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HOLOMORPHIC ALMOST PERIODIC FUNCTIONS ON COVERINGS OF STEIN MANIFOLDS

ALEXANDER BRUDNYI AND DAMIR KINZEBULATOV

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Abstract. In this paper we introduce and study algebras of holomorphic almost periodic functions on regular coverings of Stein manifolds. These functions embrace two famous theories: Bohr's holomorphic almost periodic functions on tube domains and von Neumann's almost periodic functions on groups. For such algebras we obtain results on approximation by analogs of exponential polynomials, on holomorphic extension from almost periodic complex submanifolds and describe the structure of their maximal ideal spaces.

Résumé. Dans cet article on introduit et étudie des algèbres de fonctions holomorphes presque périodiques sur des revêtements reguliers des variétés de Stein. Ces fonctions unifient deux célèbres théories: les fonctions holomorphes presque périodiques sur des domaines tubulaires (d'après Bohr) et des fonctions presque périodiques sur des groupes (d'après von Neumann). Pour ces algèbres on obtient des résultats sur l'approximation par des analogues des polynômes exponentiels et sur le prolongement holomorphe des sous-variétés presque périodiques complexes. On décrit aussi la structure de leurs espaces des idéaux maximaux.

1. **Introduction.** In the 1920s H. Bohr [Bo] created his famous theory of almost periodic functions, which shortly acquired numerous applications to various areas of mathematics, from harmonic analysis to differential equations. Two branches of this theory were particularly rich on deep and interesting results: holomorphic almost periodic functions on a complex strip [Bo], [Lv] (and later on tube domains [FR]) and J. von Neumann's almost periodic functions on groups [N]. The holomorphic almost periodic functions on a complex strip $\Sigma = \mathbb{R} + i(a,b)$ arise as limits of exponential polynomials

$$z \mapsto \sum_{k=1}^{m} c_k e^{i\lambda_k z}, \quad z \in \Sigma, \quad c_k \in \mathbb{C}, \quad \lambda_k \in \mathbb{R},$$

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while the almost periodic functions on a group G can be obtained as limits of linear combinations of the form

$$t \mapsto \sum_{k=1}^{m} c_k \sigma_{ij}^k(t), \quad t \in G, \quad c_k \in \mathbb{C}, \quad \sigma^k = (\sigma_{ij}^k),$$

where σ^k $(1 \le k \le m)$ are finite-dimensional irreducible unitary representations of G.

In this paper we introduce a new class of holomorphic almost periodic functions that embraces both von Neumann's and holomorphic tube-domain theories, holomorphic almost periodic functions on a Galois covering of a complex manifold. In the case such a manifold is Stein, we obtain results on holomorphic almost periodic extension from almost periodic sets including Hartogs-type and CR extension theorems, on approximation of almost periodic functions by analogs of exponential polynomials and on the structure of maximal ideal spaces of some algebras of holomorphic almost periodic functions. For the covering being a tube domain our definition gives the classical Bohr holomorphic almost periodic functions; most of our results on holomorphic almost periodic extensions are new even in this setting.

The classical approach to holomorphic almost periodic functions on a tube domain $\mathbb{R}^n + i\Omega$, where $\Omega \subset \mathbb{R}^n$ is open and convex, relies heavily upon the fact that it is a trivial bundle with base Ω and fibre $\mathbb{R}^n$ (*cf.* the characterization of almost periodic functions in terms of their Jessen functions defined on Ω, see [JT], [Rn] and [FR], the proofs based on some results on almost periodic functions on $\mathbb{R}$, see [Ln], *etc.*). In our approach we consider the tube domain as an Abelian covering over a relatively complete Reinhardt domain (*e.g.*, over an annulus in $\mathbb{C}$ if $n = 1$) which enables us to regard holomorphic almost periodic functions as holomorphic sections of a certain holomorphic Banach vector bundle on this domain or as 'holomorphic' functions on a fibrewise Bohr compactification of the covering, a fibre bundle sharing some properties of a complex manifold.

This point of view allows to enrich the variety of techniques used in the theory of holomorphic almost periodic functions (such as Fourier analysis-type arguments for $n = 1$, *cf.* [Bo], [Ln], [Lv], arguments based on the properties of Monge–Ampère currents for $n > 1$, *cf.* [FRR], *etc.*) by the methods of Banach-valued complex analysis [Bu], [L] and by algebro-geometric methods such as Cartan Theorem B for coherent sheaves.

Similar methods based on the Stone-Čech compactification of fibres of a Galois covering of a complex manifold and on an analogous Banach vector bundle construction were developed in [Br1], [Br2], [Br3], [Br4] in connection with corona-type problems for some algebras of bounded holomorphic functions on coverings of bordered Riemann surfaces and integral representation of holomorphic functions of slow growth on coverings of Stein manifolds. The Bohr compactification $b\mathbb{R}^n + i\Omega$ of tube domain $\mathbb{R}^n + i\Omega$ was used in [Fav1], [Fav2] in the context of the problem of a characterization of zero sets of holomorphic almost periodic functions among all almost periodic divisors.

It is interesting to note that already in his monograph [Bo], H. Bohr uses equally often the aforementioned "trivial fibre bundle" and "Abelian covering" points of view on a complex strip.

Now let us recall definitions of holomorphic almost periodic functions on tube domains and of almost periodic functions on groups.

Following S. Bochner, a holomorphic function f on a tube domain $\mathbb{R}^n + i\Omega$ is called almost periodic if the family of its translates $\{z \mapsto f(z+s)\}_{s\in\mathbb{R}}$ is relatively compact in the topology of uniform convergence on tube subdomains $\mathbb{R}^n + i\Omega'$, $\Omega' \Subset \Omega$. The Frechet algebra of holomorphic almost periodic functions, endowed with the above topology, is denoted by $\mathrm{APH}(\mathbb{R}^n + i\Omega)$. Analogously, one defines holomorphic almost periodic functions on a tube domain with the open relatively compact base $\Omega^c \subset \mathbb{R}^n$ as holomorphic functions continuous in the closure $\mathbb{R}^n + i\bar{\Omega}^c$ and such that their families of translates are relatively compact in the topology of uniform convergence on $\mathbb{R}^n + i\bar{\Omega}^c$. The Banach algebra of such holomorphic almost periodic functions is denoted by $\mathrm{APH}(\mathbb{R}^n + i\Omega^c)$.

The following result, called the *approximation theorem*, is the cornerstone of Bohr's theory.

