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# NUCLEARITY THROUGH ABSORBING EXTENSIONS

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Abstract. Let $\mathcal{A}$ be a unital, separable, simple $C^*$-algebra. Denote by $G := U\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$ the unitary group of the multiplier algebra of $\mathcal{A}\otimes\mathcal{K}$, given the strict topology. Then the following conditions are equivalent:

(1) $\mathcal{A}$ is a nuclear $C^*$-algebra.
(2) $G$ is an amenable topological group.
(3) $G$ is an extremely amenable topological group.
(4) The Kasparov extension of $\mathcal{A}\otimes\mathcal{K}$ is absorbing.
(5) The Lin and Kasparov extensions of $\mathcal{A}\otimes\mathcal{K}$ are approximately unitarily equivalent (with unitaries coming from $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$).
(6) The Kasparov extension of $S\mathcal{A}\otimes\mathcal{K}$ is absorbing.
(7) The suspended Lin extension and the Kasparov extension, of $S\mathcal{A}\otimes\mathcal{K}$, are approximately unitarily equivalent (with unitaries coming from $\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})$).
(8) Every purely large extension of $\mathcal{A}\otimes\mathcal{K}$ is absorbing.
(9) Every properly purely large extension of $\mathcal{A}\otimes\mathcal{K}$ is absorbing.

Résumé. Soit $\mathcal{A}$ une $C^*$-algèbre unifère, séparable et simple et soit $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ l'algèbre des multiplicateurs de $\mathcal{A}\otimes\mathcal{K}$. Dénotons par $G$ le groupe unitaire de $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ muni de la topologie stricte. Nous démontrons plusieurs caractérisations équivalentes de la nucléarité de $\mathcal{A}$. En particulier, nous prouvons l'équivalence des conditions suivantes:

(1) $\mathcal{A}$ est une $C^*$-algèbre nucléaire.
(2) $G$ est un groupe topologique moyennable.
(3) L'extension de Kasparov de $\mathcal{A}\otimes\mathcal{K}$ est absorbante.

1. **Introduction.** We say that a topological group $G$ is *amenable* if every affine continuous action of $G$ on a compact convex set has a fixed point. We note that this notion applies to non-locally compact (as well as locally compact) groups. For example, if $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space, then the unitary group of $\mathcal{H}$, given the strong operator topology, is a non-locally compact amenable topological group.

In recent years, there have been many studies of the relationship between properties of an operator algebra and amenability of an associated unitary group. De la Harpe showed that a von Neumann algebra $M$ is amenable if and only if the unitary group $U(M)$ of $M$, given the strong operator topology, is an amenable topological group ([8]). Paterson showed that a unital $C^*$-algebra $\mathcal{A}$ is nuclear if and only if its unitary group $U(\mathcal{A})$, given the relative weak topology, is

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amenable ([23]). (Giordano and Pestov have generalized these results in certain interesting directions related to the interesting property of extreme amenability (a very strong type of amenability). See [12] and [13].)

We note that the topological groups mentioned above are quite interesting since they often exhibit phenomena that never appear in the locally compact case. For example, consider the unitary group $U(\mathcal{H})$ on a separable infinite dimensional Hilbert space $\mathcal{H}$, given the strong operator topology. By de la Harpe's result ([8]), $U(\mathcal{H})$ is an amenable group (actually, it is extremely amenable!). But also by a result of Bekka ([1]), $U(\mathcal{H})$ has property $T$. This is rather interesting since, in the locally compact case, amenability and property $T$ are opposite properties. Among other things, a locally compact group that is both amenable and has property $T$ must be compact.

In this paper, we explore amenability for the unitary group of a multiplier algebra. The multiplier algebra $\mathcal{M}(\mathcal{B})$ of a $C^*$-algebra $\mathcal{B}$ is the largest $C^*$-algebra inside which the $C^*$-algebra $\mathcal{B}$ is an essential ideal. Naturally, it is a basic object in the extension theory of $\mathcal{B}$ (see, for example, [3] and [27]), and it plays an interesting role in a number of places (among other things, the classification program for simple nuclear $C^*$-algebras). Most interest is in the case that $\mathcal{B}$ is a stable $C^*$-algebra since this allows the class of extensions of $\mathcal{B}$, by a fixed $C^*$-algebra, to have a semigroup structure (leading, among other things, to KK-theory).

