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ON CYCLES AND PRODUCTS OF IDEALS AND CORRESPONDING INDEFINITE QUADRATIC FORMS

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Abstract. Let $k \geq 2$ be an integer and let $D = k^2 + k + 1$ be a positive non-square integer. In this work, we derive some properties (including cycles) of ideals $I_1 = [k, k-1+\sqrt{D}]$, $I_2 = [k+1, k+\sqrt{D}]$ and their product I. In the last section, we consider the indefinite binary quadratic forms F_{I_1}, F_{I_2} and F_I of discriminant $\Delta = 4D$ which correspond to I_1, I_2 and I, respectively and we formulate the cycle of F_{I_1} and F_{I_2}.

Résumé. Soit $k \geq 2$ un entier tel que $D = k^2 + k + 1$ ne soit pas le carré d'un entier. Dans ce travail, on obtient quelques propriétés (incluant des cycles) des idéaux $I_1 = [k, k-1+\sqrt{D}]$, $I_2 = [k+1, k+\sqrt{D}]$ et de leur produit I. Dans le dernier paragraphe, on considère les formes quadratiques binaires indéfinies F_{I_1}, F_{I_2} et F_I de discriminant $\Delta = 4D$ qui correspondent respectivement aux idéaux I_1, I_2 et I à fin de formuler le cycle de F_{I_1} et F_{I_2}.

1. **Introduction.** A real binary quadratic form (or just a form) F is a polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. F is an integral form if and only if $a, b, c \in \mathbb{Z}$, and is called indefinite if and only if $\Delta(F) > 0$. An indefinite form $F = (a, b, c)$ of discriminant Δ is said to be reduced if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{1.1}$$

Most properties of quadratic forms can be given by the aid of the extended modular group $\overline{\Gamma}$ (see [11]). Gauss (1777–1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$\begin{aligned} gF(x, y) = {} & (ap^2 + bpq + cq^2)x^2 + (2apu + bpv + buq + 2cqv)xy \\ & + (au^2 + buv + cv^2)y^2 \end{aligned} \tag{1.2}$$

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for $g = \left(\begin{smallmatrix} p & q \\ u & v \end{smallmatrix}\right) = [p; q; u; v] \in \overline{\Gamma}$. Hence two forms F and G are called equivalent if and only if there exists a $g \in \overline{\Gamma}$ such that $gF = G$. If $\det g = 1$, then F and G are called properly equivalent, and if $\det g = -1$, then F and G are called improperly equivalent. If a form F is improperly equivalent to itself, then it called ambiguous. Let $\rho(F)$ denotes the normalization (it means that replacing F by its normalization) of $(c, -b, a)$. To be more explicit, we set

$$\rho^i(F) = (c, -b + 2cr_i, cr_i^2 - br_i + a),$$

where

$$r_i = r_i(F) = \begin{cases} \operatorname{sign}(c) \left\lfloor \frac{b}{2|c|} \right\rfloor & \text{for } |c| \geq \sqrt{\Delta} \\ \operatorname{sign}(c) \left\lfloor \frac{b+\sqrt{\Delta}}{2|c|} \right\rfloor & \text{for } |c| < \sqrt{\Delta} \end{cases}$$

for $i \geq 0$. The number r_i is called the reducing number and the form $\rho^i(F)$ is called the reduction of F. Further, if F is reduced, then $\rho^i(F)$ is also reduced by (1.1). In fact, ρ^i is a permutation of the set of all reduced indefinite forms. Now consider the transformation $\tau(F) = \tau(a, b, c) = (-a, b, -c)$. Then the cycle of F is the sequence $\big((\tau\rho)^i(G)\big)$ for $i \in \mathbb{Z}$, where $G = (A, B, C)$ is a reduced form with $A > 0$ that is equivalent to F. The cycle of F can be derived by the following theorem.

Theorem 1.1 ([5]). *Let $F = (a, b, c)$ be a reduced indefinite quadratic form of discriminant Δ. Then the cycle of F is a sequence $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ of length l, where $F_0 = F = (a_0, b_0, c_0)$,*

$$s_i = |s(F_i)| = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor \tag{1.3}$$

and

$$F_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1}) = \big(|c_i|, -b_i + 2s_i|c_i|, -(a_i + b_i s_i + c_i s_i^2)\big)$$

for $1 \leq i \leq l - 2$.