THEOREM 1.1 (H. Bohr). *The exponential polynomials*

$$\mathrm{APH}_0(\mathbb{R}^n + i\Omega) := \mathrm{span}_{\mathbb{C}}\{e^{i\langle\lambda,z\rangle} : z \in \mathbb{R}^n + i\Omega\}_{\lambda\in\mathbb{R}^n},$$

$$\mathrm{APH}_0(\mathbb{R}^n + i\Omega^c) := \mathrm{span}_{\mathbb{C}}\{e^{i\langle\lambda,z\rangle} : z \in \mathbb{R}^n + i\Omega^c\}_{\lambda\in\mathbb{R}^n}$$

are dense in $\mathrm{APH}(\mathbb{R}^n + i\Omega)$ *and* $\mathrm{APH}(\mathbb{R}^n + i\Omega^c)$*, respectively.*

Here $\langle\lambda,\cdot\rangle$ *is the linear functional on* $\mathbb{C}^n$ *corresponding to* $\lambda \in \mathbb{C}^n = (\mathbb{C}^n)^*$.

J. von Neumann's almost periodic functions on groups, originally introduced in connection with Hilbert's fifth problem on characterization of Lie groups among all topological groups, are defined as follows.

Let G be a topological group, $C_b(G)$ be the algebra of bounded continuous $\mathbb{C}$-valued functions on G endowed with sup-norm. A function $f \in C_b(G)$ is called almost periodic if its translates $\{t \mapsto f(st)\}_{s\in G}$ and $\{t \mapsto f(ts)\}_{s\in G}$ are relatively compact in $C_b(G)$. It was later proved in [Ma] that the relative compactness of either the left of the right family of translates already gives the von Neumann definition of almost periodicity.

Let $\mathrm{AP}(G) \subset C_b(G)$ be the uniform subalgebra of all almost periodic functions on G, and $\mathrm{AP}_0(G)$ be the linear hull over $\mathbb{C}$ of matrix entries of finite-dimensional irreducible unitary representations of G.

THEOREM 1.2 (J. von Neumann). $\mathrm{AP}_0(G)$ *is dense in* $\mathrm{AP}(G)$*.*

In what follows, we assume that the finite-dimensional irreducible unitary representations of G separate points of G. Following von Neumann, such groups are called *maximally almost periodic*. Equivalently, G is maximally almost periodic iff it admits a monomorphism into a compact topological group. Any residually finite group, *i.e.*, a group such that the intersection of all its finite index normal subgroups is trivial, belongs to this class. In particular, finite groups, free groups, finitely generated nilpotent groups, pure braid groups, fundamental groups of three dimensional manifolds are maximally almost periodic.

In the paper we study holomorphic almost periodic functions on a regular (*i.e.*, Galois) covering $p\colon X \to X_0$ of a Stein manifold X_0 such that the deck transformation group $\pi_1(X_0)/p_*\pi_1(X)$ is maximally almost periodic; here $\pi_1(X)$ is the fundamental group of X_0. (For definitions and basic facts of the theory of Stein manifolds, see, *e.g.*, [GR].) Our results admit generalizations to some algebras of holomorphic functions on X invariant with respect to the action of group $\pi_1(X_0)/p_*\pi_1(X)$.

The detailed proofs of the presented results are contained in [BK2].

2. Holomorphic almost periodic functions. Let X be a complex manifold with a free and properly discontinuous left holomorphic action of a discrete maximally almost periodic group G. Then the orbit space $X_0 := X/G$ is also a complex manifold and the projection $p\colon X \to X_0$ is holomorphic. Moreover, $p\colon X \to X_0$ can be viewed as a principal bundle over X_0 with discrete fibre G. In particular, if X is connected, then $p\colon X \to X_0$ is a regular covering of X_0 with deck transformation group G.

By $\mathcal{O}(X)$ we denote the algebra of holomorphic functions on X.

Definition 2.1. A function $f \in \mathcal{O}(X)$ is almost periodic if each point $x \in X$ has a G-invariant neighbourhood $W_x \Subset X$ such that the family of translates $\{z \mapsto f(g\cdot z) : z \in W_x\}_{g\in G}$ is relatively compact in the topology of uniform convergence on W_x.

This definition is a variant of the definition in [W] (with G being the group of all holomorphic automorphisms of X). It is equivalent to the following one.

Let $F_x := p^{-1}(x)$, $x \in X_0$. Since F_x is discrete and G acts on F_x freely and transitively, by Definition 2.1 we can define the algebra $\mathrm{AP}(F_x)$ of continuous almost periodic functions on F_x so that $\mathrm{AP}(F_x) \cong \mathrm{AP}(G)$ (*cf.* the above-cited result in [Ma]).

Definition 2.2. A function $f \in \mathcal{O}(X)$ is almost periodic if each point in X_0 has a neighbourhood $W_0 \Subset X_0$ such that the restriction $f|_{p^{-1}(W_0)}$ is bounded and $f|_{F_x} \in \mathrm{AP}(F_x)$ for all $x \in X_0$.

The algebra of almost periodic holomorphic functions on X, endowed with the topology of uniform convergence on subsets $p^{-1}(W_0)$, $W_0 \Subset X_0$, is denoted by $\mathcal{O}_{\mathrm{AP}}(X)$.

If X is a connected Stein manifold, then $\mathcal{O}_{\mathrm{AP}}(X)$ separates points on X. (It follows from the maximal almost periodicity of group G and Theorem 3.5 below.) Note also that if G is finite then $\mathcal{O}_{\mathrm{AP}}(X) = \mathcal{O}(X)$.

We also consider almost periodic functions on closed sets of the form $\bar{D} := p^{-1}(\bar{D}_0) \subset X$, where D_0 is a relatively compact subdomain of X_0.

DEFINITION 2.3. A function $f \in \mathcal{O}(D) \cap C(\bar{D})$ is almost periodic if it is bounded on D and $f|_{F_x} \in \mathrm{AP}(F_x)$ for all $x \in \bar{D}_0$.

The algebra of almost periodic holomorphic functions on $\bar{D}$, endowed with sup-norm $\|\cdot\|$, is denoted by $\mathcal{A}_{\mathrm{AP}}(D)$. It is immediate from the definition that $\mathcal{O}_{\mathrm{AP}}(X)$ is a Frechet algebra and $\mathcal{A}_{\mathrm{AP}}(D)$ is a Banach algebra (*i.e.*, they are complete).

In what follows, we assume that the complex manifold X is connected, X_0 is a Stein manifold (this implies that X is also Stein, see [St]) and domain $D_0 \Subset X_0$ is strictly pseudoconvex; here and below all strictly pseudoconvex domains have C^2-smooth boundary (we refer to [GR] for the corresponding definitions). Since for each point $x \in \partial D$ there exist a neighbourhood $U \subset X$ of x and a neighbourhood $U_0 \subset X_0$ of $p(x) \in \partial D_0$ such that $p|_U : U \to U_0$ is a biholomorphism, the set $D = p^{-1}(D_0)$ is strictly pseudoconvex in X.

