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ON ALGEBRAS OF HOLOMORPHIC FUNCTIONS WITH SEMI-ALMOST PERIODIC BOUNDARY VALUES

On the occasion of the 35th anniversary of the Yu. Brudnyi theory of functions seminar at Yaroslavl University and the 2nd anniversary of the P. Milman student seminar at the University of Toronto

ALEXANDER BRUDNYI AND DAMIR KINZEBULATOV

Presented by Pierre Milman, FRSC

Abstract. We study the algebras of bounded holomorphic functions on the unit disk whose boundary values, having, in a sense, the weakest possible discontinuities, belong to the algebra of semi-almost periodic functions on the unit circle. The latter algebra contains as a special case an algebra introduced by Sarason in connection with some problems in the theory of Toeplitz operators. We show that such algebras have the Grothendieck approximation property, prove the corona theorem for them and formulate some results on the structure of their maximal ideal spaces. Also, we extend the notion of the Bohr–Fourier spectrum to holomorphic semi-almost periodic functions and prove that under certain assumptions on their spectra the corresponding algebras are projective free and their maximal ideal spaces have trivial Čech cohomology groups.

Résumé. On étudie les algèbres des fonctions holomorphes bornées sur le disque unité dont les valeurs au bord ayant, dans us certain sens, des discontinuités les plus faibles possible, appartiennent á l'algèbre de fonctions semi-presque périodique sur le circle unité. Cette dernière contient, en particulier, une algèbre introduite par Sarason en relation avec certains problèmes de la théorie des opérateurs de Toeplitz. On montre que ces algèbres ont la propriéte d'approximation de Grothendieck; on prouve le theorème corona pour celles-ci et on formule quelques résultats sur la structure de leurs espaces idéaux maximaux. On étend aussi la notion du spectre de Bohr-Fourier à des fonctions holomorphiques semi-presques périodiques et on prouve que sous certaines hypothèses sur leur spectres, tout module projectif des algèbres correspondants est libre et leurs espaces idéaux maximaux ont des cohomologies triviales de Čech.

1. Introduction

In this paper we study Banach algebras of *holomorphic semi-almost periodic functions*, i.e., bounded holomorphic functions on the unit

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disk $\mathbb{D} \subset \mathbb{C}$ whose boundary values belong to the algebra $\mathrm{SAP}(\partial\mathbb{D}) \subset L^\infty(\partial\mathbb{D})$ of *semi-almost periodic functions* on the unit circle $\partial\mathbb{D}$. The latter algebra contains as a special case an algebra introduced by Sarason [15] in connection with some problems in the theory of Toeplitz operators. Our primary interest in holomorphic semi-almost periodic functions is motivated by the problem of description of the weakest possible boundary discontinuities of functions in $H^\infty(\mathbb{D})$, the Hardy space of bounded holomorphic functions on $\mathbb{D}$. (Recall that a function $f \in H^\infty(\mathbb{D})$ has radial limits almost everywhere on $\partial\mathbb{D}$, the limit function $f|_{\partial\mathbb{D}} \in L^\infty(\partial\mathbb{D})$, and f can be recovered from $f|_{\partial\mathbb{D}}$ by means of the Cauchy integral formula.) In the most general form this problem is as follows.

> Given a continuous function $\Phi\colon \mathbb{C} \to \mathbb{C}$ to describe the minimal Banach subalgebra $H^\infty_\Phi(\mathbb{D}) \subset H^\infty(\mathbb{D})$ containing all invertible elements $f \in H^\infty(\mathbb{D})$ such that $\Phi(f)|_{\partial\mathbb{D}}$ is piecewise Lipschitz having finitely many first-kind discontinuities.

Clearly, each $H^\infty_\Phi(\mathbb{D})$ contains the *disk-algebra* $A(\mathbb{D})$ (i.e., the algebra of holomorphic functions continuous up to the boundary). Moreover, if $\Phi(z) = z$, $\mathrm{Re}(z)$, or $\mathrm{Im}(z)$, then the Lindelöf theorem (see *e.g.,* [9]) implies that $H^\infty_\Phi(\mathbb{D}) = A(\mathbb{D})$. In contrast, if Φ is constant on a closed simple curve which does not encompass $0 \in \mathbb{C}$, then $H^\infty_\Phi(\mathbb{D}) = H^\infty(\mathbb{D})$. (This result is obtained by consequent applications of the Carathéodory conformal mapping theorem, the Mergelyan theorem, and the Marshall theorem; see [9].) In [4] we studied the case of $\Phi(z) = |z|$ and showed that $H^\infty_\Phi(\mathbb{D})$ coincides with the algebra of holomorphic semi-almost periodic functions $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$. In the present paper we continue the investigation. Despite the fact that our results concern the particular choice of $\Phi(z) = |z|$, the methods developed here and in [4] can be applied further to a more general class of functions Φ.

