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TORSION IN THE K_0-GROUP OF A RECURSIVE SUBHOMOGENEOUS ALGEBRA

SANDRO MOLINA-CABRERA

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Abstract. We show that the K_0-group of an inductive limit of recursive subhomogeneous algebras with compact metrizable spaces of dimension at most one as local spectra is torsion free. This result implies that the K_0-group of a unital simple AH algebra which is the inductive limit of recursive subhomogeneous algebras, with compact metrizable spaces of dimension at most one as local spectra, is torsion free. This proves that Li's reduction theorem for the dimension of the local spectra of unital simple AH algebras cannot be improved, in other words, that the dimension of the local spectra of unital simple AH algebras cannot be further reduced from two to one, even when we use subhomogeneous algebras. This also shows that if a reduction theorem for the dimension of the local spectra of simple inductive limits of recursive subhomogeneous algebras exists, then, after the reduction, the local spectra of the building blocks cannot always be one dimensional.

Résumé. Nous démontrons que le K_0-groupe d'une limite inductive des algèbres sous-homogènes récursives, dont les spectres locaux consistent en des espaces compacts métrisables de dimension au plus un, n'a pas de torsion. Ce résultat implique que les K_0-groupes d'une algèbre AH simple et avec l'unité qui est la limite des algèbres sous-homogènes rećursives, dont les spectres locaux consistent en des espaces compacts métrisables de dimension au plus un, n'a pas de torsion. Cela prouve que le théorème de Li de la réduction pour la dimension des spectres locaux des algèbres AH simples et avec l'unité ne peut pas être améliorée, en d'autres termes, que la dimension des spectres locaux des algèbres AH simples et avec l'unité ne peut pas encore être réduit de deux à un, même quand on utilise des algèbres sous-homogènes. Cela montre aussi que si un théorème de réduction pour la dimension des spectres locaux d'une limite inductive simple des algèbres sous-homogènes récursives existe, alors, après la réduction, les spectres locaux des blocs de construction ne peuvent pas être toujours de dimension un.

A C^*-algebra A is called an ASH (approximately sub-homogeneous) algebra if it is the inductive limit of a sequence $(A_n)_{n=1}^{+\infty}$ of C^*-algebras where A_n

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is a subalgebra of $\bigoplus_{j=1}^{s_n} M_{n_j}(C(X_{n_j}))$ with X_{n_j} a compact metrizable space, and where n_j, s_n are positive integers. In particular, an AH (approximately homogeneous) algebra is a C^*-algebra which is the inductive limit of a sequence $(A_n)_{n=1}^{+\infty}$ of C^*-algebras where $A_n = \bigoplus_{j=1}^{s_n} p_{n_j} M_{n_j}(C(X_{n_j}))p_{n_j}$, with X_{n_j} a compact metrizable space, p_{n_j} a projection in $M_{n_j}(C(X_{n_j})$, and n_j, s_n positive integers. The spaces X_{n_j} are known as the local spectra of the algebra. Unital simple AH algebras have been intensively studied during the past two decades, with special emphasis on their classification. See [2] and [3]. In particular, unital simple AH algebras with no dimension growth, that is, with $\sup_{n_j}\{\dim(X_{n_j})\} < +\infty$, have been classified up to isomorphism using their scaled ordered K-groups $(K_0(A), K_0(A)^+, [1_A]_0, K_1(A))$, their space of tracial states $T(A)$, and the natural pairing between $K_0(A)$ and $T(A)$.

One of the main techniques used in the classification of unital simple AH algebras consists in rewriting the algebra as another inductive limit in which the dimensions of the local spectra are small. Gong showed that a unital simple AH algebra with metrizable, compact local spectra and with no dimension growth can be written as an inductive limit where the local spectra consist of spaces of dimension at most three [4]. Using this, George Elliott, Guihua Gong, and Liangqing Li proved a classification theorem assuming the AH algebras had this form [3]. Later, Liangqing Li, using homogeneous C^*-algebras and dimension drop C^*-algebras (which are subhomogeneous), showed that a simple AH algebra with metrizable, compact local spectra and with slow dimension growth can be rewritten as an inductive limit where the dimension of the local spectra is at most two and proved a classification theorem assuming the algebras had this particular form [7]. Gong and Li noted that the dimension cannot be reduced to one if we only use dimension drop subhomogeneous algebras together with homogeneous algebras, since it is not possible to generate torsion in K_0 using dimension drop algebras and homogeneous algebras with one-dimensional local spectra as building blocks. It is natural to ask if we can reduce the dimension of the local spectra from two to one if we use more general subhomogeneous C^*-algebras.