EXAMPLE 2.4. (1) Tube domain $T := \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ with $\Omega \subset \mathbb{R}^n$ being open and convex; it can be shown that T is a pseudoconvex subset of $\mathbb{C}^n$. For

$$p(z) := (e^{2\pi i z_1}, \dots, e^{2\pi i z_n}), \quad z = (z_1, \dots, z_n) \in T,$$

we denote $T_0 := p(T) \subset \mathbb{C}^n$. Then $p\colon T \to T_0$ is a regular covering with deck transformation group $\mathbb{Z}^n$ and (open) pseudoconvex base T_0 (*i.e.*, T_0 is a Stein manifold). Analogously, we define covering $p\colon T^s \to T_0^s$, where $T^s := \mathbb{R}^n + i\Omega^c$, $\Omega^c \Subset \mathbb{R}^n$ is open and strictly convex with C^2-smooth boundary. Then T_0^s is strictly pseudoconvex in $\mathbb{C}^n$.

It is also easy to see that T_0 and T_0^s are relatively complete Reinhardt domains in $\mathbb{C}^n$ (see [S]).

Algebras $\mathrm{APH}(\mathbb{R}^n + i\Omega)$ and $\mathrm{APH}(\mathbb{R}^n + i\Omega^c)$ defined in the Introduction are isomorphic to $\mathcal{O}_{\mathrm{AP}}(T)$ and $\mathcal{A}_{\mathrm{AP}}(T^s)$, respectively. Inclusions $\mathrm{APH}(\mathbb{R}^n + i\Omega) \subset \mathcal{O}_{\mathrm{AP}}(T)$ and $\mathrm{APH}(\mathbb{R}^n + i\Omega^c) \subset \mathcal{A}_{\mathrm{AP}}(T^s)$ are obvious. The opposite inclusions are obtained from Theorem 3.1 and Example 3.3 below (see the argument in [BK1] for their proof).

(2) Let X_0 be a non-compact Riemann surface, and let $p\colon X \to X_0$ be a regular covering with a maximally almost periodic deck transformation group G (*e.g.*, X_0 is hyperbolic and $X = \mathbb{D}$, the open unit disk, is its universal covering; then $G = \pi_1(X_0)$ is, not necessarily finitely generated, free group). The functions in $\mathcal{O}_{\mathrm{AP}}(X)$ arise, *e.g.*, as linear combinations over $\mathbb{C}$ of matrix entries of fundamental solutions of certain linear differential equations on X. Indeed, a unitary representation $\sigma\colon G \to U_n$ can be obtained as the monodromy of the system $dF = \omega F$ on X_0, where ω is a holomorphic 1-form on X_0 with values in the space of $n \times n$ complex matrices $M_n(\mathbb{C})$ (see, *e.g.*, [For]). In particular, the pulled back system $dF = (p^*\omega)F$ on X admits a global solution $F \in \mathcal{O}\big(X, \mathrm{GL}_n(\mathbb{C})\big)$ such that $F \circ g^{-1} = F\sigma(g)$ $(g \in G)$. By definition, a linear combination of matrix entries of F is an element of $\mathcal{O}_{\mathrm{AP}}(X)$.

3. Approximation: Extension from periodic sets.

3.1. Approximation theorems. Let R_G be the set of finite dimensional irreducible unitary representations of group G. For $\sigma \in R_G$ by $\mathcal{O}_\sigma(X)$ and $\mathcal{A}_\sigma(D)$ we denote the $\mathbb{C}$-linear hulls of coordinates of vector-valued functions f in $\mathcal{O}(X, \mathbb{C}^n)$ and $\mathcal{A}(D, \mathbb{C}^n)$, respectively, having the property that $f \circ g = \sigma(g) f$ for all $g \in G$. Further, let $\mathcal{O}_0(X)$ and $\mathcal{A}_0(D)$ be $\mathbb{C}$-linear hulls of spaces $\mathcal{O}_\sigma(X)$ and $\mathcal{A}_\sigma(D)$, respectively, with σ varying over the set R_G.

THEOREM 3.1. *The following statements are true:*
(1) $\mathcal{O}_0(X)$ *is dense* $\mathcal{O}_{\mathrm{AP}}(X)$*,*
(2) $\mathcal{A}_0(D)$ *is dense* $\mathcal{A}_{\mathrm{AP}}(D)$*.*

In the case of almost periodic holomorphic functions on a tube domain (*cf.* Example 2.4(1)) Theorem 3.1 implies Theorem 1.1, see Example 3.3.

In fact, for algebra $\mathcal{A}_{\mathrm{AP}}(D)$ a stronger version of the approximation theorem is valid.

Recall that a Banach space A is said to have the *approximation property* if for any $\varepsilon > 0$ and any relatively compact subset $K \subset A$ there exists a finite rank bounded linear (*approximating*) operator $S \in \mathcal{L}(A, A)$ such that $\|x - Sx\|_A < \varepsilon$ for all $x \in K$.

For example, algebra $\mathrm{AP}(G)$ has the approximation property with approximating operators in $\mathcal{L}\big(\mathrm{AP}(G), \mathrm{AP}_0(G)\big)$.

If T is a compact Hausdorff topological space, A is a closed subspace of $C(T)$, B is a Banach space (here and below all Banach spaces are complex) and $A_B \subset C_B(T)$ is the space of continuous B-valued functions f such that for any $\varphi \in B^*$ one has $\varphi(f) \in A$, then A has the approximation property if and only if $B \otimes A$ is dense in A_B, see [G].

