right A_θ-module, it corresponds to a projection in some $M_k(A_\theta)$. We also denote it by $V_\theta(m,l\,;k)$. Let $\mathrm{Tr}_\theta := \tau_\theta\otimes \mathrm{Tr}$ be the unnormalized trace on $M_k(A_\theta)$ where Tr is the usual trace on $M_k(\mathbb{C})$. The following lemma is based on a proof by K. Kodaka [5, Lemma 7].

Lemma 3.1. *If q is a proper projection in A_θ such that $\tau_\theta(q)=k(c\theta+d)$ where k is a natural number and c, d are integers such that $\gcd(c,d)=1$, then $qA_\theta q\cong M_k(A_{\frac{a\theta+b}{c\theta+d}})$ for any $a,b\in\mathbb{Z}$ such that $ad-bc=\pm1$.*

Proof. By [9, Theorem 1.4], $\mathrm{Tr}_\theta(V_\theta(d,c\,;k))=k(c\theta+d)$. Let e_1 be a rank one projection in $M_k(\mathbb{C})$. Then qA_θ is isomorphic to $(q\otimes e_1)M_k(A_\theta)$ as a module. Since $\mathrm{Tr}_\theta(q\otimes e_1)=k(c\theta+d)=\mathrm{Tr}_\theta(V_\theta(c,d\,;k))$, qA_θ is isomorphic to $V_\theta(c,d\,;k)$ as a module by [5, Lemma 6]. Since qA_θ is the $qA_\theta q$-A_θ-equivalence bimodule, $qA_\theta q\cong M_k(A_{\frac{a\theta+b}{c\theta+d}})$ for any $a,b\in\mathbb{Z}$ such that $ad-bc=\pm1$ by [9, Theorem 1.1, Corollary 2.6]. □

Let

$$\begin{aligned} S_1 = \Big\{ &\frac{-K(2d-1)\pm\sqrt{K^2-4K}}{2cK}\ ; \\ &K,c,d\in\mathbb{Z},\ K\geq 5,\ \gcd(c,d)=1,\ \frac{Kd^2-Kd+1}{c}\in\mathbb{Z}\Big\}, \\ S_2 = \Big\{ &\frac{2-K(2d-1)-\sqrt{K^2+4}}{2cK}\ ; \\ &K,c,d\in\mathbb{Z},\ K\neq 0,\ \gcd(c,d)=1,\ \frac{Kd^2-Kd-2d+1}{c}\in\mathbb{Z}\Big\}. \end{aligned}$$

Lemma 3.2. *If an irrational rotation algebra A_θ has a locally trivial inclusion, then $\theta\in S_1\cup S_2$.*

Proof. There exists a projection q such that $qA_\theta q\cong(1-q)A_\theta(1-q)$. By [8, Proposition 1.3], there exist integers c, d and a natural number k such that $\tau_\theta(q)=k(c\theta+d)$ and $\gcd(c,d)=1$. Since $qA_\theta q\cong(1-q)A_\theta(1-q)$, $c\neq0$. By Lemma 3.1, $qA_\theta q\cong M_k(A_{\frac{a\theta+b}{c\theta+d}})$ for any $a,b\in\mathbb{Z}$ such that $ad-bc=\pm1$. Fix $a,b\in\mathbb{Z}$ such that $ad-bc=1$. Since $M_k(A_{\frac{a\theta+b}{c\theta+d}})$ is isomorphic to a subalgebra of A_θ with a common unit, $k=1$ by Lemma 2.1. By Lemma 3.1 and $qA_\theta q\cong(1-q)A_\theta(1-q)$, there exist integers s, t such that $s(1-d)+tc=\pm1$ and $\frac{a\theta+b}{c\theta+d}=\frac{s\theta+t}{-c\theta+1-d}$. Therefore we have

$$(s+a)c\theta^2+((s+a)d+(t+b)c-a)\theta+(t+b)d-b=0$$

We shall consider the case $s(1-d)+tc=1$. Define $K := s+a$. Since $ad-bc = 1$, $s(1-d)+tc=1$, and $c \neq 0$, we have $t+b = \frac{(s+a)d-s}{c} \in \mathbb{Z}$. Therefore elementary calculations show $\theta = \frac{-K(2d-1)\pm\sqrt{K^2-4K}}{2cK}$ and $\frac{Kd^2-Kd+1}{c} \in \mathbb{Z}$. Since $0 < \tau_\theta(q) < 1$, $0 < c\theta + d = \frac{K\pm\sqrt{K^2-4K}}{2K} < 1$. Hence $K \geq 5$. Consequently, $\theta \in S_1$. In the case $s(1-d)+tc = -1$, similar calculations show that $\theta \in S_2$. □

