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A DE RHAM THEOREM FOR L^∞ FORMS AND HOMOLOGY ON SINGULAR SPACES

To the second anniversary of P. Milman's working seminar for his graduate students and postdocs

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Presented by Pierre Milman, FRSC

Abstract. We introduce a notion of a smooth L^∞ form on singular (semialgebraic) spaces X in $\mathbb{R}^n$. An L^∞ form is the data of a stratification Σ of X and a collection of smooth forms ω on the nonsingular strata with matching tangential components on the adjacent strata and bounded size (in the metric induced from $\mathbb{R}^n$). We prove Stokes' Theorem and Poincaré's Lemma for L^∞ forms. As a result we obtain a De Rham type theorem establishing a natural isomorphism between the singular cohomology and the cohomology of smooth L^∞ forms.

Résumé. On introduit la notion d'une forme L^∞ pour des espaces singuliers semialgébriques. Une forme lisse L^∞ est la donnée d'une stratification et d'une famille de forme lisses sur les strates coincidant le long des strates adjacentes. On prouve la formule de Stokes et le lemme de Poincaré pour les formes L^∞. On en déduit un théorème de type De Rham établissant un isomorphisme naturel entre la cohomologie des formes L^∞ et la cohomologie singulière.

1. **Introduction** In a recent paper J.-P. Brasselet and M. J. Pflaum [2] proved a De Rham type theorem for Whitney functions. Namely, they showed that the cohomology of the complex of Whitney differential forms naturally coincides with the singular cohomology of a subanalytic space.

In the present work we study the cohomology of the co-chain complex of the so-called smooth L^∞ forms (see Definition 2.7), which is intrinsically defined for the singular space in question.

The singular spaces considered in this paper are semialgebraic subsets of $\mathbb{R}^n$ but the results can be generalized to subanalytic sets. Let $X \subset \mathbb{R}^n$ be a semialgebraic set. A stratification of X is a collection of smooth (semialgebraic) locally closed manifolds (strata) in X such that the boundary of each stratum is a union of strata.

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We introduce a version of De Rham theory for L^∞ differential forms on X and prove the Stokes' formula and a De Rham type theorem for it. An L^∞ differential form on X is, roughly speaking, a data of a stratification Σ of X and a collection of smooth forms on the nonsingular strata such that the tangential components of the forms on the adjacent strata match and the size of the form is bounded (in the metric induced from $\mathbb{R}^n$). We consider two L^∞ forms to be equivalent if there exists a stratification of X on which the restrictions of the two forms coincide. The exterior derivative of such forms is defined by the exterior derivatives of their "components" on each stratum. The L^∞ forms endowed with the L^∞ exterior derivative form a co-chain complex. The main theorems of this paper is the Stokes Theorem and a De Rham type theorem (Theorem 2.14). Namely, we prove that the map

$$\omega \overset{\psi}{\mapsto} \text{“}c \mapsto \int_c \omega\text{”} \ ,$$

from the complex of L^∞ forms to the space of semialgebraic singular co-chains, is a natural map of chain complexes that induces an isomorphism on cohomology.

A detailed version of our results will appear in [9].

2. **The Main Results** All of the considered subsets and mappings will be assumed to be semialgebraic, except the differential forms which will be smooth on each stratum. Generalities on semialgebraic sets can be found in [1]. We begin by introducing the basic notations and definitions.

Notation 2.1. Let $X \subset \mathbb{R}^n$. Denote by $\overline{X}$ the closure of X and by $\partial X := \overline{X} - X$ the boundary of X.

Suppose that $f\colon X \to Y$ is a map of topological spaces. We write $f(x) \to y$ as $x \to a$ to denote $\lim_{x\to a} f(x) = y$.

Definitions 2.2. A stratified space (X, Σ) is a set $X \subset \mathbb{R}^n$ together with a partition (stratification) Σ of X into locally closed orientable manifolds (strata) such that the boundary of each stratum is a union of strata in Σ. If S and S' are two strata in Σ such that $S' \subset \partial S$, then we write $S' \le S$. A *refinement* of Σ is a stratification Σ' such that each stratum of Σ is a union of strata of Σ'; we write $\Sigma' \prec \Sigma$.