Multiplier algebras are also interesting from the point of view of the structure theory of $C^*$-algebras (for example, the question of whether an extension of a separable stable $C^*$-algebra by a separable stable $C^*$-algebra gives a stable extension algebra; see [19] and [22].) Among other things, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are separable $C^*$-algebras such that $\mathcal{M}(\mathcal{B}_1) \cong \mathcal{M}(\mathcal{B}_2)$ then $\mathcal{B}_1 \cong \mathcal{B}_2$ ([2]).

Still another point of view is that the multiplier algebra $\mathcal{M}(\mathcal{B})$ of a $C^*$-algebra $\mathcal{B}$ is similar to a von Neumann algebra (specifically, similar to the enveloping von Neumann algebra $\mathcal{B}^{**}$ of $\mathcal{B}$). Hence, it is natural to study the unitary group of $\mathcal{M}(\mathcal{B})$ in analogy with (de la Harpe's work on) the unitary group of a von Neumann algebra.

Like von Neumann algebras, multiplier algebras have more than one interesting topology. The multiplier algebra $\mathcal{M}(\mathcal{B})$ has a natural norm topology and another natural topology called the "strict topology". (The strict topology on $\mathcal{M}(\mathcal{B})$ is given by the seminorms $\{\|\cdot\|_b\}_{b\in\mathcal{B}}$ where for all $m \in \mathcal{M}(\mathcal{B})$, $\|m\|_b := \|mb\| + \|bm\|$. Recall that $\mathcal{B}$ is an ideal of $\mathcal{M}(\mathcal{B})$.) The strict topology on $\mathcal{M}(\mathcal{B})$ is similar to the ultrastrong* topology on a von Neumann algebra. Indeed, in the case that $\mathcal{B} = \mathcal{K}$, the algebra of compact operators, $\mathcal{M}(\mathcal{K}) = \mathbb{B}(\mathcal{H})$ and the strict topology is the ultrastrong* topology on $\mathbb{B}(H)$. We note that on the unitary group of a von Neumann algebra, the ultrastrong, ultrastrong*, strong and weak operator topologies and also the weak* topology coincide. Hence, it is natural (and interesting from more than one point of view) to study the unitary group of $\mathcal{M}(\mathcal{B})$, given the strict topology.

We also note that if $\mathcal{B}$ is stable then the unitary group of $\mathcal{M}(\mathcal{B})$ cannot be amenable in the norm topology. This is because a unital $C^*$-algebra whose unitary group is amenable in the norm topology must have a tracial state; but if $\mathcal{B}$ is stable then $\mathcal{M}(\mathcal{B})$ must contain a unital copy of the Cuntz algebra $O_2$ and hence $\mathcal{M}(\mathcal{B})$ cannot have a tracial state.

Our first result in this paper is the following:

THEOREM 1.1 (*cf.* Theorem 3.4). *Suppose that $\mathcal{A}$ is a simple, unital, separable $C^*$-algebra. Then $\mathcal{A}$ is nuclear if and only if the unitary group of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, given the strict topology, is amenable.* [1]

We note that the "if" direction of Theorem 1.1 follows from de la Harpe's (or Paterson's) result since the strict topology on $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is stronger than the weak* topology induced from $(\mathcal{A}\otimes\mathcal{K})^{**} \supseteq \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. (See the proof of (b) implies (a) in Theorem 3.4.)

Another interesting notion for topological groups is the notion of extreme amenability. A topological group $G$ is said to be extremely amenable if every continuous action of $G$ on a compact Hausdorff topological space has a fixed point. This is a very strong notion of amenability. Indeed, it is so strong that no non-trivial locally compact group can be extremely amenable ([26]). However, Gromov and Milman showed that $U(\mathcal{H})$ (the unitary group on a separable infinite dimensional Hilbert space, given the strong operator topology) is extremely amenable ([15]). Moreover, Giordano and Pestov have shown that many interesting non-locally compact groups (*e.g.*, the unitary group of an infinite-dimensional injective factor given the strong operator topology) also are extremely amenable ([12], [13]).

