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MEYER'S FUNCTION AND THE WORD METRIC ON THE HYPERELLIPTIC MAPPING CLASS GROUP

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ABSTRACT. In this short note, we bound the value of Meyer's function of the hyperelliptic mapping class group Δ_g by a constant times the distance to the identity, measured in any word metric on Δ_g. We also construct an example which shows this bound is asymptotically sharp.

RÉSUMÉ. Dans cette note courte, nous avons borné la valeur de la fonction de Meyer de l'hyperelliptic qui dresse une carte de groupe de classe par un temps constant la distance à l'identité, mesuré dans tout mot métrique sur le groupe. Nous construisons aussi un exemple qui montre que ce lien est brusquement asymptotique.

1. **Introduction.** Meyer's function of the hyperelliptic mapping class group is a kind of secondary invariant associated with the signature cocycle [5]. It is related to several interesting invariants in algebraic or geometric context, including Hirzebruch's signature defect, the Atiyah–Patodi–Singer eta-invariant and its adiabatic limit, the special value of the Shimizu L-function, the von Neumann rho-invariant, the Casson invariant and so on (see [1], [8] for example).

The purpose of this note is to show an elementary property of Meyer's function $\phi\colon \Delta_g \to \mathbb{Q}$. To this end, we first review the notion of word norm of a group.

For a given finitely generated group G and a finite set S of generators for G, we denote by $\|\cdot\|$ the induced word norm on G. Namely, $\|x\|$ is the length of the shortest word in $S^{\pm 1}$ which equals x. Different choices of finite generating sets for G give word norms whose ratios are bounded by a constant.

THEOREM 1. *There exists a constant $C > 0$ so that $|\phi(x)| \leq C\|x\|$ for every $x \in \Delta_g$. This bound is sharp in the sense that there exists an infinite set $\{x_n\} \subset \Delta_g$ and a constant $K > 0$ so that $|\phi(x_n)| \geq K\|x_n\|$ for all n.*

Our motivation comes from a relation between Meyer's function and the Casson invariant of integral homology 3-spheres [6], and a recent result due to Broaddus–Farb–Putman [3] which gives a bound of the value of the Casson

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invariant by a constant times the distance-squared to the identity, measured in any word metric on the Torelli group.

2. Signature cocycle and Meyer's function. Let Σ_g be a closed oriented smooth surface of genus g. The mapping class group $\mathcal{M}_g$ is defined to be the group of path components of $\mathrm{Diff}_+ \Sigma_g$, which is the group of orientation preserving diffeomorphisms of Σ_g equipped with the C^∞-topology.

For $x, y \in \mathcal{M}_g$, let $E_{x,y}$ be the oriented Σ_g-bundle over the thrice-punctured 2-sphere with monodromies x, y and xy on the boundary components so that $\partial E_{x,y} = M_x \cup M_y \cup -M_{xy}$, where $M_x = \Sigma_g \times I/(p, 0) \sim \big(x(p), 1\big)$. We then define $\tau(x, y)$ to be the signature $\sigma(E_{x,y})$ of the 4-manifold $E_{x,y}$. From the Novikov additivity, it gives a group 2-cocycle of $\mathcal{M}_g$, and is called the signature cocycle. As shown in [5], τ can be calculated by using just homological information. In particular, the range of $|\tau|$ is bounded by $2g$.

We fix an involution $\iota \in \mathcal{M}_g$ with $2g + 2$ fixed points. The centralizer $\Delta_g = \{x \in \mathcal{M}_g \mid x\iota = \iota x\}$ is called the hyperelliptic mapping class group of Σ_g with respect to ι. The conjugacy class of Δ_g in $\mathcal{M}_g$ does not depend on a choice of ι. It is known that $\Delta_g = \mathcal{M}_g$ if $g = 1, 2$ and $\Delta_g \neq \mathcal{M}_g$ if $g \geq 3$.

Since $H^*(\Delta_g, \mathbb{Q}) = 0$ for $* = 1, 2$ (see [4]), there exists a uniquely defined map

$$\phi\colon \Delta_g \to \mathbb{Q}$$

such that $\delta\phi = \tau$, which is called Meyer's function of the hyperelliptic mapping class group. See [8] for a through exposition of Meyer's function.

3. Proof of Theorem 1. Let S be a finite generating set of Δ_g (see [2]) and write $x = s_1 \cdots s_n$ ($s_i \in S^{\pm 1}$) for a given element $x \in \Delta_g$, where $\|x\| = n$. The relation $\delta\phi = \tau$ on Δ_g implies

$$\phi(x) = \phi(s_1 \cdots s_n) = \sum_{i=1}^{n} \phi(s_i) - \sum_{i=1}^{n-1} \tau(s_i, s_{i+1} \cdots s_n).$$

Since S is finite, there exists a constant $C_1 > 0$ so that $|\phi(s)| \leq C_1$ for every $s \in S^{\pm 1}$. Thus we have

$$|\phi(x)| \leq \sum_{i=1}^{n} |\phi(s_i)| + \sum_{i=1}^{n-1} |\tau(s_i, s_{i+1} \cdots s_n)| \leq C_1 n + 2g(n-1) \leq (2g + C_1)\|x\|.$$

If we put $C = 2g + C_1 > 0$, we obtain the first assertion.

We now consider the latter claim. Let $\{\zeta_1, \ldots, \zeta_{2g+1}\}$ be the finite generating set of Δ_g due to Birman and Hilden, and put $y = (\zeta_1 \cdots \zeta_{2h})^{4h+2}$ for $1 \leq h \leq g - 1$. This is a BSCC-map of genus h, a Dehn twist along a bounding simple closed curve on Σ_g. In particular, y acts trivially on $H_1(\Sigma_g, \mathbb{Z})$, so that

$\tau(y, y^i) = 0$ for any i. It is also known that $\phi(y) = -4h(g-h)/(2g+1)$ (see [6, Example 2.6]).

If $\|y\| = K_1$, then clearly $\|y^n\| \leq K_1 n$. We then obtain

$$|\phi(y^n)| = \Big|n\phi(y) - \sum_{i=1}^{n-1} \tau(y, y^i)\Big| = \Big|-\frac{4}{2g+1}h(g-h)n\Big| \geq \frac{4h(g-h)}{(2g+1)K_1}\|y^n\|.$$

If we put $K = \frac{4h(g-h)}{(2g+1)K_1} > 0$, we get the desired infinite set $\{y^n \mid n \geq 1\} \subset \Delta_g$. Because y has infinite order in Δ_g (in fact, y is an element of the Torelli group).

Remark 2. The first assertion of Theorem 1 holds for any quasi-homomorphism (namely, its deviation from being a homomorphism is bounded) of a given finitely generated group.

Remark 3. Let $\Phi\colon \Delta_g \to \mathbb{R}$ be the homogenized quasi-homomorphism defined by $\Phi(x) = \lim_{n\to\infty} \frac{\phi(x^n)}{n}$. Proposition 2.1 in [7] shows that the difference $\Phi - \phi\colon \Delta_g \to \mathbb{R}$ is described in terms of the von Neumann rho-invariant $\rho^{(2)}(\hat{M}_x)$ for the $\mathbb{Z}$-covering $\mathbb{Z} \to \hat{M}_x \to M_x$ associated with a surjective homomorphism $\pi_1 M_x \to \pi_1 S^1 \cong \mathbb{Z}$. Hence the value of the von Neumann rho-invariant is also bounded by a constant times the distance to the identity.

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