A *tangent bundle* of (X, Σ) is $TX := \bigcup_{x\in X}\{x\} \times T_xX \subset \mathbb{R}^n \times \mathbb{R}^n$ with the subspace topology, where $T_xX := T_xS$, $x \in S \in \Sigma$. Similarly, define the exterior product of the tangent bundle by

$$\wedge^k TX := \bigcup_{x\in X}\{x\} \times \wedge^k T_xX \subset \wedge^k(T\mathbb{R}^n).$$

A *stratified k-form* on X is a pair (ω, Σ), where Σ is a stratification of X and $\omega = (\omega_S)_{S\in\Sigma}$, where ω_S is a continuous differential k-form on $S \in \Sigma$, such that the graph of $\omega\colon \wedge^k TX \to \mathbb{R}$, $(x, \xi) \mapsto \omega(x; \xi)$ is closed in $\wedge^k TX \times \mathbb{R}$.

We say that such form ω is smooth if ω_S is smooth for every S. The *exterior derivative* of a smooth stratified form (ω, Σ) is then defined as $(d\omega_S)_{S\in\Sigma}$ and denoted by $(d\omega, \Sigma)$.

The *weak exterior derivative* of a stratified form (ω, Σ) is defined as $(\overline{d}\omega_S)_{S\in\Sigma}$ (whenever exists) and denoted by $(\overline{d}\omega, \Sigma)$. Let us recall that a k-form ω_S on a smooth orientable manifold S is *weakly differentiable* if there exists a $(k+1)$-form ω' on S such that for any smooth $(n-k-1)$-form φ on S, with compact support, we have

$$\int_S \omega' \wedge \varphi = (-1)^{k+1} \int_S \omega \wedge d\varphi.$$

We say that ω' is the *weak exterior derivative of* ω. When no confusion may arise, we write ω, $d\omega$ and $\overline{d}\omega$ instead of (ω, Σ), $(d\omega, \Sigma)$ and $(\overline{d}\omega, \Sigma)$.

If (ω, Σ) is a stratified form then a refinement $\Sigma' \prec \Sigma$ induces a stratified form, (ω, Σ') in a canonical way. Furthermore, given a stratified k-form (ω, Σ) on X, we say that ω is *bounded* if $|\omega_S| \leq C$ for all $S \in \Sigma$ and some positive constant C, where the norm $|\cdot|$ is induced from $\mathbb{R}^n$. The set of all smooth stratified and bounded k-forms on X with stratified exterior derivatives is denoted by $\widetilde{\Omega}^k(X)$.

Bounded stratified forms have the following continuity property.

LEMMA 2.3. *If $\omega = (\omega_S, \Sigma)$ is a stratified and bounded k-form then for any pair $(S, S') \in \Sigma \times \Sigma$ with $S' \leq S$, any sequence of points $p_n \in S$ such that $\lim_{n\to\infty} p_n = p \in S'$, and any sequence of multivectors $\xi_n \in \wedge^k T_{p_n} S$ with*

$$\lim_{n\to\infty} \xi_n = \xi \in \wedge^k T_p S',$$

we have

$$\lim_{n\to\infty} \omega(p_n; \xi_n) = \omega(p; \xi).$$

PROOF. Since ω is bounded, there exists a subsequence (p_{n_j}, ξ_{n_j}) and a real number a such that $\lim_{j\to\infty} \omega(p_{n_j}; \xi_{n_j}) = a$. But since the graph of ω is closed, it follows that $a = \omega(p, \xi)$. □

DEFINITION 2.4. Let (X, Σ) and (Y, Σ') be two stratified sets in $\mathbb{R}^n$. A continuous map $h\colon X \to Y$ is *semi-differentiable* (see [7]) if

(i) for any $S \in \Sigma$ there is a stratum $S' \in \Sigma'$ such that h maps S to S' in a smooth way;
(ii) The differentials dh_i, $i = 1, \ldots, n$ of the components of h are stratified and bounded forms on (X, Σ).