Mollin [3] considered the arithmetic of ideals in his book. Let $D \neq 1$ be a square free integer and let $\Delta = \frac{4D}{r^2}$, where $r = 2$ if $D \equiv 1 \pmod 4$, and $r = 1$, otherwise. If we set $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, then $\mathbb{K}$ is called a quadratic number field of discriminant Δ and O_Δ is the ring of integers of the quadratic field $\mathbb{K}$ of discriminant Δ. Let $I = [\alpha, \beta]$ denote the $\mathbb{Z}$-module $\alpha\mathbb{Z} \oplus \beta\mathbb{Z}$, *i.e.*, the additive abelian group, with basis elements α and β consisting of $\{\alpha x + \beta y : x, y \in \mathbb{Z}\}$. Note that $O_\Delta = [1, \frac{1+\sqrt{D}}{r}]$. In this case $w_\Delta = \frac{r-1+\sqrt{D}}{r}$ is called the principal surd. Every principal surd $w_\Delta \in O_\Delta$ can be uniquely expressed as $w_\Delta = x\alpha + y\beta$, where $x, y \in \mathbb{Z}$ and $\alpha, \beta \in O_\Delta$. We call α, β an integral basis for $\mathbb{K}$. If $\frac{\alpha\overline{\beta} - \beta\overline{\alpha}}{\sqrt{\Delta}} > 0$, then α and β are called ordered basis elements. Recall that two basis of an ideal are ordered if and only if they are equivalent under an element of $\overline{\Gamma}$. If I has ordered basis elements, then we say that I is simply ordered. If I is ordered, then

$$F(x,y)=\frac{N(\alpha x+\beta y)}{N(I)}$$

is a quadratic form of discriminant Δ. In this case we say that F belongs to I and write $I \to F$. Conversely let us assume that

$$G(x,y)=Ax^2+Bxy+Cy^2=d(ax^2+bxy+cy^2)$$

be a quadratic form, where $d=\pm\gcd(A,B,C)$ and $b^2-4ac=\Delta$. If $B^2-4AC>0$, then we get $d>0$, and if $B^2-4AC<0$, then we choose d such that $a>0$. If

$$I=[\alpha,\beta]=\begin{cases}[a,\frac{b-\sqrt{\Delta}}{2}] & \text{for } a>0,\\ [a,\frac{b-\sqrt{\Delta}}{2}]\sqrt{\Delta} & \text{for } a<0 \text{ and } \Delta>0,\end{cases}$$

then I is an ordered O_Δ-ideal. Note that if $a>0$, then I is primitive and if $a<0$, then $\frac{I}{\sqrt{\Delta}}$ is primitive. Thus to every form G, there corresponds an ideal I to which G belongs and we write $G \to I$. Hence we have a correspondence between ideals and quadratic forms.

Theorem 1.2. *If $I=[a,b+cw_\Delta]$, then I is a non-zero ideal of O_Δ if and only if $c|b$, $c|a$ and $ac|N(b+cw_\Delta)$* [3].

Let δ denote a real quadratic irrational integer with trace $t=\delta+\overline{\delta}$ and norm $n=\delta\overline{\delta}$. Given a real quadratic irrational $\gamma\in\mathbb{Q}(\delta)$, there are rational integers P and Q such that $\gamma=\frac{P+\delta}{Q}$ with $Q|(\delta+P)(\overline{\delta}+P)$. Hence for each $\gamma=\frac{P+\delta}{Q}$ there is a corresponding $\mathbb{Z}$-module

$$I_\gamma=[Q,P+\delta] \tag{1.4}$$

(in fact, this module is an ideal by Theorem 1.2 and an indefinite quadratic form

$$F_\gamma(x,y)=Q(x+\gamma y)(x+\overline{\gamma}y)=Qx^2+(t+2P)xy+\Big(\frac{n+tP+P^2}{Q}\Big)y^2 \tag{1.5}$$

of discriminant $\Delta=t^2-4n$. The ideal I_γ in (1.4) is said to be reduced if and only if

$$P+\delta>Q \quad \text{and} \quad -Q<P+\overline{\delta}<0 \tag{1.6}$$

and is said to be ambiguous if and only if it contains both $\frac{P+\delta}{Q}$ and $\frac{P+\overline{\delta}}{Q}$, so if and only if $\frac{2P}{Q}\in\mathbb{Z}$.