Let $b\mathbb{D}$ be the maximal ideal space of the algebra $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$, i.e., the set of all nonzero homomorphisms $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D}) \to \mathbb{C}$ equipped with the *Gelfand topology.* The disk $\mathbb{D}$ is naturally embedded into $b\mathbb{D}$. In [4] we proved that $\mathbb{D}$ is dense in $b\mathbb{D}$ (the so-called *corona theorem* for $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$). We also described the topological structure of $b\mathbb{D}$. In the present paper we refine and extend some of these results. In particular, we introduce Bohr–Fourier coefficients and spectra of functions from $\mathrm{SAP}(\partial\mathbb{D})$, describe Čech cohomology groups of $b\mathbb{D}$, and establish projective freeness of certain subalgebras of $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$. Recall that a commutative ring R with identity is called *projective free* if every finitely generated projective R-module is free. Equivalently, R is projective free if and only if every square idempotent matrix F with entries in R (i.e., such that $F^2 = F$) is conjugate over R to a matrix of the form

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

where I_k stands for the $k \times k$ identity matrix. Every field $\mathbb{F}$ is trivially projective-free. Quillen and Suslin proved that if R is projective free, then the rings of polynomials $R[x]$ and formal power series $R[[x]]$ over R are projective free as well (see [12]). Grauert proved that the ring $\mathcal{O}(\mathbb{D}^n)$ of holomorphic functions on the unit polydisk $\mathbb{D}^n$ is projective free [10]. It was shown in [6] that the triviality of any complex vector bundle of finite rank over the connected maximal ideal space of a unital semi-simple commutative complex Banach algebra is sufficient for its projective freeness. We employ this result to show that the subalgebras of $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$ whose elements have their spectra in non-negative or non-positive semi-groups are projective free. Note that if a unital semi-simple commutative complex Banach algebra A is projective free, then it is *Hermite*, i.e., every finitely generated stably free A-module is free. Equivalently, A is Hermite if and only if any $k \times n$ matrix, $k < n$, with entries in A having rank k at each point of the maximal ideal space of A can be extended to an invertible $n \times n$ matrix with entries in A, see [7]. (Here the values of elements of A at points of the maximal ideal space are defined by means of the Gelfand transform.)

Finally, we prove that $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$ has the *approximation property*, i.e., each compact linear operator on $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$ can be approximated in operator norm by linear operators of finite rank. (This result strengthens the approximation theorem of [4].) Although it is strongly believed that the class of spaces with the approximation property includes practically all spaces which appear naturally in analysis, it is not known yet even for the space $H^\infty(\mathbb{D})$ (see [2] for some results in this direction). The first example of a space which fails to have the approximation property was constructed by Enflo [8]. Since Enflo's work, several other examples of such spaces were constructed (for the references see [14]). Many problems of Banach space theory admit especially simple solutions if one of the spaces under consideration has the approximation property. One of such problems is the problem of determination whether given two Banach algebras $A \subset C(X)$, $B \subset C(Y)$ (X and Y are compact Hausdorff spaces) their *slice algebra*

$$S(A,B) := \{f \in C(X \times Y) : f(\cdot, y) \in A \text{ for all } y \in Y, f(x, \cdot) \in B \text{ for all } x \in X\}$$

coincides with $A \otimes B$, the closure in $C(X \times Y)$ of the symmetric tensor product of A and B. For instance, this is true if either A or B have the approximation property. The latter is an immediate consequence of Grothendieck theorem (see Theorem 3.4).

The paper is organized as follows. Section 2 is devoted to the algebra of semi-almost periodic functions $\mathrm{SAP}(\partial\mathbb{D})$. In Section 3 we formulate our main results on the algebra of holomorphic semi-almost periodic functions $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$. The detailed proofs of all statements are contained in [5].

2. Semi-almost periodic functions

We first recall the notion of a Bohr almost periodic function on $\mathbb{R}$. In what follows, by $C_b(\mathbb{R})$ we denote the algebra of bounded continuous functions on $\mathbb{R}$ endowed with sup-norm.

DEFINITION 2.1 ([1]). A function $f \in C_b(\mathbb{R})$ is said to be *almost periodic* if the family of its translations $\{S_\tau f\}_{\tau\in\mathbb{R}}$, $S_\tau f(x):=f(x+\tau)$, $x\in\mathbb{R}$, is relatively compact in $C_b(\mathbb{R})$.

Let $\mathrm{AP}(\mathbb{R})$ be the Banach algebra of almost periodic functions endowed with sup-norm. The basic characteristics of an almost periodic function $f\in\mathrm{AP}(\mathbb{R})$ are its Bohr–Fourier coefficients $a_\lambda(f)$ and the spectrum $\mathrm{spec}(f)$, defined in terms of the mean value

$$M(f):=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}f(x)dx\in\mathbb{C}.$$

Specifically,

$$a_\lambda(f):=M(fe^{-i\lambda x}),\quad \lambda\in\mathbb{R}.$$

Then $a_\lambda(f)\neq 0$ for at most countably many values of λ; these values form the spectrum $\mathrm{spec}(f)$ of f. In particular, if $f=\sum_{l=1}^{\infty}c_le^{i\lambda_lx}$ ($c_l\neq 0$, $\sum_{l=1}^{\infty}|c_l|<\infty$), then $\mathrm{spec}(f)=\{\lambda_1,\lambda_2,\dots\}$.