THEOREM 3.2. $\mathcal{A}_{\mathrm{AP}}(D)$ *has the approximation property with approximating operators in* $\mathcal{L}\big(\mathcal{A}_{\mathrm{AP}}(D), \mathcal{A}_0(D)\big)$*.*

EXAMPLE 3.3. (1) Consider the covering $p\colon T \to T_0$ of Example 2.4(1), where T is a tube domain. Let us show that the subspace of exponential polynomials

$$z \mapsto \sum_{k=1}^{m} c_k e^{i\langle \lambda_k, z\rangle}, \quad z \in T, \quad c_k \in \mathbb{C}, \quad \lambda_k \in \mathbb{R}^n,$$

is dense in $\mathcal{O}_0(T)$ (a similar statement with an analogous proof is valid for algebra $\mathcal{A}_{\mathrm{AP}}(D)$). Indeed, since group $G := \mathbb{Z}^n$ is free Abelian, the irreducible unitary representations of G are one-dimensional and given by the formulas $\sigma_\lambda(l) := e^{i\langle \lambda, l\rangle}$, $l \in \mathbb{Z}^n$ for $\lambda \in \mathbb{R}^n$. We define

$$e_\lambda(z) := e^{i\langle \lambda, z\rangle}, \quad z \in T.$$

Then $e_\lambda \in \mathcal{O}_{\sigma_\lambda}(T)$ and for any $h \in \mathcal{O}_{\sigma_\lambda}(T)$ there exists a function $\tilde{h} \in \mathcal{O}(T_0)$ such that $h/e_\lambda = p^*\tilde{h}$. By definition, for an $f \in \mathcal{O}_0(T)$ there exist functions $h_k \in \mathcal{O}_{\sigma_{\lambda_k}}(T)$ $(1 \le k \le m)$ such that

$$f(z) = \sum_{k=1}^{m} c_k h_k(z), \quad z \in T,\ c_k \in \mathbb{C}.$$

Therefore,

$$f(z) = \sum_{k=1}^{m} c_k (p^*\tilde{h}_k)(z) e_{\lambda_k}(z), \quad z \in T. \tag{3.1}$$

Since the base T_0 of the covering is a relatively complete Reinhardt domain, functions $\tilde{h}_k$ admit expansions into Laurent series (see, *e.g.*, [S])

$$\tilde{h}_k(z) = \sum_{|\ell|=-\infty}^{\infty} b_\ell z^\ell, \quad z \in T_0, \quad b_\ell \in \mathbb{C},$$

where $\ell = (\ell_1, \dots, \ell_n)$, $|\ell| = \ell_1 + \cdots + \ell_n$. Therefore, $p^*\tilde{h}_k$ admit approximations by finite sums

$$\sum_{|\ell|=-M}^{M} b_j e^{2\pi i \langle \ell, z\rangle}, \quad z \in T,$$

uniformly on subsets $p^{-1}(W_0) \subset T$, $W_0 \Subset T_0$. Together with (3.1) this implies the required.

(2) In the setting of Example 2.4(2), given an irreducible unitary representation $\sigma\colon G \to U_n$ we consider a function $F \in \mathcal{O}\big(X, \mathrm{GL}_n(\mathbb{C})\big)$ such that $F \circ g^{-1} = F\sigma(g)$, $g \in G$, and a function $f \in \mathcal{O}(X, \mathbb{C}^n)$ such that $f \circ g = \sigma(g) f$ (*i.e.*, with coordinates lying in $\mathcal{O}_\sigma(X)$). It follows that there exists a function $\tilde{f} \in \mathcal{O}(X_0, \mathbb{C}^n)$ such that $Ff = p^*\tilde{f}$. Therefore all functions in $\mathcal{O}_\sigma(X)$ are obtained as $\mathbb{C}$-linear combinations of coordinates of vector-valued functions of the form $F^{-1}(p^*\tilde{f})$ with $\tilde{f} \in \mathcal{O}(X_0, \mathbb{C}^n)$. Note that entries of F^{-1} are the same as for $(F^t)^{-1}$ satisfying $(F^t)^{-1} \circ g^{-1} = (F^t)^{-1}\big(\sigma(g)^t\big)^{-1}$, $g \in G$, where $(\sigma^t)^{-1}$ is an irreducible unitary representation of G as well.

Let $[R_G]$ be the set of equivalence classes of irreducible representations $G \to U_n$, $n \in \mathbb{N}$. For each class $[\sigma] \in [R_G]$ representing $\sigma\colon G \to U_n$ we fix an $F_{[\sigma]} \in \mathcal{O}\big(X, \mathrm{GL}_n(\mathbb{C})\big)$ satisfying $F_{[\sigma]} \circ g^{-1} = F_{[\sigma]}\sigma(g)$, $g \in G$. Further, consider an at most countable subset $A \subset \mathcal{O}(X_0)$ such that the complex algebra generated by A is dense in $\mathcal{O}(X_0)$ (*e.g.*, in the case $X_0 = \{z \in \mathbb{C} : |w(z)| < 1$, w is holomorphic in a neighbourhood of $X_0\} \Subset \mathbb{C}$ is an analytic polyhedron, we may take $A = \{w\}$). Then the products of matrix entries of $F_{[\sigma]}$, $[\sigma] \in [R_G]$, with functions in p^*A may be viewed as analogs of exponential polynomials of Example 3.3(1). The $\mathbb{C}$-linear hull generated by these functions is dense in $\mathcal{O}_\sigma(X)$ (A similar result is valid for $\mathcal{A}_{\mathrm{AP}}(D)$.)

3.2. Extension from periodic sets. A subset $Y := p^{-1}(Y_0)$, $Y_0 \subset X_0$, will be called *periodic*.

We consider a complete path metric d on X_0 with an associated $(1,1)$-form obtained, *e.g.*, as the restriction of the Hermitian $(1,1)$-form on $\mathbb{C}^{2n+1}$ to X_0, where X_0 is regarded as a closed submanifold of $\mathbb{C}^{2n+1}$, see the embedding theorem for Stein manifolds in [GR]. For a simply connected subset $W_0 \subset X_0$ we naturally identify $p^{-1}(W_0)$ with $W_0 \times G$.

Definition 3.4. A function $f \in C(Y)$ is called almost periodic if it is bounded and uniformly continuous on subsets $p^{-1}(W_0) = W_0 \times G$ for simply connected $W_0 \Subset Y_0$ with respect to pseudo-metric $\tilde{d}\big((x_1, g), (x_2, g)\big) := d(x_1, x_2)$ on $p^{-1}(W_0)$, and $f|_{p^{-1}(x)} \in \mathrm{AP}(F_x)$ for all $x \in Y_0$.

If, in addition Y_0 is a complex analytic subspace of X_0 and $f \in \mathcal{O}(Y)$ satisfies the above properties, it is called holomorphic almost periodic (in this case uniform continuity with respect to $\tilde{d}$ can be derived from uniform boundedness of f).

By $C_{\mathrm{AP}}(Y)$ and $\mathcal{O}_{\mathrm{AP}}(Y)$ we denote the algebras of continuous and holomorphic almost periodic functions on Y.