Let $[m_0;\overline{m_1,m_2,\ldots,m_{l-1}}]$ denote continued fraction expansion of $\gamma=\frac{P+\delta}{Q}$ with period length $l=l(I)$. Then the cycle of I_γ is $I_\gamma=I_\gamma^0\sim I_\gamma^1\sim\cdots\sim I_\gamma^{l-1}$ of length l, where

$$m_i=\Big\lfloor\frac{P_i+\delta}{Q_i}\Big\rfloor,\quad P_{i+1}=m_iQ_i-P_i,\quad\text{and}\quad Q_{i+1}=\frac{\delta^2-P_{i+1}^2}{Q_i},\tag{1.7}$$

for $i\geq 0$ (see also [1, 2, 4, 6, 7]).

Let $\delta=\sqrt{D}$ and let $I_1=[Q_1,P_1+\sqrt{D}]$ and $I_2=[Q_2,P_2+\sqrt{D}]$ be two ideals. Then the product of I_1 and I_2 is an ideal $I=(S)[Q,P+\sqrt{D}]$, where

$$\begin{aligned}S&=\gcd(Q_1,Q_2,P_1+P_2),\\ Q&=\frac{Q_1Q_2}{S^2},\\ P&\equiv\frac{1}{S}\Big[UQ_2P_1+VQ_1P_2+\frac{W}{2}(P_1P_2+D)\Big]\pmod{2Q},\end{aligned}\tag{1.8}$$

and U, V and W are integers such that

$$UQ_2+VQ_1+\frac{W}{2}(P_1+P_2)=S.$$

2. Cycles and product of ideals.

In [8–10, 12, 13], we considered some properties of quadratic irrationals $\gamma=\frac{P+\delta}{Q}$, quadratic ideals I_γ and indefinite quadratic forms F_γ. In the present paper, we aim to derive some properties of ideals

$$I_1=[k,k-1+\sqrt{k^2+k+1}]\quad\text{and}\quad I_2=[k+1,k+\sqrt{k^2+k+1}]\tag{2.1}$$

and their product $I=I_1I_2$, where $k\geq 2$ is an integer.

THEOREM 2.1. *Let I_1 and I_2 be the ideals in (2.1). Then*

(1) *I_1 is reduced for every $k\geq 2$.*
(2) *I_1 is ambiguous if and only if $k=2$.*
(3) *If $k=3t+1$ for an integer $t\geq 1$, then the continued fraction expansion of γ_1 is $[\overline{1,1,6t+2,1,1,2t}]$, and the cycle of I_1 is*

$$\begin{aligned}&I_1^0=[3t+1,3t+\sqrt{9t^2+9t+3}]\sim I_1^1=[3t+2,1+\sqrt{9t^2+9t+3}]\sim\\ &I_1^2=[1,3t+1+\sqrt{9t^2+9t+3}]\sim I_1^3=[3t+2,3t+1+\sqrt{9t^2+9t+3}]\sim\\ &I_1^4=[3t+1,1+\sqrt{9t^2+9t+3}]\sim I_1^5=[3,3t+\sqrt{9t^2+9t+3}]\end{aligned}$$

of length 6.

(4) *The ideal I_2 is reduced for every $k\geq 2$.*

(5) *The ideal* I_2 *is not ambiguous for every* $k \geq 2$.
(6) *If* $k = 3t + 1$ *for an integer* $t \geq 1$, *then the continued fraction expansion of* γ_2 *is* $[\,\overline{1, 1, 2t, 1, 1, 6t + 2}\,]$, *and the cycle of* I_2 *is*

$$I_2^0 = [3t + 2, 3t + 1 + \sqrt{9t^2 + 9t + 3}] \sim I_2^1 = [3t + 1, 1 + \sqrt{9t^2 + 9t + 3}] \sim$$

$$I_2^2 = [3, 3t + \sqrt{9t^2 + 9t + 3}] \sim I_2^3 = [3t + 1, 3t + \sqrt{9t^2 + 9t + 3}] \sim$$

$$I_2^4 = [3t + 2, 1 + \sqrt{9t^2 + 9t + 3}] \sim I_2^5 = [1, 3t + 1 + \sqrt{9t^2 + 9t + 3}]$$

of length 6.