One of the main results of the theory of almost periodic functions states that each function $f\in\mathrm{AP}(\mathbb{R})$ can be uniformly approximated by functions of the form $\sum_{l=1}^{m}c_le^{i\lambda_lx}$ with $\lambda_l\in\mathrm{spec}(f)$.

Let $\Gamma\subset\mathbb{R}$ be a unital additive semi-group (i.e., $0\in\Gamma$). It follows easily from the above approximation result that the space $\mathrm{AP}_\Gamma(\mathbb{R})$ of almost periodic functions with spectra in Γ forms a unital Banach subalgebra of $\mathrm{AP}(\mathbb{R})$.

Next, we recall the definition of a semi-almost periodic function on $\partial\mathbb{D}$ introduced in [4]. In what follows, we consider $\partial\mathbb{D}$ with the counterclockwise orientation. For $t_0\in[0,2\pi)$ let

$$\gamma_{t_0}^k(s):=\{e^{i(t_0+kt)}:\ 0\le t<s<2\pi\},\quad k\in\{-1,1\}$$

be two open arcs having e^{it_0} as the right and the left endpoints (with respect to the orientation), respectively.

Definition 2.2 ([4]). A function $f\in L^\infty(\partial\mathbb{D})$ is called *semi-almost periodic* on $\partial\mathbb{D}$ if for any $t_0\in[0,2\pi)$ and any $\varepsilon>0$ there exist a number $s=s(t_0,\varepsilon)\in(0,\pi)$ and functions $f_k\colon\gamma_{t_0}^k(s)\to\mathbb{C}$, $k\in\{-1,1\}$, such that functions

$$t\mapsto f_k\big(e^{i(t_0+kse^t)}\big),\quad -\infty<t<0,\quad k\in\{-1,1\},$$

are restrictions of some almost periodic functions from $\mathrm{AP}(\mathbb{R})$, and

$$\sup_{z\in\gamma_{t_0}^k(s)}|f(z)-f_k(z)|<\varepsilon,\quad k\in\{-1,1\}.$$

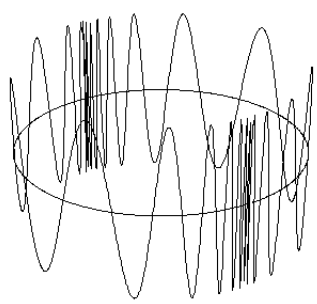
By $\mathrm{SAP}(\partial\mathbb{D})$ we denote the Banach algebra of semi-almost periodic functions on $\partial\mathbb{D}$ endowed with sup-norm. It is easy to see that the set of points of discontinuity of a function in $\mathrm{SAP}(\partial\mathbb{D})$ is at most countable. For S being a closed subset of $\partial\mathbb{D}$ we denote by $\mathrm{SAP}(S)$ the Banach algebra of semi-almost periodic functions on $\partial\mathbb{D}$ that are continuous on $\partial\mathbb{D}\setminus S$. (Note that Sarason's algebra introduced in [15] is isomorphic to $\mathrm{SAP}(\{z_0\})$, $z_0\in\partial\mathbb{D}$.)

Example 2.3 ([4]). A function g defined on $\mathbb{R}\sqcup(\mathbb{R}+i\pi)$ is said to belong to the space $\mathrm{AP}(\mathbb{R}\sqcup(\mathbb{R}+i\pi))$ if the functions $g(x)$ and $g(x+i\pi)$, $x\in\mathbb{R}$, belong to $\mathrm{AP}(\mathbb{R})$. The space $\mathrm{AP}(\mathbb{R}\sqcup(\mathbb{R}+i\pi))$ is a function algebra (with respect to sup-norm).

Given $s_0\in\partial\mathbb{D}$ consider the map $\varphi_{s_0}\colon\partial\mathbb{D}\setminus\{-s_0\}\to\mathbb{R}$, $\varphi_{s_0}(s):=\frac{2i(s_0-s)}{s_0+s}$, and define a linear isometric embedding $L_{s_0}\colon\ \mathrm{AP}(\mathbb{R}\sqcup(\mathbb{R}+i\pi))\to L^\infty(\partial\mathbb{D})$ by the formula

$$(L_{s_0}g)(s):=(g\circ\mathrm{Log}\circ\varphi_{s_0})(s),$$

where $\mathrm{Log}(z):=\ln|z|+i\,\mathrm{Arg}(z)$, $z\in\mathbb{C}\setminus\mathbb{R}_-$, and $\mathrm{Arg}\colon\mathbb{C}\setminus\mathbb{R}_-\to(-\pi,\pi)$ stands for the principal branch of the multi-function arg. Then the range of L_{s_0} is a subspace of $\mathrm{SAP}(\{-s_0,s_0\})$ and the graph of each $L_{s_0}g$ (for a real-valued g) has the following form