Theorem 3.5. *Let M_0 be a closed complex submanifold of X_0 (so that $M := p^{-1}(M_0)$ is a closed complex submanifold of X).*

(1) *For every $f \in \mathcal{O}_{\mathrm{AP}}(M)$ there exists a function $F \in \mathcal{O}_{\mathrm{AP}}(X)$ such that $F|_M = f$.*

Suppose that M_0 intersects transversely the boundary of a strictly pseudoconvex domain D_0. Let $\mathcal{A}_{\mathrm{AP}}(D \cap M)$, $D = p^{-1}(D_0)$, be the algebra of holomorphic almost periodic functions in $D \cap M$ continuous on $\bar{D} \cap M$ endowed with the sup-*norm.*

(2) *There exists a bounded linear operator $E_M^D \in \mathcal{L}\big(\mathcal{A}_{\mathrm{AP}}(D \cap M), \mathcal{A}_{\mathrm{AP}}(D)\big)$ such that*

$$(E_M^D f)|_{D\cap M} = f \quad \text{for all } f \in \mathcal{A}_{\mathrm{AP}}(D \cap M).$$

Theorem 3.6. *Let $S_0 \subset X_0$, $S := p^{-1}(S_0)$, be a piecewise smooth real hypersurface. Suppose $f \in \mathcal{O}(X)$ is such that $f|_S \in C_{\mathrm{AP}}(S)$ and f is bounded on subsets $W = p^{-1}(W_0)$, $W_0 \Subset X_0$. Then $f \in \mathcal{O}_{\mathrm{AP}}(X)$.*

The classical result of Bohr's theory of holomorphic almost periodic functions on a strip $\Sigma := \mathbb{R} + i(a, b) \subset \mathbb{C}$ states that if a function $f \in \mathcal{O}(\Sigma)$ is bounded on closed substrips and is almost periodic on a horizontal line, then $f \in \mathcal{O}_{\mathrm{AP}}(\Sigma)$ (see, *e.g.*, [Ln]). It follows from Theorem 3.6 that the horizontal line can be replaced here by any real periodic piecewise smooth curve. In fact, a similar result holds if S_0 is a uniqueness set for the space $\mathcal{O}(X_0)$, *e.g.*, if S_0 is a generic CR submanifold of X_0 of real codimension $\le n$.

We illustrate Theorem 3.6 by the following result on (bounded) holomorphic extension of almost periodic CR functions.

Suppose $n = \dim(X_0) \geq 2$. Let $S_0 \subset X_0$ be a C^3-smooth real hypersurface, *i.e.*, the zero set of a function $\rho \in C^3(Q_0)$, $Q_0 \subset X_0$ is open, $\nabla\rho \neq 0$ on Q_0. Denote $S := p^{-1}(S_0)$. We say that a function $f \in C_{\mathrm{AP}}(S)$ satisfies tangential CR equations on S if

$$\int_S f\bar{\partial}_S\omega = 0$$

for all C^∞-smooth $(n, n-2)$-forms ω on X having compact supports in Q_0; here $\bar{\partial}_S$ is the tangential $\bar{\partial}$-operator. If $f \in C_{\mathrm{AP}}(S) \cap C^1(S)$, then the tangential CR equations are equivalent to the equation $\bar{\partial}\tilde{f} \wedge \bar{\partial}p^*\rho = 0$ on S; here $\tilde{f}$ is a C^1-smooth extension of f to a neighbourhood of S.

Recall that in local complex coordinates $(z_1, \ldots, z_n)$ on a neighbourhood of $s_0 \in S_0$ in X_0 the *Levi form* $L_{s_0,\rho}$ of ρ at s_0 is given by the formula

$$L_{s_0,\rho}(w, \bar{w}) = \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \bar{\partial} z_k}(s_0) w_j \bar{w}_k, \quad w = \sum_{k=1}^{n} w_k \frac{\partial}{\partial z_k} \in T^{1,0}_{s_0}(S_0),$$

where

$$T^{1,0}_{s_0}(S_0) = \Big\{ w = (w_1, \ldots, w_n) \in T^{1,0}_{s_0}(\mathbb{C}^n) : \sum_{k=1}^{n} \frac{\partial\rho}{\partial z_k}(s_0) w_k = 0 \Big\}$$

is the complex tangent space of S_0 at s_0. Note that $L_{s_0,\rho} = aL_{s_0,\tilde{\rho}}$, $a \in \mathbb{R} \setminus \{0\}$, for a different defining function $\tilde{\rho}$ of S_0 (we refer to [Bog] for the corresponding definitions and results). In the following result, independent of the choice of the defining function ρ of S_0, we refer to a form $L_{s_0,\rho}$ as the Levi form for S_0 at s_0.

Corollary 3.7. *Suppose that the Levi form for S_0 at s_0 has eigenvalues of opposite signs for all $s_0 \in S_0$. Let $f \in C_{\mathrm{AP}}(S)$ satisfy the tangential CR equations on S. Then there exists an open set $V_0 \subset X_0$ containing S_0 and a function $F \in \mathcal{O}_{\mathrm{AP}}(V)$, where $V := p^{-1}(V_0)$, such that $F|_S = f$.*

By the uniqueness property of holomorphic functions it suffices to prove this statement locally. Now, for a simply connected coordinate neighbourhood $U_0 \subset X_0$ of a point $s_0 \in S_0$ the hypothesis of the corollary implies (*cf.* [Bog, Ch. 15]) that there exists a neighbourhood $V_0 \Subset U_0$, $V_0 \cap S_0 \Subset S_0$, of s_0 such that any function $h \in C(S_0 \cap U_0)$ satisfying $\int_{S_0} h\bar{\partial}_{S_0}\omega_0 = 0$ for all C^∞-smooth $(n, n-2)$-forms ω_0 on X_0 having compact supports in U_0 admits a (unique) extension $H \in \mathcal{O}(V)$ so that

$$\sup_{V_0} |H| \leq \sup_{S_0 \cap V_0} |h|.$$

We identify naturally $p^{-1}(U_0)$ with $\bigsqcup_{g\in G} U_0 \times \{g\}$ and apply the above result to CR functions $f_g := f|_{S\cap(U_0\times\{g\})}$, $g \in G$ (here each f_g can be regarded, in the above identification, as a continuous CR function on $S_0 \cap U_0$). Then there exists a (unique) function $F \in \mathcal{O}(V)$, $V := p^{-1}(V_0)$, such that $F|_{S\cap V} = f|_{S\cap V}$ and

$$\sup_V |F| \leq \sup_{S\cap V} |f|. \tag{3.2}$$

From Theorem 3.6 we then obtain that $F \in \mathcal{O}_{\mathrm{AP}}(V)$ as required.

We use a result in [Br3] and an argument similar to that in the proof of Theorem 3.6 to obtain the following Hartogs-type theorem.