In this paper, we exhibit another large class of extremely amenable groups:

THEOREM 1.2 (*cf.* Theorem 3.4). *Let $\mathcal{A}$ be a unital simple separable $C^*$-algebra. Then $\mathcal{A}$ is nuclear if and only if the unitary group of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, given the strict topology, is extremely amenable.*

Since we are dealing with multiplier algebras, there naturally are connections with the theory of extensions. In extension theory, an important class of extensions are the *absorbing extensions*. (Absorbing extensions are defined in Section 2. Separable $C^*$-algebras always have such extensions. See [25].) Absorbing extensions play an interesting role in a number of places. Among other things, they give a clean characterization of KK-theory ([18]). They are also useful in the classification program for simple nuclear $C^*$-algebras (*e.g.*, stable uniqueness

[1] In Theorem 1.1, separability is required since we use the theory of absorbing extensions. See [3, Theorem 15.12.4] or [11, Theorem 6]. On the other hand, simplicity is not really needed in Theorem 1.1. Indeed, in Theorem 3.4, the equivalence of statements (a), (b), (c), (d) and (f) do not require simplicity of $\mathcal{A}$. Simplicity is only required to ensure that the Lin extension is nuclearly absorbing. See, for example, [11, Theorem 17].

Also, in Theorem 1.1 (as well as Theorems 1.2, 1.3, 1.4 and 3.4) one can replace the unitality of $\mathcal{A}$ by the weaker assumption that $\mathcal{A}$ has a nonzero projection. This is to ensure that $\mathcal{A}\otimes\mathcal{K}$ has an approximate unit consisting of projections, which is an ingredient in the proof of the main result. (See the argument of Theorem 3.4.)

theorems—see, for example, [6], [20] and [21]). Most recently, the theory of absorbing extensions has been shown to have even more direct connections to the structure theory of $C^*$-algebras (see [19]), and also the classification of interesting classes of $C^*$-algebras with unique nontrivial ideal (*e.g.*, $C^*$-algebras coming from certain dynamical systems (Matsumoto algebras associated to primitive aperiodic substitutional subshifts); see [4], [9], [22]).

Two classes of extensions which have been rather useful to classification theory and extension theory are Lin's extensions and Kasparov's extensions (the definitions are in the next section). These extensions are absorbing in the case that the relevant $C^*$-algebras are nuclear (see, for example, [6], [11], [20] and [21]). It turns out that even more is true:

THEOREM 1.3 (*cf.* Theorem 3.4). *Let $\mathcal{A}$ be a simple, unital, separable $C^*$-algebra. Then the following conditions are equivalent:*
(i) *$\mathcal{A}$ is a nuclear $C^*$-algebra.*
(ii) *The Kasparov extension, of $\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$, is absorbing.*
(iii) *The Lin and Kasparov extensions, of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$, are approximately unitarily equivalent, with unitaries coming from $\mathcal{M}(A \otimes \mathcal{K})$.*[2]

Finally, the previous results indicate that characterizations of absorbing extensions, for nuclear $C^*$-algebras, give characterizations of nuclearity using absorption. Among other things, nuclearity is equivalent to the existence of sufficiently many absorbing extensions. Combining the preceding result with work of Elliott and Kucerovsky ([11]), we get the following result (definitions of terms given in Sections 2 and 4):

THEOREM 1.4 (*cf.* Theorem 4.2). *Let $\mathcal{A}$ be a simple, unital, separable $C^*$-algebra. Then the following conditions are equivalent:*
(i) *$\mathcal{A}$ is a nuclear $C^*$-algebra.*
(ii) *Every purely large extension of $\mathcal{A} \otimes \mathcal{K}$ is absorbing.*
(iii) *Every properly purely large extension of $\mathcal{A} \otimes \mathcal{K}$ is absorbing.*

We end this introduction with the following question:

QUESTION 1.5. *Let $\mathcal{A}$ be a unital separable simple nuclear $C^*$-algebra. Let $G := U\big(\mathcal{M}(\mathcal{A} \otimes \mathcal{K})\big)$ denote the unitary group of the multiplier algebra of the stabilization of $\mathcal{A}$, with the strict topology. Then does $G$ have property $T$?*

## 2. Absorbing extensions.

In this section, we recall some basic facts about absorbing extensions. A nonunital extension $\tau$ of a stable $C^*$-algebra $\mathcal{B}$ by a $C^*$-algebra $\mathcal{A}$ is said to be *absorbing* if $\tau +_{BDF} \rho$ is equivalent to $\tau$ for every trivial extension $\rho$ of $\mathcal{B}$ by $\mathcal{A}$. Here, the sum is the BDF-sum and equivalence is via unitaries coming from $\mathcal{M}(\mathcal{B})$. A unital extension $\tau$ is said to be *absorbing*

[2] It is an interesting open question whether, for unital separable simple $\mathcal{A}$, nuclearity of $\mathcal{A}$ is equivalent to the Lin extension of $\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$ being absorbing.

if the same property holds, except that the trivial extensions $\rho$ are restricted to those that have a lifting to a unital $*$-homomorphism from $\mathcal{A}$ to $\mathcal{M}(\mathcal{B})$.