Recursive subhomogeneous algebras were introduced by N. C. Phillips in [8] (we recall the definition below). They include dimension drop algebras, splitting interval algebras and homogeneous algebras, C^*-algebras that have been used as building blocks. See [5], [7], and [8]. Considering this, it is natural to ask if we can further reduce the dimension of the local spectra of AH algebras from two to one if we use recursive subhomogeneous algebras as building blocks, and in particular, we can ask if the K_0-group of a recursive subhomogeneous algebra with one-dimensional compact metrizable spaces is torsion free. On the other hand, the class of recursive subhomogeneous algebras is a natural place to prove analogous results first established for homogeneous algebras. Thus, it is reasonable to use the classification of unital simple AH algebras as a model for the classification of inductive limits of recursive subhomogeneous algebras. We show in this note that the K_0-group of an inductive limit of recursive subhomogeneous algebras with compact, metrizable spaces of dimension one as local spectra has no torsion. Hence it is impossible to produce torsion in the K_0-group of an inductive limit of recursive subhomogeneous algebras, or even in the K_0-group of a unital simple AH algebra, using recursive subhomogeneous building blocks with one-dimensional spectra. In addition, if a reduction theorem exists for recursive subhomogeneous algebras, then, after reduction, the local spectra of the building blocks cannot be assured to be one-dimensional.

Definition 1. Let A, B, and C be C^*-algebras. Let $\phi\colon A \to C, \psi\colon B \to C$ be $*$-homomorphisms. The pullback of A and B along C is the C^*-algebra

$$A \oplus_C B = \{(a,b) \in A \times B : \phi(a) = \psi(b)\}.$$

We recall the definition of a *recursive subhomogeneous algebra*. We refer the reader to [8] for further information. First, let us denote by M_k the C^*-algebra of $k \times k$ matrices with complex entries. Let $M_k(C(X))$ be the C^*-algebra of M_k-valued continuous functions over a topological space X. We recall that $M_k(C(X))$ can be identified with $C(X, M_k)$ in a canonical way.

Definition 2. A recursive subhomogeneous algebra is a C^*-algebra defined as follows.

(i) If X is a compact Hausdorff space, then $M_k(C(X))$ is a recursive subhomogeneous algebra.

(ii) If A is a recursive subhomogeneous algebra, X a compact Hausdorff space, X_0 a closed subset of X, $\phi\colon A \to M_k(C(X_0))$ any $*$-homomorphism, and $\psi\colon M_k(C(X)) \to M_k(C(X_0))$ the restriction homomorphism, then the pullback

$$A \oplus_{M_k(C(X_0))} M_k(C(X)) = \{(a,f) \in A \oplus M_k(C(X)) : \phi(a) = \psi(f)\}$$

is a recursive subhomogeneous algebra.

In (ii), the choice $X_0 = \varnothing$ is allowed (and thus $\phi = 0$ is allowed) and hence the pullback could be an ordinary direct sum. Also, it is convenient to allow the zero C^*-algebra to be a recursive subhomogeneous algebra.

For a compact Hausdorff space X_k, set $C_k = M_{n(k)}(C(X_k))$ and set $C_k^0 = M_{n(k)}(C(X_{k,0}))$ for a (possibly empty) closed subset $X_{k,0}$ of X_k. It follows from the definition that a recursive subhomogeneous algebra has a *decomposition* of the form

$$[\cdots[[C_0 \oplus_{C_1^0} C_1] \oplus_{C_2^0} C_2] \cdots] \oplus_{C_l^0} C_l,$$

with $l < \infty$, and where the maps $C_k \to C_k^0$ are always assumed to be the restriction maps. A recursive subhomogeneous algebra can have many different decompositions, but it always has a decomposition of the form given above.

We associate this decomposition with the following data:

- its *length* l;

- the *k-th stage algebra* ($k < l$)

$$R_k = [\cdots [[C_0 \oplus_{C_1^0} C_1] \oplus_{C_2^0} C_2] \cdots] \oplus_{C_k^0} C_k;$$

- its *local spectra* $X_0, X_1, X_2, \ldots, X_l$;
- its *total space* $X = \amalg_{j=0}^{l} X_{j,0}$;
- its *interconnecting spaces* $X_{1,0}, X_{2,0}, X_{3,0}, \ldots, X_{l,0}$;
- its *topological dimension* $\dim(X) = \max_j \dim(X_j)$.

Remark 3. In [8] the spaces $X_0, X_1, X_2, \ldots, X_l$ are called the base spaces of the decomposition, but following the convention for AH algebras, we prefer to call them the local spectra. (See [8, Definition 1.2].) By the dimension of a space X we mean the covering dimension of X.