LEMMA 3.3. *If $\theta \in S_1 \cup S_2$, then A_θ has a locally trivial inclusion.*

PROOF. We assume that $\theta \in S_1$. There exist integers c, d, and K such that $\gcd(c,d)=1$, $K \geq 5$, $\frac{Kd^2-Kd+1}{c} \in \mathbb{Z}$, and $\theta = \frac{-K(2d-1)\pm\sqrt{K^2-4K}}{2cK}$. Since $K \geq 5$, $0 < c\theta+d = \frac{K\pm\sqrt{K^2-4K}}{2K} < 1$. By [8, Proposition 1.3], there exists a projection q in A_θ such that $\tau_\theta(q) = c\theta+d$. Because $\gcd(c,d)=1$, there exist integers a, b such that $ad-bc=1$. Define $s := K-a$ and $t := \frac{s(d-1)+1}{c}$. We shall show that t is an integer. Elementary calculations show $t = -b + \frac{Kd-K+a}{c}$. Hence we only need to show that $\frac{Kd-K+a}{c}$ is an integer. Since $\gcd(c,d)=1$, it is sufficient to show that $\frac{d}{c}(Kd-K+a)$ is an integer. We have $\frac{d}{c}(Kd-K+a) = b + \frac{Kd^2-Kd+1}{c}$ by simple calculations. Since $\frac{Kd^2-Kd+1}{c} \in \mathbb{Z}$, $\frac{d}{c}(Kd-K+a)$ is an integer. Therefore t is an integer. It is easy to see that $s(1-d)+tc=1$. By Lemma 3.1, we have $qA_\theta q \cong A_{\frac{a\theta+b}{c\theta+d}}$ and $(1-q)A_\theta(1-q) \cong A_{\frac{s\theta+t}{-c\theta+1-d}}$. It is easily seen that $\frac{a\theta+b}{c\theta+d} = \frac{s\theta+t}{-c\theta+1-d}$. Therefore $qA_\theta q \cong (1-q)A_\theta(1-q)$. Consequently, A_θ has a locally trivial inclusion. In the case $\theta \in S_2$, similar calculations show A_θ has a locally trivial inclusion. □

We shall determine the locally trivial inclusions of irrational rotation algebras.

THEOREM 3.4. *We have the following.*

(i) *Let $\theta = \frac{-K(2d-1)+\sqrt{K^2-4K}}{2cK}$ with $K,c,d \in \mathbb{Z}$, $K \geq 5$, $\gcd(c,d)=1$ and $\frac{Kd^2-Kd+1}{c} \in \mathbb{Z}$. Then the irrational rotation algebra A_θ has a locally trivial inclusion $A_{K\theta} \subseteq A_\theta$.*

(ii) *Let $\theta = \frac{-K(2d-1)-\sqrt{K^2-4K}}{2cK}$ with $K,c,d \in \mathbb{Z}$, $K \geq 5$, $\gcd(c,d)=1$ and $\frac{Kd^2-Kd+1}{c} \in \mathbb{Z}$. Then the irrational rotation algebra A_θ has a locally trivial inclusion $A_{K\theta} \subseteq A_\theta$.*

(iii) *Let $\theta = \frac{-K(2d-1)+2-\sqrt{K^2+4}}{2cK}$ with $K,c,d \in \mathbb{Z}$, $K \neq 0$, $\gcd(c,d)=1$ and $\frac{Kd^2-Kd-2d+1}{c} \in \mathbb{Z}$. Then the irrational rotation algebra A_θ has a locally trivial inclusion $A_{K\theta} \subseteq A_\theta$.*

(iv) *Let θ be an irrational number not of the form in* (i), (ii) *and* (iii). *Then the irrational rotation algebra A_θ has no locally trivial inclusions.*

Proof. In (i), by Lemma 3.3, A_θ has a locally trivial inclusion $A_{\frac{a\theta+b}{c\theta+d}} \subseteq A_\theta$ for any $a, b \in \mathbb{Z}$ such that $ad - bc = \pm 1$. Fix $a, b \in \mathbb{Z}$ such that $ad - bc = 1$. Define $m := \frac{Kd-K+a}{c}$. By the proof of Lemma 3.3, we have $m \in \mathbb{Z}$. Elementary calculations show that $\frac{a\theta+b}{c\theta+d} - m = K\theta$. Consequently, $A_{\frac{a\theta+b}{c\theta+d}} \cong A_{K\theta}$. The proofs of (ii) and (iii) are similar. (iv) is immediate by Lemma 3.2. □

The case studied by K. Kodaka in [5] is (iii) with $K = \pm 1$. We shall show some examples.

Example 3.5. Let θ be an algebraic integer of a real quadratic field. If the discriminant of θ is not 5, then A_θ has no locally trivial inclusions.