Proposition 2.4. *For every $s_0\in\partial\mathbb{D}$ there exists a homomorphism of Banach algebras $E_{s_0}:\mathrm{SAP}(\partial\mathbb{D})\to\mathrm{AP}(\mathbb{R}\sqcup(\mathbb{R}+i\pi))$ of norm 1 such that for each $f\in\mathrm{SAP}(\partial\mathbb{D})$ the function $f-L_{s_0}(E_{s_0}f)\in\mathrm{SAP}(\partial\mathbb{D})$ is continuous and equal to 0 at s_0. Moreover, any bounded linear operator $\mathrm{SAP}(\partial\mathbb{D})\to\mathrm{AP}(\mathbb{R}\sqcup(\mathbb{R}+i\pi))$ satisfying this property coincides with E_{s_0}.*

The functions $f_{-1,s_0}(x):=(E_{s_0}f)(x)$ and $f_{1,s_0}(x):=(E_{s_0}f)(x+i\pi)$, $x\in\mathbb{R}$, are used to define the left ($k=-1$) and the right ($k=1$) mean values $M^k_{s_0}(f)$ of a function $f\in\mathrm{SAP}(\partial\mathbb{D})$ over s_0 (cf. Remark 2.8 below). Precisely, we put

$$M^k_{s_0}(f):=M(f_{k,s_0}).$$

Proposition 2.5. *The mean value $M^k_{s_0}$ is a continuous linear functional $\mathrm{SAP}(\partial\mathbb{D})\to\mathbb{C}$.*

Similarly, we define the left ($k=-1$) and the right ($k=1$) Bohr–Fourier coefficients of f over s_0 by the formulas $a_\lambda^k(f,s_0):=a_\lambda(f_{k,s_0})$. Those values of λ for which $a_\lambda^k(f,s_0)\neq 0$ form the left ($k=-1$) and the right ($k=1$) spectrum $\mathrm{spec}_{s_0}^k(f)$ of f over s_0. Moreover, $\mathrm{spec}_{s_0}^k(f)$ is at most countable.

Let $\Sigma : S\times\{-1,1\}\to 2^{\mathbb{R}}$ be a set-valued map which associates with each $s\in S$, $k\in\{-1,1\}$ a unital semi-group $\Sigma(s,k)\subset\mathbb{R}$. By $\mathrm{SAP}_\Sigma(S)\subset\mathrm{SAP}(S)$ we denote the Banach algebra of semi-almost periodic functions f with $\mathrm{spec}_s^k(f)\subset\Sigma(s,k)$ for all $s\in S$, $k\in\{-1,1\}$.

Theorem 2.6.

(i) *A function $f\in\mathrm{SAP}_\Sigma(S)$ if and only if for each $s\in S$ and $k\in\{-1,1\}$ the almost periodic functions f_k in Definition 2.2 can be chosen from* $\mathrm{AP}_{\Sigma(s,k)}(\mathbb{R})$.

(ii) *The "total spectrum" $\bigcup_{s\in\partial\mathbb{D},\,k=\pm1}\mathrm{spec}_s^k(f)$ of a function $f\in\mathrm{SAP}(\partial\mathbb{D})$ is at most countable.*

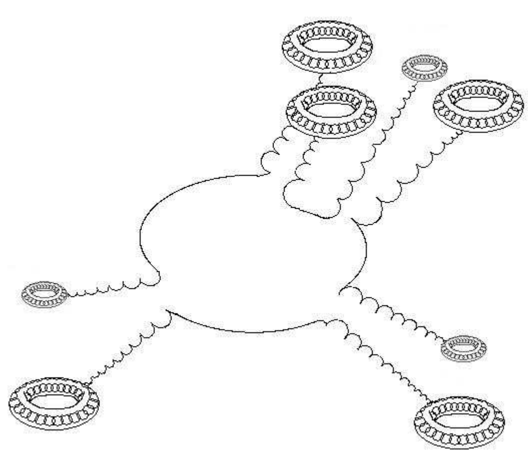
For a unital semi-group $\Gamma\subset\mathbb{R}$ by $b_\Gamma(\mathbb{R})$ we denote the maximal ideal space of algebra $\mathrm{AP}_\Gamma(\mathbb{R})$. (E.g., for $\Gamma=\mathbb{R}$ the space $b\mathbb{R}:=b_{\mathbb{R}}(\mathbb{R})$, commonly called the *Bohr compactification* of $\mathbb{R}$, is a compact abelian topological group viewed as the inverse limit of compact finite-dimensional tori. The group $\mathbb{R}$ admits a canonical embedding into $b\mathbb{R}$ as a dense subgroup.)