We now recall some definitions from [11]. Recall that an extension $0 \to \mathcal{B} \to \mathcal{C} \to \mathcal{A} \to 0$ is *nuclearly trivial* if the splitting homomorphism $\rho\colon \mathcal{A} \to \mathcal{C}$ can be chosen to be weakly nuclear as defined by Kirchberg in [17]: For every $b \in \mathcal{B}$, the map $\mathcal{A} \mapsto \mathcal{B} \subseteq \mathcal{C}\colon a \mapsto b\rho(a)b^*$ is nuclear. (Recall that a $C^*$-algebra map is nuclear if it factors approximately through finite-dimensional $C^*$-algebras, by means of completely positive contractions, in the sense of pointwise convergence in norm.) An (unital) extension of $\mathcal{B}$ by $\mathcal{A}$ is correspondingly called *nuclearly absorbing* if it absorbs every nuclearly trivial extension (with a unital lifting $*$-homomorphism from $\mathcal{A}$ to $\mathcal{M}(\mathcal{B})$) of $\mathcal{B}$ by $\mathcal{A}$ ([11]). Finally, we note that in the presence of nuclearity, absorbing is the same as nuclearly absorbing. More precisely (notation as above), if either $\mathcal{A}$ or $\mathcal{B}$ is a nuclear $C^*$-algebra then every nuclearly absorbing (unital) extension of $\mathcal{B}$ by $\mathcal{A}$ is an absorbing (unital) extension of $\mathcal{B}$ by $\mathcal{A}$.

Absorbing extensions are interesting from more than one point of view. They give a clean characterization of KK-theory ([18]), and they have been useful for classification theory ([6], [20], [21]). More recently, there have been very direct connections with the structure of $C^*$-algebras ([19], [22]) as well as the classification of certain interesting nonsimple $C^*$-algebras—including purely infinite $C^*$-algebras with a unique nontrivial ideal and Matsumoto algebras associated with primitive aperiodic substitutional subshifts (see [4], [9] and [22]).

For the purposes of this paper, we will only require the following result about trivial absorbing extensions, which is a generalization of Voiculescu's Theorem.

Theorem 2.1. *Let $\mathcal{B}$ be a separable stable $C^*$-algebra. Let $\mathcal{C}$ be a unital separable $C^*$-algebra. Suppose that either $\mathcal{B}$ or $\mathcal{C}$ is nuclear. Let $\phi, \psi\colon \mathcal{C} \to \mathcal{M}(\mathcal{B})$ be two trivial absorbing extensions such that either both $\phi$ and $\psi$ are unital or both are nonunital. Then there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{M}(\mathcal{B})$ such that*
(i) $\phi(c) - u_n\psi(c)u_n^* \in \mathcal{B}$ *for all* $n$ *and for all* $c \in \mathcal{C}$, *and*
(ii) $\|\phi(c) - u_n\psi(c)u_n^*\| \to 0$ *as* $n \to \infty$ *for all* $c \in \mathcal{C}$.

A proof of this result can be found in, among other places, [11] (see the arguments of [11, Lemma 10, Lemma 11 and Theorem 6]). We note that for Voiculescu's original theorem, $\mathcal{B}$ (above) is the algebra $\mathcal{B} = \mathcal{K}$ of compact operators.

Kasparov and Lin constructed families of extensions (which are absorbing when the appropriate $C^*$-algebras are nuclear). These extensions have been important for extension theory and classification theory (see, for example, [6], [20], [21] and [3]). For our purposes, we only need special cases of their extensions. We will use the terminology "Lin extension" and "Kasparov extension" to denote these cases.

Definition 2.1. Let $\mathcal{A}$ be a unital, simple, separable $C^*$-algebra. Let $\mathrm{id}\colon \mathcal{A} \to \mathcal{A}$ denote the identity map. Consider the unital trivial extension of

$\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$ given by the $*$-homomorphism $\mathrm{id}\otimes 1_{\mathcal{M}(\mathcal{K})}\colon \mathcal{A}\to\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, which is called the *Lin extension* of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$.