PROOF. (1) Note that $k^2 + k + 1 > 1$ since $k \geq 2$. So

$$\begin{aligned} k^2 + k + 1 > 1 &\Leftrightarrow k^2 + k + 1 > k^2 + (k - 1)^2 - 2k(k - 1) \\ &\Leftrightarrow D > Q_1^2 + P_1^2 - 2P_1Q_1 \\ &\Leftrightarrow P_1 + \sqrt{D} > Q_1. \end{aligned}$$

On the other hand $k^2 + 1 < 3k + k^2 + 1 \Leftrightarrow (k - 1)^2 < k^2 + k + 1 \Leftrightarrow P_1 - \sqrt{D} < 0$ and

$$\begin{aligned} 4k^2 - 4k + 1 > k^2 + k + 1 &\Leftrightarrow (k - 1)^2 + k^2 + 2k(k - 1) > k^2 + k + 1 \\ &\Leftrightarrow P_1^2 + Q_1^2 + 2P_1Q_1 > D \\ &\Leftrightarrow P_1 - \sqrt{D} > -Q_1. \end{aligned}$$

So we get $P_1 + \sqrt{D} > Q_1$ and $-Q_1 < P_1 - \sqrt{D} < 0$, that is, I_1 is reduced, by (1.6).

(2) Let I_1 be ambiguous. Then by definition $\frac{2P_1}{Q_1}$ must be an integer. So $\frac{2P_1}{Q_1} \in \mathbb{Z} \Leftrightarrow \frac{2k-2}{k} = 2 - \frac{2}{k} \in \mathbb{Z} \Leftrightarrow k = 2$. Conversely, let $k = 2$. Then $\frac{2P_1}{Q_1} = 1 \in \mathbb{Z}$. So I_1 is ambiguous.

(3) Recall that for $x = u + v\sqrt{d}$ with $u + v\sqrt{d}$ is a quadratic irrational, if $x > 1$ and $-1 < \overline{x} < 0$, then the continued fraction that represents x is a purely periodic continued fraction. We proved in (1) that I_1 is reduced, that is, $P_1 + \sqrt{D} > Q_1$ and $-Q_1 < P_1 - \sqrt{D} < 0$. Hence

$$P_1 + \sqrt{D} > Q_1 \Leftrightarrow \frac{P_1 + \sqrt{D}}{Q_1} > 1$$

and

$$-Q_1 < P_1 - \sqrt{D} < 0 \Leftrightarrow -1 < \frac{P_1 - \sqrt{D}}{Q_1} < 0.$$

So γ_1 has a purely periodic continued fraction. Now let $I_1 = I_1^0 = [3t + 1, 3t + \sqrt{9t^2 + 9t + 3}]$. Then by (1.7), we have $m_0 = 1$ and hence $P_1 = 1$ and $Q_1 = 3t+2$. Similarly, we obtain Table 1.

i	0	1	2	3	4	5
P_i	$3t$	1	$3t+1$	$3t+1$	1	$3t$
Q_i	$3t+1$	$3t+2$	1	$3t+2$	$3t+1$	3
m_i	1	1	$6t+2$	1	1	$2t$

Table 1

Therefore, the continued fraction expansion of γ_1 is $[\,\overline{1,1,6t+2,1,1,2t}\,]$, and the cycle of I_1 is $I_1^0=[3t+1,3t+\sqrt{9t^2+9t+3}]\sim I_1^1=[3t+2,1+\sqrt{9t^2+9t+3}]$ $\sim I_1^2=[1,3t+1+\sqrt{9t^2+9t+3}]\sim I_1^3=[3t+2,3t+1+\sqrt{9t^2+9t+3}]\sim I_1^4=$ $[3t+1,1+\sqrt{9t^2+9t+3}]\sim I_1^5=[3,3t+\sqrt{9t^2+9t+3}]$.