Let $b_\Sigma^S(\partial\mathbb{D})$ be the maximal ideal space of algebra $\mathrm{SAP}_\Sigma(S)$ and $r_\Sigma^S\colon b_\Sigma^S(\partial\mathbb{D})\to\partial\mathbb{D}$ be the mapping transpose to the embedding $C(\partial\mathbb{D})\hookrightarrow\mathrm{SAP}_\Sigma(S)$.

Theorem 2.7.

(i) *The map transpose to the restriction of homomorphism E_{s_0} to $\mathrm{SAP}_\Sigma(S)$ determines an embedding $h_\Sigma^s : b_{\Sigma(s,-1)}(\mathbb{R})\sqcup b_{\Sigma(s,1)}(\mathbb{R})\hookrightarrow b_\Sigma^S(\partial\mathbb{D})$ whose image coincides with $(r_\Sigma^S)^{-1}(s)$.*

(ii) *The restriction $r_\Sigma^S : b_\Sigma^S(\partial\mathbb{D})\setminus(r_\Sigma^S)^{-1}(S)\to\partial\mathbb{D}\setminus S$ is a homeomorphism.*



(For an m point set S and each $\Sigma(s,k)$, $s\in S$, $k\in\{-1,1\}$, being a group, the maximal ideal space $b_\Sigma^S(\partial\mathbb{D})$ is the union of $\partial\mathbb{D}\setminus S$ and $2m$ Bohr compactifications $b_{\Sigma(s,k)}(\mathbb{R})$ that can be viewed as (finite or infinite dimensional) tori.)

Remark 2.8. There is an equivalent way to define the mean value of a semi-almost periodic function. Specifically, for a semi-almost periodic function $f\in\mathrm{SAP}(\partial\mathbb{D})$, $k\in\{-1,1\}$, and a point $s_0\in S$ we define the left ($k=-1$) and the right ($k=1$) mean values of f over s_0 by

$$M_{s_0}^k(f):=\lim_{n\to\infty}\frac{1}{b_n-a_n}\int_{a_n}^{b_n} f\left(s_0\cdot e^{ike^t}\right)dt,$$

where $\{a_n\}$, $\{b_n\}$ are arbitrary sequences converging to $-\infty$ such that $\lim_{n\to\infty}$ $(b_n-a_n)=+\infty$.

The Bohr–Fourier coefficients of f over s_0 can be then defined by the formulas

$$a_\lambda^k(f,s_0):=M_{s_0}^k(fe^{-i\lambda\log_{t_0}^k}),$$

where $s_0=e^{it_0}$ and

$$\log_{t_0^k}(e^{i(t_0+kt)}):=\ln t,\quad 0<t<2\pi,\quad k\in\{-1,1\}.$$

3. **Holomorphic semi-almost periodic functions.** Let $C_b(T)$ denote the space of bounded continuous functions on the strip $T:=\{z\in\mathbb{C}\ :\ \mathrm{Im}(z)\in[0,\pi]\}$ endowed with sup-norm.

Definition 3.1 ([1]). A function $f\in C_b(T)$ is called *holomorphic almost periodic* on the strip T if it is holomorphic in the interior of T and the family of translates $\{S_xf\}_{x\in\mathbb{R}}$, $S_xf(z):=f(z+x)$, is relatively compact in $C_b(T)$.

We denote by $\mathrm{APH}(T)$ the Banach algebra of holomorphic almost periodic functions endowed with sup-norms. The mean value of a function $f\in\mathrm{APH}(T)$ is defined by the formula

$$M(f):=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} f(x+iy)dx\in\mathbb{C}$$

($M(f)$ does not depend on y, see [1]). Further, the Bohr–Fourier coefficients of f are defined by $a_\lambda(f):=M(fe^{-i\lambda z}),\lambda\in\mathbb{R}$. Then $a_\lambda(f)\neq 0$ for at most countably many values of λ and these values form the spectrum $\mathrm{spec}(f)$ of f. For instance, if $f=\sum_{l=1}^\infty c_le^{i\lambda_lz}$ ($c_l\neq 0$, $\sum_{l=1}^\infty|c_l|<\infty$), then $\mathrm{spec}(f)=\{\lambda_1,\lambda_2,\dots\}$. Similarly to the case of functions from $\mathrm{AP}(\mathbb{R})$ each $f\in\mathrm{APH}(T)$ can be uniformly approximated by functions of the form $\sum_{l=1}^m c_le^{i\lambda_lz}$ with $\lambda_l\in\mathrm{spec}(f)$.

Let $\Gamma \subset \mathbb{R}$ be a unital additive semi-group. The space $\mathrm{APH}_\Gamma(T)$ of holomorphic almost periodic functions with spectra in Γ forms a unital Banach algebra.