Lemma 4. *Let X be a simplicial complex of dimension one and let X_0 be a subcomplex of X. For any $u\colon X_0 \to U_n(\mathbb{C})$ there is a unitary $\upsilon\colon X \to U_n(\mathbb{C})$ such that $u = \upsilon|_{X_0}$. Consequently, if $\psi_*\colon K_1(M_k(C(X))) \to K_1(M_k(C(X_0)))$ is the map induced by the restriction map $\psi\colon M_k(C(X)) \to M_k(C(X_0))$, then ψ_* is surjective.*

Proof. We must show that given $u\colon X_0 \to U_k(\mathbb{C})$ there is $\upsilon\colon X \to U_k(\mathbb{C})$ such that $\upsilon|_{X_0} = u$. Since X_0 is a subcomplex of X, we can construct X from X_0 by attaching simplices of dimension zero or one. Therefore it is enough to prove the statement for two special cases: in the first one, $X_0 = \varnothing$, $X = \{pt\}$, and in the second one, $X_0 = \{0,1\}$ and $X = [0,1]$. The first case is trivial. For the second case it is enough to show that for any two unitaries u and v, there is a continuous family $\{u_t\}_t$, for t in $[0,1]$, such that $u_0 = u$ and $u_1 = v$. Since u^*v is a unitary, there is a unitary ω such that $u^*v = \omega(e^{i\lambda_1} \oplus e^{i\lambda_2} \oplus \cdots \oplus e^{i\lambda_k})\omega^*$, for some real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$. For $t \in [0,1]$, let $u_t = u\omega(e^{i\lambda_1 t} \oplus e^{i\lambda_2 t} \oplus \cdots \oplus e^{i\lambda_k t})\omega^*$. Then $t \to u_t$ is continuous and $\{u_t\}$ connects u_0 and u_1. □

Lemma 5. *Let X be a compact Hausdorff space and let F be a closed subset of X. Let $f\colon F \to U_k(\mathbb{C})$ be a continuous function. Then there is an open subset $\mathcal{O}$ containing F and a continuous function $\widetilde{f}\colon \mathcal{O} \to U_k(\mathbb{C})$ such that $\widetilde{f}|_F = f$.*

Proof. Recall that $U_k(\mathbb{C}) \subseteq M_k(\mathbb{C}) \cong \mathbb{R}^{k^2}$. By the Tietze extension theorem there is a map $g\colon X \to M_k(\mathbb{C})$ such that $g|_F = f$. Note that $Gl_k(\mathbb{C})$ is an open set of $M_k(\mathbb{C})$, and thus, if we denote $g^{-1}(Gl_k(\mathbb{C}))$ by $\mathcal{O}$, $\mathcal{O}$ is an open subset of X and it contains F, since $U_k(\mathbb{C}) \subseteq Gl_k(\mathbb{C})$. Define $u\colon Gl_k(\mathbb{C}) \to U_k(\mathbb{C})$ by $u(A) = (AA^*)^{-\frac{1}{2}}A$. Note that for A in $Gl_k(\mathbb{C})$, $u(A)^*u(A) = u(A)u(A)^* = 1$. Also $u|_{U_k(\mathbb{C})} = \mathrm{id}$. Finally, let $\widetilde{f}\colon \mathcal{O} \to U_k(\mathbb{C})$ be defined by $\widetilde{f} = u \circ g$. □

We will also need the following fact.

Lemma 6. *Assume X is a compact Hausdorff space of dimension at most 1. Then $K_0(M_k((C(X)))$ is torsion free.*

Let $C_{X_0}(X, M_k)$ denote the C^*-subalgebra of $M_k(C(X))$ consisting of all the M_k valued continuous functions over X that vanish over X_0, *i.e.,* such that $f|_{X_0} = 0$.

Proposition 7. *Let A be a C^*-algebra such that $K_0(A)$ is torsion free. Let X be a compact, metrizable space of dimension at most 1, and let X_0 be a closed subset of X. Let $\psi\colon M_k(C(X)) \to M_k(C(X_0))$ denote the restriction homomorphism, let $\phi\colon A \to M_k(C(X_0))$ be a unital $*$-homomorphism and set $B = A \oplus_{M_k(C(X_0))} M_k(C(X))$. Then $K_0(B)$ is torsion free.*

To prove Proposition 7 let us begin by denoting the projection from B to A by σ. Since the $*$-homomorphism ψ is surjective we have the following.