DEFINITION 2.5. An L^∞ map is a map $f\colon X \to Y$ such that there exists a stratification Σ_X of X and a stratification Σ_Y of Y with respect to which f is a semi-differentiable map.

Note that L^∞ maps are Lipschitz. Conversely, we have the following.

Proposition 2.6. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$. Then the map $f : X \to Y$ is Lipschitz if and only if f is an L^∞ map.*

The proof of this proposition is in [9]. We can now define a notion of an L^∞ form.

Definition 2.7. Let X be a set and define an equivalence relation on $\widetilde{\Omega}^k(X)$:

$$(\omega, \Sigma) \sim (\omega', \Sigma')$$

if there exists a stratification Σ'' that refines both Σ and Σ' such that

$$(\omega|_{\Sigma''}, \Sigma'') = (\omega'|_{\Sigma''}, \Sigma'').$$

Denote by $\Omega^k_\infty(X)$ the classes of equivalence of "$\sim$". An element of $\Omega^k_\infty(X)$ is called an $\mathbf{L}^\infty$ *form.*

Remark 2.8. The exterior algebra structure is defined on $\Omega^\bullet_\infty(X) := \bigcup_k \Omega^k_\infty(X)$ in a natural way. The sum of two L^∞ forms ω and ω' is an L^∞ form ω'' that can be constructed as follows. If (ω, Σ) and (ω, Σ') represent ω and ω', then $(\omega + \omega', \Sigma'')$ represents ω'' where Σ'' is any stratification refining both Σ and Σ'. The exterior product is defined in a similar fashion.

Pullback of L^∞ forms Let $f\colon X \to Y$ be an L^∞ map and $\omega \in \Omega^k_\infty(Y)$ be an L^∞ k-form. Then, for any stratification Σ_X of X and Σ_Y of Y for which f is semi-differentiable and (ω, Σ_Y) is a stratified smooth form, the form defined by $f|_S^*\omega$ on each $S \in \Sigma_X$ defines a smooth stratified form $f^*\omega$ called the pullback of ω.

Proposition 2.9. *In the situation above, the pullback $f^*\omega$ defines a unique L^∞ form.*

For a proof, see [9].

Integration of L^∞ forms. Let $X \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^k$ be k-dimensional oriented compact semialgebraic submanifold with corners of $\mathbb{R}^k$, where $k \leq n$ and the orientation of A is induced by the standard orientation of $\mathbb{R}^k$. Let $\sigma\colon A \to X$ be a map. We want to define the integral of an L^∞ k-form ω over σ. Let Σ_X and Σ_A be stratifications of X and A respectively such that $(\sigma^*\omega, \Sigma_A)$ is a stratified form. The integral of ω over σ is defined by

$$\int_\sigma \omega := \sum_{S \in \Sigma^k_A} \int_S \sigma^*_S \omega \ . \tag{2.1}$$

Proposition 2.10. *Let X, ω, A and σ be as above then the integral of ω over σ, as defined in* (2.1)*, is well defined (independent of the stratification).*

For a proof see [9].

Definitions 2.11. Let $X \subset \mathbb{R}^n$. Define $(C_k(X), \partial)_{k\in\mathbb{Z}_+}$ to be the chain complex generated by continuous semialgebraic singular simplices, $\sigma\colon \Delta^k \to X$, where Δ^k is the standard k simplex in $\mathbb{R}^k$, and ∂ is the standard boundary operator [13, p. 142]. Set $C^k(X) := \mathrm{Hom}(C_k(X), \mathbb{R})$ to be the complex of cochains endowed with differential $d := \partial^*$.