The functions in $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ are called *holomorphic semi-almost periodic.*

Example 3.2 ([4]). Let $f \in \mathrm{APH}(T)$, $s_0 \in \partial\mathbb{D}$. Consider the map

$$\varphi_{s_0} : \bar{\mathbb{D}} \setminus \{-s_0\} \to \bar{\mathbb{H}}_+, \quad \varphi_{s_0}(z) := \frac{2i(s_0 - z)}{s_0 + z}.$$

Here $\mathbb{H}^+$ is the upper half-plane. Then φ_{s_0} maps $\mathbb{D}$ conformally onto $\mathbb{H}_+$ and $\partial\mathbb{D} \setminus \{-s_0\}$ diffeomorphically onto $\mathbb{R}$ (the boundary of $\mathbb{H}_+$) so that $\varphi_{s_0}(s_0) = 0$.

Let T_0 be the interior of the strip T. Consider the conformal map

$$\mathrm{Log}\colon \mathbb{H}_+ \to T_0, z \mapsto \mathrm{Log}(z) := \ln|z| + i\,\mathrm{Arg}(z),$$

where $\mathrm{Arg}\colon \mathbb{C} \setminus \mathbb{R}_- \to (-\pi, \pi)$ is the principal branch of the multi-function arg. The function Log is extended to a homeomorphism of $\overline{\mathbb{H}}_+ \setminus \{0\}$ onto T. Let $g \in \mathrm{APH}(T)$. Then the function

$$(L_{s_0} g)(z) := (g \circ \mathrm{Log} \circ \varphi_{s_0})(z), \quad z \in \mathbb{D},$$

belongs to $\mathrm{SAP}(\{-s_0, s_0\}) \cap H^\infty(\mathbb{D})$.

Proposition 3.3. *Suppose that $f \in \mathrm{SAP}(S)$ is a holomorphic semi-almost periodic function. Then $\mathrm{spec}_s^{-1}(f) = \mathrm{spec}_s^1(f) =: \mathrm{spec}_s(f)$ and, moreover,*

$$a_\lambda^{-1}(f, s) = e^{\lambda\pi} a_\lambda^1(f, s) \quad \textit{for each} \quad \lambda \in \mathrm{spec}_s(f).$$

(Recall that the choice of the upper indices ± 1 is determined by the orientation of $\partial\mathbb{D}$.)

Proposition 3.3 implies that $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D}) = \mathrm{SAP}_{\Sigma'}(S) \cap H^\infty(\mathbb{D})$, where

$$\Sigma'(s, k) := \Sigma(s, -1) \cap \Sigma(s, 1) \quad \text{for } k = -1, 1.$$

In what follows we assume that

$$\Sigma(s, -1) = \Sigma(s, 1) =: \Sigma(s).$$

3.1. Analytical structure of $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ Recall that a Banach space B is said to have the *approximation property* if for every compact set $K \subset B$ and every $\varepsilon > 0$ there is an operator $T\colon B \to B$ of finite rank so that $\|Tx - x\|_B < \varepsilon$ for every $x \in K$.

Let $A \subset C(X)$ be a closed subspace, B be a Banach space and $A_B \subset C_B(X) := C(X, B)$ be the Banach space of all continuous B-valued functions

f such that $\varphi(f) \in A$ for any $\varphi \in B^*$. By $A \otimes B$ we denote completion of symmetric tensor product of A and B with respect to norm

$$\Big\|\sum_{k=1}^{m} a_k \otimes b_k\Big\| := \sup_{x\in X}\Big\|\sum_{k=1}^{m} a_k(x) b_k\Big\|_B \quad \text{with } a_k \in A,\ b_k \in B.$$

THEOREM 3.4 ([11]). *The following statements are equivalent:*

(i) *A has the approximation property;*
(ii) *$A \otimes B = A_B$ for every Banach space B.*

THEOREM 3.5. *$\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ has the approximation property.*

Our proof of Theorem 3.5 is based on the equivalence established in Theorem 3.4 and on an approximation result for Banach-valued analogues of algebra $\mathrm{SAP}(S) \cap H^\infty(\mathbb{D})$ formulated below. Specifically, for a Banach space B we define

$$\mathrm{SAP}^B_\Sigma(S) := \mathrm{SAP}_\Sigma(S) \otimes B.$$

Using the Poisson integral formula we can extend each function from $\mathrm{SAP}^B_\Sigma(S)$ to a bounded B-valued harmonic function on $\mathbb{D}$ having the same sup-norm. We identify $\mathrm{SAP}^B_\Sigma(S)$ with its harmonic extension. Let $H^\infty_B(\mathbb{D})$ be the Banach space of bounded B-valued holomorphic functions on $\mathbb{D}$ equipped with sup-norm. By $(\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D}))_B$ we denote the Banach space of all continuous B-valued functions f on the maximal ideal space $b^S(\mathbb{D})$ of algebra $\mathrm{SAP}(S) \cap H^\infty(\mathbb{D})$ such that $\varphi(f) \in \mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ for any $\varphi \in B^*$. In what follows we naturally identify $\mathbb{D}$ with a subset of $b^S(\mathbb{D})$.