Definition 2.2. Let $\mathcal{A}$ be a unital, simple, separable $C^*$-algebra and let $\mathcal{B}$ be a separable $C^*$-algebra. Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space, and let $\phi\colon \mathcal{A}\to\mathbb{B}(\mathcal{H})$ be a unital $*$-representation. Consider the unital trivial extension $\tau$ of $\mathcal{B}\otimes\mathcal{K}$ by $\mathcal{A}$ given by composing the $*$-homomorphism $1_{\mathcal{M}(\mathcal{B})}\otimes\phi\colon \mathcal{A}\to\mathcal{M}(\mathcal{B})\otimes\mathbb{B}(\mathcal{H})$ with the natural inclusion map $\mathcal{M}(\mathcal{B})\otimes\mathbb{B}(\mathcal{H})\subseteq\mathcal{M}(\mathcal{B}\otimes\mathcal{K})$. $\tau$ is called a *Kasparov extension* of $\mathcal{B}\otimes\mathcal{K}$ by $\mathcal{A}$.

The extensions of Lin and Kasparov are important for the K-theoretic classification of simple unital separable nuclear $C^*$-algebras (see [6], [20] and [21]). Among other things, the results of Lin and Kasparov imply that the Lin and Kasparov extensions are nuclearly absorbing. Since we will be using this, we will state it as a theorem. Proofs can be found in [11], [20] and [21]. ([11] contains a unified treatment.)

Theorem 2.2. *Let $\mathcal{A}$ be a unital simple separable $C^*$-algebra and let $\mathcal{B}$ be a separable $C^*$-algebra. Then we have the following:*
(i) *The Lin extension, of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$, is nuclearly absorbing.*
(ii) *The Kasparov extension, of $\mathcal{B}\otimes\mathcal{K}$ by $\mathcal{A}$, is nuclearly absorbing.*

The suspended versions of Lin's extensions are also of great interest. (See [6].)

Definition 2.3. Let $\mathcal{A}$ be a unital, separable, simple $C^*$-algebra, and let $S\mathcal{A} := C_0(0,1)\otimes\mathcal{A}$ denote the suspension of $\mathcal{A}$.

The trivial extension of $S\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$, given by $\mathcal{A}\to\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})\colon a\mapsto 1_{\mathcal{M}(C_0(0,1))}\otimes a\otimes 1_{\mathcal{M}(\mathcal{K})}$, is called the *suspended Lin extension* of $S\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$.

Theorem 17 (iii) of [11] also shows that the suspended Lin extension is nuclearly absorbing. Since we will cite this, we again state it as a theorem.

Theorem 2.3. *Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra. Then the suspended Lin extension, of $S\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$, is nuclearly absorbing.*

From Theorems 2.1, 2.2 and 2.3, we see that for a unital separable simple nuclear $C^*$-algebra $\mathcal{A}$, the Kasparov and suspended Lin extensions, of $S\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$, are approximately unitarily equivalent (with unitaries in $\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})$).

## 3. Amenability of $U\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$.

Our proof of amenability and extreme amenability involves the use of a locally dense net.

Definition 3.1. Let $G$ be a topological group, and let $\Pi$ be a uniform structure on $G$ that is compatible with the topology on $G$. Let us say that a net of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of $G$ is *locally dense* in $G$ if for every entourage $\mathcal{E}\in\Pi$ and every finite set $\{g_i\}_{i=1}^n$ in $G$, there is a $\lambda_0\in\Lambda$ such that for $\lambda\geq\lambda_0$ and for $1\leq i\leq n$, $(\{g_i\}\times H_\lambda)\cap\mathcal{E}\neq\phi$.

The next lemma will be used in proving (extreme) amenability. The proofs for amenability and extreme amenability are almost the same, and are exercises in point-set topology.

LEMMA 3.1. *Let $G$ be a topological group, and let $\Pi$ be a uniform structure on $G$ that is compatible with the topology on $G$. If $(G, \Pi)$ has a locally dense net $\{H_\lambda\}_{\lambda \in \Lambda}$ consisting of (extremely) amenable subgroups, then $G$ is (extremely) amenable.*

We need the following well-known perturbation result (the proof of which can be found in, say, [7, Lemma II.3.2]—and many other places).

LEMMA 3.2. *There exists a continuous, positive real-valued function $h$ with $h(0) = 0$ such that the following statement holds. Let $\mathcal{A}_0$ be a unital $C^*$-algebra, and let $a$ be an element of $\mathcal{A}_0$. Suppose that $\|a\|$ is within $\delta$ of $1$, and suppose that $aa^*a$ is within $\delta$ of $a$. Then $a$ is within $h(\delta)$ of a partial isometry in $\mathcal{A}_0$.*

Let $\mathcal{A}$ be a unital $C^*$-algebra. For every $b \in \mathcal{A} \otimes \mathcal{K}$ and for every $\epsilon > 0$, we can find an isometry $S \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $\|S^*bS\| < \epsilon$. From this and the argument of [11, p. 401, paragraph three], we have the following result.