THEOREM 3.8. *Let $n = \dim(X_0) \geq 2$, and $D_0 \Subset X_0$ be a subdomain with a connected piecewise-smooth boundary S_0; then $S := p^{-1}(S_0)$ is the boundary of D. Suppose that $f \in C_{\mathrm{AP}}(S)$ satisfies tangential CR equations on S. Then there exists a function $F \in \mathcal{A}_{\mathrm{AP}}(D)$ such that $F|_S = f$.*

The result can be applied, *e.g.*, to tube domains $T := \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$, $n \geq 2$, where $\Omega \Subset \mathbb{R}^n$ is a domain with piecewise-smooth boundary $\partial\Omega$, and to continuous almost periodic functions on the boundary $\partial T := \mathbb{R}^n + i\partial\Omega$ of T satisfying tangential CR equations there. Then every such a function admits a continuous extension to a function from $\mathcal{A}_{\mathrm{AP}}(T)$.

The results of Section 3.2 are valid for sufficiently small almost periodic perturbations of periodic sets of coverings $p\colon T \to T_0$ of Example 2.4(1), see [BK2] for details.

3.3. Banach vector bundle construction. Our proofs of Theorems 3.1–3.8 rely on the Banach vector bundle construction which we present for algebra $\mathcal{O}_{\mathrm{AP}}(X)$ (the construction for $\mathcal{A}_{\mathrm{AP}}(D)$ is analogous).

The regular covering $p\colon X \to X_0$ determines a principal fibre bundle with structure group G. Indeed, given a cover $\{U_\gamma\}_{\gamma\in\Gamma}$ of X_0 by simply connected open sets $U_\gamma \Subset X_0$ there exists a locally constant cocycle $c = \{c_{\delta\gamma}\colon U_\gamma \cap U_\delta \to G : \gamma, \delta \in \Gamma\}$ so that the covering $p\colon X \to X_0$ can be obtained from the disjoint union $\bigsqcup_{\gamma\in\Gamma} U_\gamma \times G$ by the identification of $(x, g) \in U_\delta \times G$ with $(x, g \cdot c_{\delta\gamma}(x)) \in U_\gamma \times G$ for all $x \in U_\gamma \cap U_\delta$, $\gamma, \delta \in \Gamma$; here the projection p is induced by projections $U_\gamma \times G \to U_\gamma$.

Further, $\mathrm{AP}(G)$ is isomorphic to $C(bG)$, where bG is the *Bohr compactification* of G. Since G is maximally almost periodic, it is a dense subgroup of the group bG, see Section 4.1. We define a holomorphic Banach vector bundle $\tilde{p}\colon CX \to X_0$ with fibre $C(bG)$ associated to the principal fibre bundle $p\colon X \to X_0$ as that obtained from the disjoint union $\bigsqcup_{\gamma\in\Gamma} U_\gamma \times C(bG)$ by the identification of $(x, f(\omega)) \in U_\delta \times C(bG)$ with $(x, f(\omega \cdot c_{\delta\gamma}(x))) \in U_\gamma \times C(bG)$ for all $x \in U_\gamma \cap U_\delta$, $\gamma, \delta \in \Gamma$; here the projection $\tilde{p}$ is induced by projections $U_\gamma \times C(bG) \to U_\gamma$.

Let $\mathcal{O}(CX)$ be the set of (global) holomorphic sections of CX. It forms a Frechet algebra with respect to the usual pointwise operations and the topology of uniform convergence on compact subsets of X_0. It follows that algebras $\mathcal{O}_{\mathrm{AP}}(X)$ and $\mathcal{O}(CX)$ are isomorphic.

Our proofs of Theorems 3.5–3.8 use a theorem in [L] asserting, in particular, that there exist holomorphic Banach vector bundles $p_1\colon E_1 \to X_0$ and $p_2\colon E_2 \to X_0$ with fibres B_1 and B_2, respectively, such that $E_2 = E_1 \oplus CX$ (the Whitney sum) and E_2 is holomorphically trivial, *i.e.*, $E_2 \cong X_0 \times B_2$. Thus, any holomorphic section of E_2 can be naturally identified with a B_2-valued holomorphic function on X_0. By $q\colon E_2 \to CX$ and $\iota\colon CX \to E_2$ we denote the corresponding quotient and embedding homomorphisms of the bundles so that $q \circ \iota = \mathrm{Id}$.

For instance, to prove Theorem 3.5(1) observe that the algebra $\mathcal{O}_{\mathrm{AP}}(M)$ is isomorphic to the algebra $\mathcal{O}(CX)|_{M_0}$ of holomorphic sections of the bundle CX over M_0. Given $\mathcal{O}(CX)|_{M_0}$ consider the subspace $\iota\big(\mathcal{O}(CX)|_{M_0}\big)$ of B_2-valued holomorphic functions on M_0 and apply to it the Banach-valued version of the extension result from [HL] asserting existence of a linear continuous extension operator $\tilde{E}_{M_0} \colon \mathcal{O}(M_0, B_2) \to \mathcal{O}(X_0, B_2)$. Finally, we define $E_M := q \circ \tilde{E}_{M_0} \circ \iota$.

4**. Extension from almost periodic complex submanifolds.** This section is devoted to holomorphic almost periodic extensions from smooth almost periodic complex submanifolds. We present an approach based on the fibrewise Bohr compactification bX of the covering $p \colon X \to X_0$ and on an analogue of the Cartan theorem B for certain sheaves on bX.

4.1. Fibrewise Bohr compactification of the covering.

4.1.1. Preliminaries on the Bohr compactification of a group.

The Bohr compactification bG of a topological group G is a compact topological group together with a homomorphism $j : G \hookrightarrow bG$ determined by the universal property

$$\begin{array}{ccc} G & \xrightarrow{\ j\ } & bG \\ {\scriptstyle \nu}\big\downarrow & \swarrow{\scriptstyle \eta} & \\ H & & \end{array}$$

here H is a compact topological group, and ν is a (continuous) homomorphism. Applying this to $H := U_n$, $n \geq 1$, we obtain that G is maximally almost periodic iff j is an embedding.

The universal property implies that there exists a bijection between finite-dimensional irreducible unitary representations of G and bG. In turn, the Peter–Weyl theorem for $C(bG)$ and von Neumann's approximation theorem for $\mathrm{AP}(G)$ (Theorem 1.2) yield that $\mathrm{AP}(G) \cong C(bG)$. Thus bG is homeomorphic to the maximal ideal space of algebra $\mathrm{AP}(G)$, and $j(G)$ is dense in bG.