PROPOSITION 3.6. *Let $f \in (\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D}))_B$. Then $f|_{\mathbb{D}} \in \mathrm{SAP}^B_\Sigma(S) \cap H^\infty_B(\mathbb{D})$.*

Let A^S_Σ be the closed subalgebra of $H^\infty(\mathbb{D})$ generated by the disk-algebra $A(\mathbb{D})$ and the functions of the form $ge^{\lambda h}$, where $\mathrm{Re}(h)|_{\partial\mathbb{D}}$ is the characteristic function of the closed arc going in the counterclockwise direction from the initial point at s to the endpoint at $-s$ such that $s \in S$, $\lambda/\pi \in \Sigma(s)$ and $g(z) := z + s$, $z \in \mathbb{D}$ (in particular, $ge^{\lambda h}$ has discontinuity at s only).

The next result combined with Proposition 3.6 and Theorem 3.4 implies Theorem 3.5.

THEOREM 3.7. *$\mathrm{SAP}^B_\Sigma(S) \cap H^\infty_B(\mathbb{D}) = A^S_\Sigma \otimes B$.*

As a corollary we obtain

COROLLARY 3.8. *$\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D}) = A^S_\Sigma$.*

We conclude this section with a result on the tangential behavior of functions from $\mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$.

THEOREM 3.9. *Let $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$ and $\{s_n\}_{n\in\mathbb{N}} \subset \partial\mathbb{D}$ converge to a point $s_0 \in \partial\mathbb{D}$. Assume that*

$$\lim_{n\to\infty} \frac{|z_n - s_n|}{|s_0 - s_n|} = 0.$$

Then for every $f \in \mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$, either the limits $\lim_{n\to\infty} f(z_n)$ and $\lim_{n\to\infty} f(s_n)$ do not exist or exist simultaneously and in the latter case they are equal.

REMARK 3.10. This result implies that the extension (by means of the Gelfand transform) of each $f \in \mathrm{SAP}(\partial\mathbb{D}) \cap H^\infty(\mathbb{D})$ to the maximal ideal space of $H^\infty(\mathbb{D})$ is constant on a nontrivial Gleason part containing a limit point of a sequence in $\mathbb{D}$ converging tangentially to $\partial\mathbb{D}$ (we refer to [9] for the corresponding definitions).

3.2. Topological structure of the maximal ideal space of $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ Let $b_\Sigma^S(\mathbb{D})$ denote the maximal ideal space of $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ and $a_\Sigma^S \colon b_\Sigma^S(\mathbb{D}) \mapsto \bar{\mathbb{D}}$ be the continuous surjective map transpose to the embedding

$$A(\mathbb{D}) \hookrightarrow \mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D}).$$

(Recall that the maximal ideal space of the disk-algebra $A(\mathbb{D})$ is homeomorphic to $\bar{\mathbb{D}}$.)

The following result is an immediate consequence of Corollary 3.8.

THEOREM 3.11. $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ *is generated by algebras* $\mathrm{SAP}_{\Sigma|_F}(F) \cap H^\infty(\mathbb{D})$ *for all possible finite subsets $F \subset S$.*

Here $\Sigma|_F$ stands for the restriction of the set-valued map Σ to F.

The inclusion

$$\mathrm{SAP}_{\Sigma|_{F_1}}(F_1) \cap H^\infty(\mathbb{D}) \subset \mathrm{SAP}_{\Sigma|_{F_2}}(F_2) \cap H^\infty(\mathbb{D}) \quad \text{if } F_1 \subset F_2$$

determines a continuous map of maximal ideal spaces $\omega_{F_1}^{F_2} \colon b_{\Sigma|_{F_2}}^{F_2}(\mathbb{D}) \to b_{\Sigma|_{F_1}}^{F_1}(\mathbb{D})$. The family $\{b_{\Sigma|_F}^F(\mathbb{D})\,;\,\omega\}_{F\subset S\,;\,\#F<\infty}$ forms the inverse limiting system. From Theorem 3.11 we obtain the following.

THEOREM 3.12. $b_\Sigma^S(\mathbb{D})$ *is the inverse limit of* $\{b_{\Sigma|_F}^F(\mathbb{D})\,;\,\omega\}_{F\subset S\,;\,\#F<\infty}$.

In what follows, by $b_\Gamma(T)$ we denote the maximal ideal space of algebra $\mathrm{APH}_\Gamma(T)$ and by $\iota_\Gamma \colon T \to b_\Gamma(T)$ the continuous map determined by evaluations at points of T.

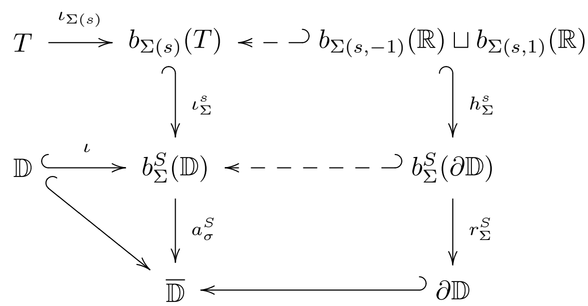
THEOREM 3.13.