(4) Note that $k^2+k>0$ since $k\geq 2$. So we have

$$
\begin{aligned}
k^2+k+1>2k^2-2k^2+2k-2k+1 &\Leftrightarrow k^2+k+1>(k+1)^2+k^2-2k(k+1)\\
&\Leftrightarrow D>Q_2^2+P_2^2-2P_2Q_2\\
&\Leftrightarrow P_2+\sqrt{D}>Q_2.
\end{aligned}
$$

Also it is easily seen that $k^2<k^2+k+1\Leftrightarrow P_2^2<D\Leftrightarrow P_2-\sqrt{D}<0$ and

$$
\begin{aligned}
3k^2+3k>0 &\Leftrightarrow k^2+k^2+2k+1+2k^2+2k>k^2+k+1\\
&\Leftrightarrow k^2+(k+1)^2+2k(k+1)>k^2+k+1\\
&\Leftrightarrow P_2^2+Q_2^2+2P_2Q_2>D\\
&\Leftrightarrow P_2-\sqrt{D}>-Q_2.
\end{aligned}
$$

Hence $P_2+\sqrt{D}>Q_2$ and $-Q_2<P_2-\sqrt{D}<0$. Therefore I_2 is reduced.

(5) $\frac{2P_2}{Q_2}=\frac{2k}{k+1}=2-\frac{2}{k+1}$ is not an integer since $k\geq 2$. Therefore I_2 is not ambiguous.

(6) Since I_2 is reduced, γ_2 has a purely periodic continued fraction. Let $I_2=I_2^0=[3t+2,3t+1+\sqrt{9t^2+9t+3}]$. Then $m_0=1$ and hence $P_1=1$ and $Q_1=3t+1$. Similarly we obtain Table 2.

i	0	1	2	3	4	5
P_i	$3t+1$	1	$3t$	$3t$	1	$3t+1$
Q_i	$3t+2$	$3t+1$	3	$3t+1$	$3t+2$	1
m_i	1	1	$2t$	1	1	$6t+2$

Table 2

So the continued fraction expansion of γ_2 is $[\overline{1,1,2t,1,1,6t+2}]$ and the cycle of I_2 is $I_2^0=[3t+2,3t+1+\sqrt{9t^2+9t+3}]\sim I_2^1=[3t+1,1+\sqrt{9t^2+9t+3}]\sim$

$I_2^2=[3,3t+\sqrt{9t^2+9t+3}]\sim I_2^3=[3t+1,3t+\sqrt{9t^2+9t+3}]\sim I_2^4=[3t+2,1+\sqrt{9t^2+9t+3}]\sim I_2^5=[1,3t+1+\sqrt{9t^2+9t+3}]$. □

From this theorem we can give the following result.

Corollary 2.2. *I_1^2 and I_1^5 are the ambiguous ideals in the cycle of I_1 and I_2^2 and I_2^5 are the ambiguous ideals in the cycle of I_2.*

Now consider the product of I_1 and I_2.

Theorem 2.3. *The product of I_1 and I_2 is $I=[k^2+k,k^2+k-1+\sqrt{k^2+k+1}]$.*

Proof. Applying (1.8), we get

$$S=\gcd(Q_1,Q_2,P_1+P_2)=\gcd(k,k+1,2k-1)=1$$

and $Q=\frac{Q_1Q_2}{S^2}=\frac{k(k+1)}{1}=k^2+k$. Hence

(2.2)

$$\begin{aligned}P&\equiv\frac{1}{S}\Big[UQ_2P_1+VQ_1P_2+\frac{W}{2}(P_1P_2+D)\Big]\pmod{2Q}\\&\equiv\Big[U(k+1)(k-1)+V(k)(k)+\frac{W}{2}(k^2-k+k^2+k+1)\Big]\pmod{2(k^2+k)}\\&\equiv\Big[U(k^2-1)+Vk^2+\frac{W}{2}(2k^2+1)\Big]\pmod{2(k^2+k)},\end{aligned}$$

where U, V, W are integers such that

$$U(k+1)+V(k)+\frac{W}{2}(2k-1)=1. \tag{2.3}$$

One solution of (2.3) is $(U,V,W)=(1-k,k,0)$. So (2.2) becomes

$$\begin{aligned}P&\equiv\Big[(1-k)(k^2-1)+k(k^2)+\frac{0}{2}(2k^2+1)\Big]\pmod{2(k^2+k)}\\&\equiv[k^2+k-1]\pmod{2(k^2+k)}\\&=k^2+k-1.\end{aligned}$$