Lemma 8. *The map σ is surjective.*

Consider the following short exact sequences:

$$0 \longrightarrow C_{X_0}(X, M_k) \xrightarrow{\iota} M_k(C(X)) \xrightarrow{\psi} M_k(C(X_0)) \longrightarrow 0$$

and

$$0 \longrightarrow C_{X_0}(X, M_k) \xrightarrow{\delta} B \xrightarrow{\sigma} A \longrightarrow 0,$$

where δ is obtain by composing the $C_{X_0}(X, M_k) \hookrightarrow M_k(C(X))$ with the map $M_k(C(X)) \to B$ defined by $f \mapsto (0, f)$. Combining these two short exact sequences gives the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{X_0}(X, M_k) & \xrightarrow{\delta} & B & \xrightarrow{\sigma} & A & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow \\
0 & \longrightarrow & C_{X_0}(X, M_k) & \xrightarrow{\iota} & M_k(C(X)) & \xrightarrow{\psi} & M_k(C(X_0)) & \longrightarrow & 0.
\end{array}$$

Since this diagram is commutative, in K-theory it gives

$$\begin{array}{ccccccc}
K_1(A) & \xrightarrow{\partial_1} & K_0(C_{X_0}(X, M_k)) & \xrightarrow{\delta_*} & K_0(B) & \xrightarrow{\sigma_*} & K_0(A) \\
\downarrow{\scriptstyle \phi_*} & & \| & & \downarrow & & \downarrow \\
K_1(M_k(C(X_0))) & \xrightarrow{\partial_1'} & K_0(C_{X_0}(X, M_k)) & \xrightarrow{\iota_*} & K_0(M_k(C(X))) & \xrightarrow{\psi_*} & K_0(M_k(C(X_0))).
\end{array}$$

From this we obtain the following short exact sequence:

$$0 \longrightarrow K_0(C_{X_0}(X, M_k))/\operatorname{Im}(\partial_1) \xrightarrow{\delta_*} K_0(B) \xrightarrow{\sigma_*} \operatorname{Im}(\sigma_*) \longrightarrow 0.$$

If $K_0(C_{X_0}(X, M_k))/\operatorname{Im}(\partial_1)$ and $\operatorname{Im}(\sigma_*)$ are torsion free, then it will follow that the group $K_0(A \oplus_{C(X_0,M_k)} M_k(C(X)))$ is torsion free. Note that $\operatorname{Im}(\sigma_*)$ is torsion free, since it is a subgroup of the group $K_0(A)$ that we assume to be torsion free. Therefore, to prove that the quotient group $K_0(C_{X_0}(X, M_k))/\operatorname{Im}(\partial_1)$ is torsion free, it suffices to prove that $K_0(C_{X_0}(X, M_k))$ is torsion free and that ∂_1 is identically zero. However, by commutativity, $\partial_1' = 0$ implies $\partial_1 = 0$. Now consider the sequence

$$K_1(M_k(C(X))) \xrightarrow{\psi_*} K_1(M_k(C(X_0))) \xrightarrow{\partial_1'} K_0(C_{X_0}(X, M_k)) \longrightarrow K_0(M_k(C(X))).$$

If ∂_1' is identically zero, then $K_0(C_{X_0}(X, M_k))$ is isomorphic to a subgroup of $K_0(M_k(C(X)))$ that is torsion free by Lemma 6. Hence to prove Proposition 7, it suffices to prove that ∂_1' is identically zero. By exactness, it is enough to show that ψ_* is surjective.

Recall that a compact metric space X is the inverse limit of a sequence $(X_j)_{j=1}^{+\infty}$ of simplicial complexes with the dimension of each complex less than or equal to the dimension of X. (See [1]). Define $X_{j,0}$ to be the image of X_0 under the canonical map from X to X_j and consider the following commutative diagram:

$$\begin{array}{ccccccc}
X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots & \longleftarrow & X \\
\uparrow & & \uparrow & & & & \uparrow \\
X_{1,0} & \longleftarrow & X_{2,0} & \longleftarrow & \cdots & \longleftarrow & X_0
\end{array}$$

Passing to K-theory we get

$$\begin{array}{ccccccc}
K_1(M_k(C(X_1))) & \longrightarrow & K_1(M_k(C(X_2))) & \longrightarrow & \cdots & \longrightarrow & K_1(M_k(C(X))) \\
\downarrow & & \downarrow & & & & \downarrow \\
K_1(M_k(C(X_{1,0}))) & \longrightarrow & K_1(M_k(C(X_{2,0}))) & \longrightarrow & \cdots & \longrightarrow & K_1(M_k(C(X_0))).
\end{array}$$

Since the diagram is commutative, if we show that each map

$$K_1(M_k(C(X_j))) \to K_1(M_k(C(X_{j,0})))$$

is surjective, where X_j is a simplicial complex and $X_{j,0}$ is a closed subset of X_j, then it will follow that $\psi_* \colon K_1(M_k(C(X))) \to K_1(M_k(C(X_0)))$ is surjective. Hence we can assume that X is a simplicial complex and X_0 is a closed subset of X. Let us now finish the proof of Proposition 7.