Example 3.6. Let $\theta = \frac{5+\sqrt{5}}{10}$. Then A_θ has locally trivial inclusions $A_\theta \subseteq A_\theta$ and $A_{5\theta} \subseteq A_\theta$.

Example 3.7. Let $\theta = \frac{3+\sqrt{3}}{6}$. Then A_θ has a locally trivial inclusion $A_{6\theta} \subseteq A_\theta$.

4. **The index of the locally trivial inclusions** In this section, we compute the index of the locally trivial inclusions of irrational rotation algebras. We refer the reader to Y. Watatani [10] for the basic facts of C*-index theory.

Throughout this section, we assume that q is a projection in A_θ such that $qA_\theta q \cong (1-q)A_\theta(1-q)$. Let φ denote an isomorphism of $qA_\theta q$ onto $(1-q)A_\theta(1-q)$. We may assume $\tau_\theta(q) > 1/2$. Let $B := \{x + \varphi(x) \,;\, x \in qA_\theta q\}$.

We define a conditional expectation $E\colon A_\theta \to B$ by

$$E(x) := \frac{1}{2}\big(qxq + (1-q)x(1-q) + \varphi(qxq) + \varphi^{-1}((1-q)x(1-q))\big).$$

Then by an easy computation, we see that E is faithful.

The following lemma is well known.

Lemma 4.1. *Let q_1, q_2 be projections in A_θ. If $\tau_\theta(q_1) \geq \tau_\theta(q_2)$, then there exists a unitary element w in A_θ such that $q_1 \geq w^*q_2w$.*

We shall show a key lemma.

Lemma 4.2. *There exist a projection q_0 in $(1-q)A_\theta(1-q)$, a natural number n, orthogonal projections $q_1, \ldots, q_n$ in $qA_\theta q$, and unitary elements $w_1, \ldots, w_n$ in A_θ such that $q = q_1 + q_2 + \cdots + q_n$ and $q_i = w_i^*(1-q)w_i, (0 < i < n)$ and $q_n = w_n^* q_0 w_n$.*

Proof. Since we assume $\tau_\theta(q) > 1/2$, $\tau_\theta(q) > \tau_\theta(1-q)$. It is easy to see that there exists a unique natural number k such that $0 < \tau_\theta(q) - k\tau_\theta(1-q) < \tau_\theta(1-q)$. Define $n := k+1$. By Lemma 4.1, there exist a projection q_1 in $qA_\theta q$ and a unitary element w_1 in A_θ such that $q_1 = w_1^*(1-q)w_1$. Similarly for $(1 < i < n)$

since $\tau_\theta(1-q) < \tau_\theta(q) - (i-1)\tau_\theta(1-q) = \tau_\theta(q - q_1 - \cdots - q_{i-1})$, there exist a projection q_i in $(q - q_1 - \cdots - q_{i-1})A_\theta(q - q_1 - \cdots - q_{i-1})$ and a unitary element w_i in A_θ such that $q_i = w_i^*(1-q)w_i$. Define $q_n := q - q_1 - \cdots - q_{n-1}$. Since $\tau_\theta(1-q) > \tau_\theta(q_n)$, there exist a projection q_0 in $(1-q)A_\theta(1-q)$ and a unitary element w_n such that $q_n = w_n^* q_0 w_n$. Therefore we obtain the conclusion. □

The lemma above and simple calculations show the following lemma.

Lemma 4.3. *We have the following.*

(i) *The family* $\{(u_1, u_1^*), \ldots, (u_{n+3}, u_{n+3}^*)\} :=$

$$\Big\{(\sqrt{2}\,q, \sqrt{2}\,q), (\sqrt{2}\,(1-q), \sqrt{2}\,(1-q)), (\sqrt{2}\,(1-q)w_1 q, \sqrt{2}\,q w_1^*(1-q)),$$
$$(\sqrt{2}\,w_1^*(1-q), \sqrt{2}\,(1-q)w_1), \ldots, (\sqrt{2}\,w_{n-1}^*(1-q), \sqrt{2}\,(1-q)w_{n-1}),$$
$$(\sqrt{2}\,w_n^* q_0, \sqrt{2}\,q_0 w_n)\Big\}$$

is a quasi-basis for E.

(ii) *Index* $E = 4$.

We obtain the following theorem by Lemma 4.3 and [10, Theorem 2.12.3].

Theorem 4.4. *Let* $B \subseteq A_\theta$ *be a locally trivial inclusion. Then* $[A_\theta : B]_0 = 4$.

Remark 4.5. The index of the locally trivial inclusions of irrational rotation algebras is equal to the minimal index (due to F. Hiai [3]) in the case of subfactors.

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