(i) *For each* $s \in S$ *there exists an embedding* $i^s_\Sigma \colon b_{\Sigma(s)}(T) \hookrightarrow b^S_\Sigma(\mathbb{D})$ *whose image is* $(a^S_\Sigma)^{-1}(s)$ *such that the pullback* $(i^s_\Sigma)^*$ *maps* $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ *surjectively onto* $\mathrm{APH}_{\Sigma(s)}(T)$*. Moreover, the composition of the restriction map to* $\mathbb{R} \sqcup (\mathbb{R} + i\pi)$ *and* $(i^s_\Sigma \circ \iota_{\Sigma(s)})^*$ *coincides with the restriction of homomorphism* E_s *to* $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ *(see Proposition 2.4).*

(ii) *The restriction* $a^S_\Sigma \colon b^S_\Sigma(\mathbb{D}) \setminus (a^S_\Sigma)^{-1}(S) \to \bar{\mathbb{D}} \setminus S$ *is a homeomorphism.*

Since $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$ separates the points on $\mathbb{D}$, the evaluation at points of $\mathbb{D}$ determines a natural embedding $\iota \colon \mathbb{D} \hookrightarrow b^S_\Sigma(\mathbb{D})$.

One has the following commutative diagram of maximal ideal spaces considered in the present paper, where the "dashed" arrows stand for embeddings in the case $\Sigma(s,-1) = \Sigma(s,1)$ are groups for all $s \in S$, and for continuous maps otherwise.



THEOREM 3.14 (Corona Theorem). $\iota(\mathbb{D})$ *is dense (in the Gelfand topology) in* $b^S_\Sigma(\mathbb{D})$ *if and only if each* $\Sigma(s)$, $s \in S$, *is a group.*

Let K^S_Σ be the Šilov boundary of algebra $\mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$, that is, the minimal closed subset of $b^S_\Sigma(\mathbb{D})$ such that for every $f \in \mathrm{SAP}_\Sigma(S) \cap H^\infty(\mathbb{D})$

$$\sup_{z \in \mathbb{D}} |f(z)| = \max_{\varphi \in K^S_\Sigma} |f(\varphi)|,$$

where f is assumed to be extended to $b^S_\Sigma(\mathbb{D})$ by means of the Gelfand transform. For a non-trivial semigroup $\Gamma \subset \mathbb{R}$ by $\mathrm{cl}_\Gamma(\mathbb{R} + i\pi)$ and $\mathrm{cl}_\Gamma(\mathbb{R})$ we denote closures of $\iota_\Gamma(\mathbb{R} + i\pi)$ and $\iota_\Gamma(\mathbb{R})$ in $b_\Gamma(T)$. One can easily show that these closures are homeomorphic to $b_{\widehat{\Gamma}}(\mathbb{R})$, where $\widehat{\Gamma}$ is the minimal subgroup of $\mathbb{R}$ containing Γ.

THEOREM 3.15.

$$K^S_\Sigma = \Big(\bigcup_{s \in S} i^s_\Sigma \big(\mathrm{cl}_{\Sigma(s)}(\mathbb{R}) \cup \mathrm{cl}_{\Sigma(s)}(\mathbb{R} + i\pi) \big) \Big) \cup \partial\mathbb{D} \setminus S.$$

REMARK 3.16. If each $\Sigma(s)$, $s \in S$, is a group, then the Šilov boundary K^S_Σ is naturally homeomorphic to the maximal ideal space $b^S_\Sigma(\partial\mathbb{D})$ of algebra $\mathrm{SAP}_\Sigma(S)$, cf. Theorem 2.7.

Next, we formulate a result on the Čech cohomology groups of $b_\Sigma^S(\mathbb{D})$.

Theorem 3.17.
(i) *The Čech cohomology groups*

$$H^k\big(b_\Sigma^S(\mathbb{D}),\mathbb{Z}\big)\cong\bigoplus_{s\in S}H^k\big(b_{\Sigma(s)}(\mathbb{R}),\mathbb{Z}\big),\quad k\geq 1.$$

(ii) *Suppose that each $\Sigma(s)$ is a subset of $\mathbb{R}_+$ or $\mathbb{R}_-$. Then $H^k\big(b_\Sigma^S(\mathbb{D}),\mathbb{Z}\big)=0$, $k\geq 1$, and $\mathrm{SAP}_\Sigma(S)\cap H^\infty(\mathbb{D})$ is projective free.*

In the proof of part (ii) we use the main result of [3].

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Department of Mathematics and Statistics
University of Calgary
Calgary, AB T2N 1N4
email: albru@math.ucalgary.ca

Department of Mathematics
University of Toronto
Toronto, ON M5S 2E4
email: dkinz@math.toronto.edu