We extend integration of L^∞ forms over all chains as follows. Let $c = \sum_{j=1}^L a_j\sigma_j \in C_k(X)$, where σ_j is a singular simplex, $a_j \in \mathbb{R}$ for all j, and suppose that $\omega \in \Omega^k_\infty(X)$. Define

$$\int_c \omega := \sum_{j=1}^{L} a_j \int_{\sigma_j} \omega \ .$$

The De Rham Theorem

Definition 2.12. Let $X \subset \mathbb{R}^n$ be a set. Define $H^k(X)$ to be the cohomology of $C^k(X)$ and $H^k_\infty(X)$ to be the cohomology of $\Omega^k_\infty(X)$.

Remark 2.13. Semialgebraic homology was studied by many authors (see [6] for a list of references). In 1981 Hans Delfs [3] proved that semialgebraic homology is isomorphic to simplicial homology of a semialgebraic set over a real closed field. In 1996 Woerheide [14] showed that homology theory of singular definable simplices in a o-minimal structure satisfies Eilnberg-Steenrod axioms and therefore coincides with the standard singular homology theory. Complete proofs of comparison theorems for o-minimal homology (in particular for semialgebraic sets over $\mathbb{R}$) can be found in a recent paper by Edmundo and Wortheide [4].

The main result of this article is

Theorem 2.14 (De Rham Theorem). *Let $X \subset \mathbb{R}^n$ be a compact set then the map*

$$\psi\colon \Omega^k_\infty(X) \to C^k(X) \quad , \psi(\omega)c := \int_c \omega$$

induces an isomorphism on cohomology.

3. Stokes' Theorem

In this section we introduce the Stokes' formula for semialgebraic chains and L^∞ forms. Stokes' formula for singular spaces was previously considered in the literature, for example, in [8] Stokes' formula is proven for subanalytic leaves. Here and in [9] we give a different proof of this fact.

THEOREM 3.1 (Stokes' Formula). *Let $X \subset \mathbb{R}^n$ and $c \in C^k(X)$. For any smooth L^∞ $(k-1)$-form ω on X:*

$$\int_c d\omega = \int_{\partial c} \omega \ .$$

A detailed proof of this theorem can be found in [9]. Here we give only a sketch of a proof.

Stratify Δ^k and X such that the image of each stratum of Δ^k is a stratum of X and ω is smooth on every stratum in X . It is enough to prove the formula for each stratum S of X, i.e.,

$$\int_S d\omega = \int_{\partial S} \omega \ .$$

Let S be a stratum in X. We approximate S from inside by a family of smooth manifolds S_ε, $\varepsilon > 0$, and show that

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} d\omega = \int_S d\omega \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{\partial S_\varepsilon} \omega = \int_{\partial S} \omega \ .$$

Clearly,

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} d\omega = \int_S d\omega \ .$$

So, we only have to show that

$$\lim_{\varepsilon \to 0} \int_{\partial S_\varepsilon} \omega = \int_{\partial S} \omega \ . \tag{3.1}$$

By using a version of the "wing lemma" we may find a subset $B \subset \partial S$, $\dim B \leq k-2$, such that $\overline{S}$ looks like a manifold with boundary near points $a \in \partial S - B$. For any $\delta > 0$ set

$$A_\delta := \{x \in \partial S : \text{dist}\,(x, B \cup \ (k-2)\text{-skeleton of } \partial S) > \delta\},$$
$$B_\delta := \{x \in S : \text{dist}\,(x, B \cup (k-2)\text{-skeleton of } \partial S) \leq \delta\}.$$

Note that

$$\int_{\partial S} \omega = \int_{A_\delta} \omega + \int_{\partial S \cap B_\delta} \omega \ .$$

The stratum S near A_δ is a manifold with boundary. Of course set $U_\delta := \overline{S} - B_\delta$ is a neighborhood of A_δ in $\overline{S}$ and $\partial S_\varepsilon \subset U_\delta \cup B_\delta$ for every $\varepsilon > 0$. We split

$$\int_{\partial S_\varepsilon} \omega = \int_{\partial S_\varepsilon \cap U_\delta} \omega + \int_{\partial S_\varepsilon \cap B_\delta} \omega \ .$$