Therefore, the product of I_1 and I_2 is $I=[k^2+k,k^2+k-1+\sqrt{k^2+k+1}]$. □

Example 2.4. Let $t=5$. Then $\gamma_1=\frac{15+\sqrt{273}}{16}=[\,\overline{1,1,32,1,1,10}\,]$ and the cycle of $I_1=[16,15+\sqrt{273}]$ is

$$I_1^0=[16,15+\sqrt{273}]\sim I_1^1=[17,1+\sqrt{273}]\sim I_1^2=[1,16+\sqrt{273}]\sim$$
$$I_1^3=[17,16+\sqrt{273}]\sim I_1^4=[16,1+\sqrt{273}]\sim I_1^5=[3,15+\sqrt{273}].$$

Similarly $\gamma_2 = \frac{16+\sqrt{273}}{17} = [\overline{1,1,10,1,1,32}]$ and the cycle of $I_2 = [17, 16+\sqrt{273}]$ is

$$I_2^0 = [17, 16+\sqrt{273}] \sim I_2^1 = [16, 1+\sqrt{273}] \sim I_2^2 = [3, 15+\sqrt{273}] \sim$$
$$I_2^3 = [16, 15+\sqrt{273}] \sim I_2^4 = [17, 1+\sqrt{273}] \sim I_2^5 = [1, 16+\sqrt{273}].$$

The product of I_1 and I_2 is $I = [272, 271+\sqrt{273}]$.

From Theorem 2.3, we can give the following result.

Theorem 2.5. *The ideal I is not reduced and is not ambiguous for every $k \geq 2$.*

Proof. Note that $k^4 + 2k^3 - 2k^2 - 3k > 0$ since $k \geq 2$. Hence

$$\begin{aligned} k^4 + 2k^3 - 2k^2 - 3k > 0 &\Leftrightarrow k^4 + 2k^3 - k^2 - 2k + 1 > k^2 + k + 1 \\ &\Leftrightarrow k^2 + k - 1 > \sqrt{k^2 + k + 1} \\ &\Leftrightarrow P - \sqrt{D} > 0 \end{aligned}$$

which is a contradiction to (1.6). So I is not reduced.

If I is ambiguous, then $\frac{2P}{Q}$ must be an integer. But clearly $\frac{2P}{Q} = \frac{2(k^2+k-1)}{k^2+k} = 2 - \frac{2}{k^2+k}$ is not an integer since $k \geq 2$. Therefore I is not ambiguous. □

3. Corresponding indefinite quadratic forms F_{I_1}, F_{I_2} and F_I.

In the previous section, we consider some properties of I_1, I_2 and also their product I. In this section, we obtain some properties of indefinite quadratic forms F_{I_1}, F_{I_2} and F_I which correspond to I_1, I_2 and I, respectively. For the ideals I_1 and I_2, we have the following indefinite quadratic forms

$$F_{I_1} = (k, 2k-2, -3) \quad \text{and} \quad F_{I_2} = (k+1, 2k, -1) \tag{3.1}$$

by (1.5). Hence we can give the following theorem.

Theorem 3.1. *Let F_{I_1} and F_{I_2} be the quadratic forms in* (3.1). *Then*
(1) *F_{I_1} is reduced and ambiguous for every $k \geq 2$.*
(2) *If $k = 3t+1$ for $t \geq 1$, then the cycle of F_{I_1} is*

$$F_{I_1^0} = (3t+1, 6t, -3) \sim F_{I_1^1} = (3, 6t, -3t-1) \sim$$
$$F_{I_1^2} = (3t+1, 2, -3t-2) \sim F_{I_1^3} = (3t+2, 6t+2, -1) \sim$$
$$F_{I_1^4} = (1, 6t+2, -3t-2) \sim F_{I_1^5} = (3t+2, 2, -3t-1)$$

of length 6.