The right action of G on itself extends uniquely to a (right) action R of G on bG given by the formula $R(g)(\omega) := \omega \cdot j(g^{-1})$, $\omega \in bG$, $g \in G$.

Let $\Upsilon \subset bG$ be a set of representatives of equivalence classes $bG/j(G)$. Each element $\xi \in \Upsilon$ determines a homomorphism (monomorphism if G is maximally almost periodic) $j_\xi \colon G \to bG$, $j_\xi(g) := \xi \cdot j(g^{-1})$, $g \in G$. In particular, if $\xi = 1$, then $j_\xi = j$. Since $j(G)$ is dense in bG and bG is a group, $j_\xi(G)$ is dense in bG for all $\xi \in \Upsilon$; it is also clear that $j_\xi(G) \cap j_{\xi'}(G) = \varnothing$ for $\xi \neq \xi'$, and bG is covered by subsets $j_\xi(G)$, $\xi \in \Upsilon$.

4.1.2. Fibrewise Bohr compactification of the covering.

We retain notation of Section 3.3. Consider the fibre bundle $p_b\colon bX \to X_0$ with fibre bG associated to the bundle $p\colon X \to X_0$. By definition, bX is obtained from the disjoint union $\bigsqcup_{\gamma\in\Gamma} U_\gamma \times bG$ by the identification of $(x,\omega) \in U_\gamma \times bG$ with $\big(x, \omega\cdot c_{\delta\gamma}(x)\big) \in U_\delta \times bG$. The bundle bX will be called the fibrewise Bohr compactification of X.

Let $\xi \in \Upsilon$. Each embedding j_ξ (recall that G is maximally almost periodic) induces local embeddings $U_\gamma \times G \hookrightarrow U_\gamma \times bG$ which, in turn, induce a global embedding

$$\iota_\xi\colon X \hookrightarrow bX.$$

Thus $bX = \bigsqcup_{\xi\in\Upsilon} \iota_\xi(X)$ and each $\iota_\xi(X)$ is dense in bX. We define the (fibrewise) Bohr compactification $b\bar{D}$ of the covering $p\colon \bar{D} \to \bar{D}_0$, $D_0 \Subset X_0$ is a domain, analogously. Spaces bX and $b\bar{D}$ with a strictly pseudoconvex D_0 are maximal ideal spaces of algebras $\mathcal{O}_{\rm AP}(X)$ and $\mathcal{A}_{\rm AP}(D)$, respectively, see Section 5.

Definition 4.1. A function $f \in C(bX)$ is called holomorphic if $\iota_\xi^* f \in \mathcal{O}(X)$ for all $\xi \in \Upsilon$. We denote by $\mathcal{O}(bX)$ the algebra of holomorphic functions on bX endowed with the topology of uniform convergence on compact subsets of bX.

A function $f \in C(b\bar{D})$ is called holomorphic if $\iota_\xi^* f \in \mathcal{O}(D) \cap C(\bar{D})$ for all $\xi \in \Upsilon$. The algebra of these functions equipped with sup-norm is denoted by $\mathcal{A}(bD)$.

In fact, we prove that it suffices to require that the above f satisfies $\iota_\xi^* f \in \mathcal{O}(X)$ or $\mathcal{O}(D) \cap C(\bar{D})$ for only one $\xi \in \Upsilon$ to get $f \in \mathcal{O}(bX)$ or $\mathcal{A}(bD)$.

Theorem 4.2. *$\mathcal{O}_{\rm AP}(X) \cong \mathcal{O}(bX)$ and $\mathcal{A}_{\rm AP}(D) \cong \mathcal{A}(bD)$.*

4.1.3. Holomorphic sheaves and holomorphic maps on bX.

Since each ι_ξ is a continuous map, $\iota_\xi^{-1}(U) \subset X$ is open for an open $U \subset bX$. For such U a function $f \in C(U)$ is called *holomorphic* if $\iota_\xi^* f \in \mathcal{O}\big(\iota_\xi^{-1}(U)\big)$ for all $\xi \in \Upsilon$. Let $\mathcal{O}(U)$ denote the algebra of holomorphic on U functions. Clearly a function $f \in C(bX)$ belongs to $\mathcal{O}(bX)$ if and only if each point in bX has a neighbourhood U such that $f|_U \in \mathcal{O}(U)$.

For $U \subset bX$ open, by ${}_U\mathcal{O}$ we denote the sheaf of germs of holomorphic functions on U.

Given a sheaf of modules $\mathcal{F}$ on bX over the sheaf of rings ${}_{bX}\mathcal{O}$ by $H^k(bX,\mathcal{F})$, $k \ge 0$, we denote the Čech cohomology groups of bX with values in $\mathcal{F}$. As usual, $H^0(bX,\mathcal{F})$ is naturally identified with the module $\Gamma(bX,\mathcal{F})$ of global sections of $\mathcal{F}$. We say that the sheaf $\mathcal{F}$ is *coherent* if each point in bX has a neighbourhood of the form $U := p_b^{-1}(U_0)$ with $U_0 \subset X_0$ open over which there exists an exact sequence of sheaves

$$0 \longrightarrow ({}_{bX}\mathcal{O})^{k_m}|_U \longrightarrow \cdots \longrightarrow ({}_{bX}\mathcal{O})^{k_1}|_U \longrightarrow ({}_{bX}\mathcal{O})^{k_0}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

Theorem 4.3 (Cartan Theorem B). *If $\mathcal{F}$ is coherent, then $H^k(bX, \mathcal{F}) = 0$, $k \geq 1$.*

Let M be a complex manifold. A map $G \in C(M, bX)$ is called holomorphic if the inverse image sheaf $G^*{}_{bX}\mathcal{O}$ is a subsheaf of ${}_M\mathcal{O}$. The collection of all holomorphic maps $M \to bX$ is denoted by $\mathcal{O}(M, bX)$.

The next result reflects the fact that the holomorphic structure of bX is concentrated in 'horizontal' layers $\iota_\xi(X)$ only.

Theorem 4.4. *Suppose that M is a connected complex manifold. For each $G \in \mathcal{O}(M, bX)$ there exists $\xi \in \Upsilon$ such that $G(M) \subset \iota_\xi(X)$.*

Let U, V be open subsets of bX. A map $F \in C(U, V)$ is called holomorphic if the inverse image sheaf $F^*{}_V\mathcal{O}$ is a subsheaf of ${}_U\mathcal{O}$. We denote the collection of holomorphic maps $U \to V$ by $\mathcal{O}(U, V)$.

A holomorphic map $F \in \mathcal{O}(U, V)$ is *biholomorphic* if it possesses a holomorphic inverse.