Observe that

$$\int_{A_\delta} \omega = \lim_{\varepsilon \to 0} \int_{\partial S_\varepsilon \cap U_\delta} \omega \ .$$

To complete the proof of (3.1) we only have to show that $\int_{\partial S_\varepsilon \cap B_\delta} \omega$ and $\int_{\partial S \cap B_\delta} \omega$ are small in terms of δ (uniformly in ε). We show that by using the well known Cauchy–Crofton formula [5] cf. Section 2.1 of [12].

THEOREM 3.2 (Cauchy–Crofton formula). *Let $A \subset \mathbb{R}^n$, set*

$$K_j^P(A) := \{q \in P : \#\left(\pi_P^{-1}(q) \cap A\right) = j\},$$

where π_P is an orthogonal projections from $\mathbb{R}^n$ to P and $\#S$ is the cardinality of a set S. Then,

$$\mu_k(A) = \int_{P \in \mathbb{G}_n^k} \sum_{j=1}^{\infty} j \mu_k(K_j^P(A)) d\gamma(P) \ , \tag{3.2}$$

on the Grassmanian, $\mathbb{G}_n^k$ of k-dimensional linear subspaces of $\mathbb{G}_k^{(k-1)}$.

Substituting $B_\delta \cap \partial S_\varepsilon$ in formula (3.2) we obtain

$$\mu_{k-1}(B_\delta \cap \partial S_\varepsilon) = \int_{P \in \mathbb{G}_n^{k-1}} \sum_{j=1}^{N(\delta,\varepsilon)} j \mu_{k-1}(K_j^P(B_\delta \cap \partial S_\varepsilon)) d\gamma(P) \ ,$$

where

$$N(\delta, \varepsilon) := \max\{\#\left(\pi_P^{-1}(q) \cap B_\delta \cap \partial S_\varepsilon\right) < \infty : P \in \mathbb{G}_n^{k-1}, q \in P\} \ .$$

There exists $N_0 \in \mathbb{N}$ such that $N(\varepsilon, \delta) < N_0$ as a consequence of uniform boundness principle for families of semialgebraic sets. Observe that

$$K_j^P(B_\delta \cap \partial S_\varepsilon) \subset \pi_P(B_\delta) \ .$$

But since B_δ is a set of points in S at a distance of at most δ from a set of dimension at most $k-2$, it follows that $\mu_{k-1}(\pi_P(B_\delta)) \to 0$ as $\delta \to 0$. Similar computation shows that $\int_{\partial S \cap B_\delta} \omega$ tends to 0 as $\delta \to 0$. □

4. The Poincaré Lemma

The main ingredient of any proof of a De Rham type Theorem is its local version, namely, the Poincaré Lemma.

THEOREM 4.1 (Poincaré Lemma). *Let ω be a smooth closed L^∞ k-form on $X \subset \mathbb{R}^n$ and $p \in X$. Then, there exists a neighborhood U_p of p in X, and an L^∞ $(k-1)$-form γ defined on U_p such that $\omega = d\gamma$ in U_p.*

We split the proof into Lemma 4.2 and Proposition 4.3.

LEMMA 4.2. *Let ω be a closed smooth L^∞ k-form on $X \subset \mathbb{R}^n$ and $p \in X$. Then, there exists a neighborhood U_p of p in X and a weakly differentiable L^∞ $(k-1)$-form γ defined on U_p such that $\omega = \overline{d}\gamma$ on U_p.*

PROPOSITION 4.3. *Let ω be a closed L^∞ k-form on X, $p \in X$ and U a neighborhood of p in X. Let Σ_U be a stratification of U given by Lemma 4.2 such that (ω, Σ_U) is a stratified form. Then, there exists a sequence of L^∞ forms on U, $\gamma^0, \dots, \gamma^n$ that satisfy the following properties.*