(3) F_{I_2} *is reduced and ambiguous for every* $k \geq 2$.
(4) *If* $k = 3t+1$ *for* $t \geq 1$, *then the cycle of* F_{I_2} *is*

$$\begin{aligned}
&F_{I_2^0} = (3t+2, 6t+2, -1) \sim F_{I_2^1} = (1, 6t+2, -3t-2) \sim \\
&F_{I_2^2} = (3t+2, 2, -3t-1) \sim F_{I_2^3} = (3t+1, 6t, -3) \sim \\
&F_{I_2^4} = (3, 6t, -3t-1) \sim F_{I_2^5} = (3t+1, 2, -3t-2)
\end{aligned}$$

of length 6.

Proof. (1) Note that $k > 0$. So $4k+4 > -8k+4 \Leftrightarrow 4k^2+4k+4 > 4k^2-8k+4 \Leftrightarrow \sqrt{4k^2+4k+4} > 2k-2 \Leftrightarrow \sqrt{\Delta} > b_1$. Further $4k+4 > 0$ since $k \geq 2$. So $4k+4 > 0 \Leftrightarrow 4k^2+4k+4 > 4k^2 \Leftrightarrow \sqrt{4k^2+4k+4} > 2k$, that is, $\sqrt{4k^2+4k+4} - 2k > 0$. On the other hand $3k-5 > 0$. So

$$\begin{aligned}
4k(3k-5) > 0 &\Leftrightarrow 12k^2 - 20k + 4 > 4 \\
&\Leftrightarrow 16k^2 - 20k + 4 > 4k^2 + 4 \\
&\Leftrightarrow 4k - 2 > \sqrt{4k^2+4k+4} \\
&\Leftrightarrow 2k - 2 > |\sqrt{4k^2+4k+4} - 2k| \\
&\Leftrightarrow b_1 > |\sqrt{\Delta} - 2|a_1||.
\end{aligned}$$

Hence we get $\left|\sqrt{\Delta} - 2|a_1|\right| < b_1 < \sqrt{\Delta}$, that is, F_{I_1} is reduced by (1.1).

Now let $g = [p;q;u;v] \in \overline{\Gamma}$. Then for $F_{I_1} = (k, 2k-2, -3)$, we have

$$\begin{aligned}
kp^2 + (2k-2)pq - 3q^2 &= k \\
2kpu + (2k-2)pv + (2k-2)uq - 6qv &= 2k-2 \qquad (3.2)\\
ku^2 + (2k-2)uv - 3v^2 &= -3
\end{aligned}$$

by (1.2). It is easily seen that (3.2) has a solution for $p = 1$, $q = 0$, $u = 0$ and $v = -1$, that is, $gF_{I_1} = F_{I_1}$ for $g = [1;0;0;-1] \in \overline{\Gamma}$. Note that $\det g = -1$. Therefore, F_{I_1} is improperly equivalent to itself and hence is ambiguous.

(2) Let $F_{I_1^0} = (3t+1, 6t, -3)$. Then by (1.3), we get $s_0 = 2t$ and hence $F_{I_1^1} = (3, 6t, -3t-1)$. Similarly we can obtain Table 3.

So the cycle of F_{I_1} is $F_{I_1^0} = (3t+1, 6t, -3) \sim F_{I_1^1} = (3, 6t, -3t-1) \sim F_{I_1^2} = (3t+1, 2, -3t-2) \sim F_{I_1^3} = (3t+2, 6t+2, -1) \sim F_{I_1^4} = (1, 6t+2, -3t-2) \sim F_{I_1^5} = (3t+2, 2, -3t-1)$.

i	0	1	2	3	4	5
a_i	$3t+1$	3	$3t+1$	$3t+2$	1	$3t+2$
b_i	$6t$	$6t$	2	$6t+2$	$6t+2$	2
c_i	-3	$-3t-1$	$-3t-2$	-1	$-3t-2$	$-3t-1$
s_i	$2t$	1	1	$6t+2$	1	1

Table 3

(3) It can be proved in the same way that (1) was proved.