LEMMA 9. *The map ψ_* is surjective.*

PROOF. To show that ψ_* is surjective, we will take advantage of Lemma 4 by "inserting" a simplicial complex Y between X and X_0 in order to factor ψ_*, using a technique of Liangqing Li [6]. Let $[f]$ be an element of $K_1(M_k(C(X_0)))$ represented by a function $f\colon X_0 \to U_n(\mathbb{C})$. To show that ψ_* is surjective it is enough to show that there is a function from X to $U_n(\mathbb{C})$ such that its restriction to X_0 is f. Let us show the existence of such a function.

By Lemma 5, with $F = X_0$, there is an open set $\mathcal{O}$ of X such that $\mathcal{O}$ contains X_0 and there is a function $\widetilde{f}\colon \mathcal{O} \to U_n(\mathbb{C})$ such that $\widetilde{f}|_{X_0} = f$. Enlarge X_0 to a simplicial subcomplex Y of X such that $Y \subseteq \mathcal{O}$ as follows. Since both X_0 and $X \setminus \mathcal{O}$ are compact, $\operatorname{dist}(X_0, X \setminus \mathcal{O}) = \delta > 0$. Take a subdivision of X such that each simplex of X has diameter at most $\delta/2$. Let Y denote the subcomplex of X consisting of all simplices $\triangle$, satisfying $\triangle \cap X_0 \neq \varnothing$, together with their faces. It is clear that $X_0 \subseteq Y$ and $\operatorname{dist}(y, X_0) \leqslant \delta/2$ for any $y \in Y$. This implies $Y \subseteq \mathcal{O}$. By Lemma 4, $\tilde{f}|_Y$ can be extended to a function $\tilde{\tilde{f}}\colon X \to U_n(\mathbb{C})$ with $\tilde{\tilde{f}}|_Y = \tilde{f}$. Hence, $\tilde{\tilde{f}}|_{X_0} = f$. Therefore,

$$\psi_*([\tilde{\tilde{f}}]) = [f] \text{ for } [f] \in K_1(M_k(C(X)). \qquad \square$$

PROPOSITION 10. *Let R be a recursive subhomogeneous algebra with compact, metrizable spaces of dimension at most one as local spectra. Then $K_0(R)$ is torsion free.*

PROOF. By Lemma 6, if X is a compact Hausdorff space of dimension at most one, then the group $K_0(M_k(C(X)))$ is torsion free. Now assume that the statement holds for recursive subhomoganeous algebras of decomposition length $l-1$ and assume R has a decomposition of length l. Then

$$R = [\cdots[[C_0 \oplus_{C_1^0} C_1] \oplus_{C_2^0} C_2]\cdots] \oplus_{C_l^0} C_l.$$

Consider the $(l-1)$-th stage algebra and set

$$A = [\cdots[[C_0 \oplus_{C_1^0} C_1] \oplus_{C_2^0} C_2]\cdots] \oplus_{C_{l-1}^0} C_{l-1}.$$

Then $R = A \oplus_{C_{l-1}^0} C_l$, and we apply the inductive hypothesis and Proposition 7 to finish the proof. $\square$

Recall that the inductive limit of torsion free groups is torsion free. The next result follows from this fact and from Proposition 10.

THEOREM 11. *Let A be a C^*-algebra that is the inductive limit of a sequence $(A_n)_{n=1}^{+\infty}$ of C^*-algebras, where each A_n is a recursive subhomogeneous algebra with compact, metrizable spaces of dimension at most one as local spectra. Then $K_0(A)$ is torsion free.*

Since the class of recursive subhomogeneous algebras contains the class of homogeneous algebras, we get the analogous result for unital simple AH algebras.

Corollary 12. *Let A be a unital simple AH algebra which is the inductive limit of a sequence $(A_n)_{n=1}^{+\infty}$ of C^*-algebras, where each A_n is a recursive subhomogeneous algebra with compact, metrizable spaces of dimension at most one as local spectra. Then $K_0(A)$ is torsion free.*

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Department of Mathematics,
Río Piedras Campus,
University of Puerto Rico
Box 23355
San Juan, Puerto Rico 00931-3355,
U.S.A.
email: smolinacabrera@gmail.com