(i) $\overline{d}\gamma^j = \omega$ *for all j.*

(ii) γ^j *is smooth on the l-dimensional strata, for all $l \leq j$.*

The proof of Proposition 4.3 uses a smoothing technique; for details see [9]. The idea of the proof of Lemma 4.2 is to proceed by induction on $\dim U_p$. In dimension 1 the statement is the fundamental theorem of calculus. For $\dim U_p > 1$, take a stratification Σ for which (ω, Σ) is a stratified form and construct a Lipschitz strong deformation retract $r : U_p \times I \to U_p$ to a set $N \subset U_p$ of smaller dimension such that r preserves the strata of Σ and $r^*\omega$ is still an L^∞ form. Here $I := [0,1]$. Then, $r^*\omega = \alpha + dt \wedge \beta$ where t is the parameter in I and α and β are smooth stratified forms on $U_p \times I$ that do not contain dt. Define

$$\gamma_0 := \int_0^1 \beta(x,t)dt \ .$$

A straightforward computation shows that $\overline{d}\gamma_0 = r_1^*\omega - r_0^*\omega$, where $r_t(x) := r(x,t)$. Note that $r_1^*\omega = \omega$ and $r_0^*\omega|_N = \omega|_N$. By the induction hypothesis for the set N and the form $\omega|_N$ we obtain a form γ' on N such that $\omega|_N = \overline{d}\gamma'$. But then, $r_0^*\omega = \overline{d}r_0^*\gamma'$. Thus, we get $\overline{d}(\gamma_0 + r_0^*\gamma') = \omega$.

The main difficulty in the proof is to find the deformation retract r that would preserve the strata and L^∞ forms. For that purpose we construct a Lipschitz deformation retract r as described in the following key theorem (cf. [11]).

THEOREM 4.4. *Let (X, Σ_X) be a stratified set in $\mathbb{R}^n$ and $p \in X$ then there exists a stratified neighborhood (U, Σ_U) of p in X such that the following hold.*

(i) $\Sigma_U \prec \Sigma_X \cap U$.

(ii) *There exists a Lipschitz strong deformation retract $r : U \times [0,1] \to U$ of U to a union $N \subset U$ of strata of Σ_U with $p \in N$ (and $\dim N < \dim U$) satisfying the following properties*

  (a) $r_0(x) \in N$ *and* $r_1(x) = x$ *where* $r_t(x) = r(x,t)$ *for* $t \in [0,1]$.

  (b) $r|_{S\times(0,1]}$ *is smooth and* $r(S \times (0,1]) \subset S$ *for any stratum* $S \in \Sigma_U$.

  (c) *For any* $S \in \Sigma_U$ *there exists* $S' \in \Sigma_U$ *such that* $r_0(S) \subset S'$.

  (d) $d_x r_t|_S(x) \to d_x r_0|_S(x)$ *as* $t \to 0$, *for any* $x \in S \in \Sigma_U$.

The proof is based on a technique developed in [10] and can be found in [9].

5. **De Rham Theorem** There are many possible approaches to proving a De Rham type theorem after having proved the Poincaré lemma. In [9] we give an elementary approach using Whitney's elementary [15] forms and basic linear algebra. In this note we give a sheaf theoretic proof which is shorter. See [13] for generalities on sheaf theory. Define $\mathcal{F}^k$ to be the sheaf associated with the presheaf of L^∞ k-forms. Clearly each $\mathcal{F}^k$ is a fine sheaf, and by Theorem 4.1 the sequence

$$\dots \overset{d}{\leftarrow} \mathcal{F}^j \overset{d}{\leftarrow} \dots \overset{d}{\leftarrow} \mathcal{F}^1 \overset{d}{\leftarrow} \mathcal{F}^0$$

is exact, where $\mathcal{F}^0$ is the sheaf of constant functions. Therefore, $\mathcal{F}^\bullet$ comprises a fine resolution of the locally constant sheaf and hence its cohomology naturally coincides with the singular cohomology of X.

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