(4) Let $F_{I_2^0} = (3t+2, 6t+2, -1)$. Then we get $s_0 = 6t+2$ and hence $F_{I_2^1} = (1, 6t+2, -3t-2)$. Similarly we can obtain Table 4.

i	0	1	2	3	4	5
a_i	$3t+2$	1	$3t+2$	$3t+1$	3	$3t+1$
b_i	$6t+2$	$6t+2$	2	$6t$	$6t$	2
c_i	-1	$-3t-2$	$-3t-1$	-3	$-3t-1$	$-3t-2$
s_i	$6t+2$	1	1	$2t$	1	1

Table 4

So the cycle of F_{I_2} is $F_{I_2^0} = (3t+2, 6t+2, -1) \sim F_{I_2^1} = (1, 6t+2, -3t-2) \sim F_{I_2^2} = (3t+2, 2, -3t-1) \sim F_{I_2^3} = (3t+1, 6t, -3) \sim F_{I_2^4} = (3, 6t, -3t-1) \sim F_{I_2^5} = (3t+1, 2, -3t-2)$. □

EXAMPLE 3.2. Let $t = 5$. Then $k = 16$ and hence the cycle of $F_{I_1} = (16, 30, -3)$ is

$$F_{I_1^0} = (16,30,-3) \sim F_{I_1^1} = (3,30,-16) \sim F_{I_1^2} = (16,2,-17) \sim$$

$$F_{I_1^3} = (17,32,-1) \sim F_{I_1^4} = (1,32,-17) \sim F_{I_1^5} = (17,2,-16)$$

and the cycle of $F_{I_2} = (17, 32, -1)$ is

$$F_{I_2^0} = (17,32,-1) \sim F_{I_2^1} = (1,32,-17) \sim F_{I_2^2} = (17,2,-16) \sim$$

$$F_{I_2^3} = (16,30,-3) \sim F_{I_2^4} = (3,30,-16) \sim F_{I_2^5} = (16,2,-17).$$

From Theorem 3.1, we can give the following result.

COROLLARY 3.3. *If $k = 3t+1$ for $t \geq 1$, then all forms $F_{I_1^i}$ in the cycle of F_{I_1} and all forms $F_{I_2^i}$ in the cycle of F_{I_2} are ambiguous for $0 \leq i \leq 5$.*

Recall that $I = [k^2+k, k^2+k-1+\sqrt{k^2+k+1}]$. So the corresponding quadratic form is $F_I = (k^2+k, 2k^2+2k-2, k^2+k-3)$. Then we can give the following theorem.

Theorem 3.4. *The quadratic form F_I in is not reduced but is ambiguous.*

Proof. Note that $k^3+2k^2-2k-3>0$ since $k\geq 2$. So

$$\begin{aligned} & 4k(k^3+2k^2-2k-3)>0 \\ & \qquad \Leftrightarrow 4k^4+8k^3-8k^2-12k+(4k^2+4k+4)>4k^2+4k+4 \\ & \qquad \Leftrightarrow 4k^4+8k^3-4k^2-8k+4>4k^2+4k+4 \\ & \qquad \Leftrightarrow 2(k^2+k-1)>\sqrt{4k^2+4k+4} \\ & \qquad \Leftrightarrow b>\sqrt{\Delta} \end{aligned}$$

which is a contradiction to (1.1). Consequently, F_I is not reduced.

Now let $g=[p;q;u;v]\in\overline{\Gamma}$. Then the system of equations

$$\begin{gathered} (k^2+k)p^2+2(k^2+k-1)pq+(k^2+k-3)q^2=k^2+k \\ 2(k^2+k)pu+2(k^2+k-1)(pv+uq)+2(k^2+k-3)qv=2(k^2+k-1) \\ (k^2+k)u^2+2(k^2+k-1)uv+(k^2+k-3)v^2=k^2+k-3 \end{gathered}$$

has a solution for $p=3$, $q=-2$, $u=4$ and $v=-3$. So $gF_I=F_I$ for $g=[3;-2;4;-3]\in\overline{\Gamma}$ with $\det g=-1$. Therefore F_I is ambiguous. □

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