4.2. Extension from almost periodic complex submanifolds.

Definition 4.5. A closed subset $Y \subset bX$ is called an *almost periodic complex submanifold* of bX of codimension k if for each point $y \in Y$ there exist its neighbourhood $U \subset bX$ biholomorphic to $U_0 \times K$ by means of a map $\varphi \in \mathcal{O}(U, U_0 \times K)$ with $U_0 \subset X_0$ and $K \subset bG$ open, and holomorphic functions $f_1, \dots, f_k \in \mathcal{O}(U)$ such that
(1) $Y \cap U := \{x \in U : f_1(x) = \cdots = f_k(x) = 0\}$;
(2) for every $\omega \in K$ the rank of the holomorphic map

$$\big((f_1 \circ \varphi^{-1})(\,\cdot\,, \omega), \dots, (f_k \circ \varphi^{-1})(\,\cdot\,, \omega)\big) : U_0 \to \mathbb{C}^k$$

is k at each point of U_0.

Any almost periodic complex submanifold $Y \subset bX$ of codimension k satisfies the following properties:
(1) if for some $\xi \in \Upsilon$ the set $\iota_\xi^{-1}\big(Y \cap \iota_\xi(X)\big)$ is non-empty, then it is a complex submanifold of X of codimension k;
(2) if for some $\xi \in \Upsilon$ the set $Y \cap \iota_\xi(X)$ is non-empty, then it is dense in Y.

The simplest example of such $Y \subset bX$ is a periodic complex manifold of codimension k. Other examples include sufficiently small almost periodic holomorphic perturbations of almost periodic complex submanifolds of codimension k and (in the case of tube domain T) finite unions of non-intersecting complex submanifolds of codimension k periodic (with respect to the action of group $\mathbb{R}^n$ on T by translations) and having different periods.

Definition 4.6. Let $Y \subset bX$ be an almost periodic complex submanifold of codimension k. A function $g \in C(Y)$ is called holomorphic if functions $g|_{\iota_\xi^{-1}(Y \cap \iota_\xi(X))}$, $\xi \in \Upsilon$, are holomorphic.

The algebra of functions holomorphic on Y is denoted by $\mathcal{O}(Y)$.

An almost periodic complex submanifold Y of bX is called *cylindrical* if for each $y \in Y$ the neighbourhood U in Definition 4.5 has the form $U := p_b^{-1}(U_0')$ for some open $U_0' \subset X_0$. In particular, if Y is the set of zeros of globally defined on bX holomorphic functions $f_1, \dots, f_k$ satisfying condition (2) of this definition, then Y is cylindrical. In turn, even a complex strip $X \Subset \mathbb{C}$ contains non-cylindrical almost periodic complex submanifolds (see details in [BK2]).

THEOREM 4.7. *Suppose that $Y \subset bX$ is an almost periodic cylindrical complex submanifold of codimension k and $g \in \mathcal{O}(Y)$. Then there exists a function $G \in \mathcal{O}(bX)$ such that $G|_Y = g$.*

Our proof of Theorem 4.7 is based on Theorem 4.3 and the fact that the sheaf of germs of holomorphic functions vanishing on such Y is coherent.

The difference between cylindrical and non-cylindrical almost periodic complex submanifolds is not yet well understood. In particular, it is not clear whether the analog of Theorem 4.7 is valid for all almost periodic complex submanifolds of bX.

5. **Maximal ideal spaces.** Let $\mathcal{M}_X$ and $\mathcal{M}_D$ be the maximal ideal spaces of algebras $\mathcal{O}_{\mathrm{AP}}(X)$ and $\mathcal{A}_{\mathrm{AP}}(D)$ with $D_0 \Subset X_0$ a strictly pseudoconvex subdomain, respectively, *i.e.*, the spaces of all non-zero continuous characters $\mathcal{O}_{\mathrm{AP}}(X) \to \mathbb{C}$ and $\mathcal{A}_{\mathrm{AP}}(D) \to \mathbb{C}$ endowed with weak* topology (in particular, if $\varphi \in \mathcal{M}_D$, then $\|\varphi\| = 1$ and $\mathcal{M}_D$ is a compact Hausdorff space). One has natural embeddings $\iota \colon X \hookrightarrow \mathcal{M}_X$, $x \mapsto \delta_x$, and $\iota_D \colon D \hookrightarrow \mathcal{M}_D$, $x \mapsto \delta_x$, where $\delta_x(f) := f(x)$, $f \in \mathcal{O}_{\mathrm{AP}}(X)$ or $\mathcal{A}_{\mathrm{AP}}(D)$. We will identify X with $\iota(X)$ and D with $\iota_D(D)$. In this setting the *corona problem* asks whether X and D are dense in $\mathcal{M}_X$ and $\mathcal{M}_D$, respectively. Using the developed technique we prove the following.

THEOREM 5.1. *$\mathcal{M}_X$ is homeomorphic to bX and $\mathcal{M}_D$ is homeomorphic to $b\bar{D}$.*

It follows from the construction of spaces bX and $b\bar{D}$ that the embeddings ι and ι_D coincide with embeddings $\iota_1 \colon X \hookrightarrow bX$ and $\iota_1 \colon D \hookrightarrow bD$ of Section 4.1 having dense images. Thus from Theorem 5.1 we obtain:

COROLLARY 5.2. *X is dense in $\mathcal{M}_X$ and D is dense in $\mathcal{M}_D$.*

As another consequence of Theorem 5.1 we obtain the following Bers-type result.

Let $X \to X_0$ and $Y \to Y_0$ be regular coverings of Stein manifolds with maximally almost periodic deck transformation groups, and let $D_0 \Subset X_0$ and $E_0 \Subset Y_0$ be strictly pseudoconvex subdomains. We set $D = p^{-1}(D_0)$, $E = p^{-1}(E_0)$.

Corollary 5.3. *Let* $\Psi_X : \mathcal{O}(bX) \to \mathcal{O}(bY)$ *and* $\Psi_D : \mathcal{A}(bD) \to \mathcal{A}(bE)$ *be continuous algebra homomorphisms. Then there exist holomorphic maps* $\psi_X \in \mathcal{O}(bY, bX)$ *and* $\psi_D \in \mathcal{O}(bE, bD)$ *such that* $\Psi_X f = \psi_X^* f$ *for* $f \in \mathcal{O}(bX)$ *and* $\Psi_D f = \psi_D^* f$ *for* $f \in \mathcal{A}(bD)$.

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Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
T2N 1N4
email: albru@math.ucalgary.ca

Department of Mathematics
University of Toronto
Toronto, Ontario
M5S 2E4
email: dkinz@math.toronto.edu