LEMMA 3.3. *Let $\mathcal{A}$ be a unital $C^*$-algebra. Suppose that $p$ is a projection in $\mathcal{A} \otimes \mathcal{K}$. Then $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - p$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.*

Finally, let us recall what is meant by "approximate unitary equivalence". Let $\tau, \rho\colon \mathcal{C} \to \mathcal{M}(\mathcal{B})$ be two trivial extensions of $\mathcal{B}$ by $\mathcal{C}$. We say that $\tau$ and $\rho$ are *approximately unitarily equivalent* if there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{M}(\mathcal{B})$ such that $\{u_n \tau(\,\cdot\,) u_n^*\}_{n=1}^{\infty}$ converges pointwise to $\rho(\,\cdot\,)$.

THEOREM 3.4. *Let $\mathcal{A}$ be a unital, simple, separable $C^*$-algebra. Denote by $G := U\big(\mathcal{M}(\mathcal{A} \otimes \mathcal{K})\big)$ the unitary group of the multiplier algebra of $\mathcal{A} \otimes \mathcal{K}$, with the strict topology. The following conditions are equivalent:*
(a) *$\mathcal{A}$ is a nuclear $C^*$-algebra.*
(b) *$G$ is an amenable topological group.*
(c) *$G$ is an extremely amenable topological group.*
(d) *The Kasparov extension, of $\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$, is absorbing.*
(e) *The Lin and Kasparov extensions, of $\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$, are approximately unitarily equivalent.*
(f) *The Kasparov extension, of $S\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$, is absorbing.*
(g) *The suspended Lin extension and the Kasparov extension, of $S\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$, are approximately unitarily equivalent.*

PROOF. First, it follows by Theorem 2.1, Theorem 2.2, and Theorem 2.3 that (a) implies (d), (a) implies (e), (a) implies (f), and (a) implies (g). Also, that (c) implies (b) is immediate.

Next we prove that (b) implies (a). Since the weak$^*$ topology (from $(\mathcal{A}\otimes\mathcal{K})^{**}$) on $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is weaker than the strict topology, if $G = U\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$ is amenable in the strict topology then $U\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$ is amenable in the weak$^*$ topology. Hence, since $U\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$ is weak$^*$-dense in $U\big((\mathcal{A}\otimes\mathcal{K})^{**}\big)$, the latter group is amenable in the weak$^*$ topology. By de la Harpe's result ([8]), $(\mathcal{A}\otimes\mathcal{K})^{**}$ is then an amenable von Neumann algebra. Hence, by Choi–Effros ([5]), $\mathcal{A}\otimes\mathcal{K}$ is a nuclear $C^*$-algebra. Hence, $\mathcal{A}$ is a nuclear $C^*$-algebra.

We now prove that (d) implies (c). Let $b$ be a strictly positive element of $\mathcal{A}\otimes\mathcal{K}$ such that $\|b\| \leq 1$. Denote by $d$ the metric on $G = U\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$ given by $d(V,W) = \|Vb - Wb\| + \|bV - bW\|$. Then $d$ induces the strict topology on $G$.

Let $\epsilon > 0$ be given. Let $U_1, U_2, \ldots, U_N$ be a finite set of unitaries in $G = \mathcal{U}\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$. Since $\mathcal{A}$ is unital, we may choose a countable approximate unit $\{e_k\}_{k=1}^{\infty}$ for $\mathcal{A}\otimes\mathcal{K}$ consisting of an increasing sequence of projections. For simplicity, let us assume that for each $k$, $e_k$ has the form $e_k = 1_{\mathcal{A}}\otimes p_k$, where $p_k$ is a (finite rank) projection in $\mathcal{K}$. (Hence, $\{p_k\}_{k=1}^{\infty}$ is an increasing sequence of projections in $\mathcal{K}$.) Now choose an integer $K_1$ such that the following holds:

(∗) For all $k \geq K_1$, the elements $b$, $be_k$, $e_kb$ and $e_kbe_k$ are all within $\delta_1$ of each other.

Since $\{e_k\}_{k=1}^{\infty}$ is an approximate unit for $\mathcal{A}\otimes\mathcal{K}$, by Lemma 3.2, choose $K_2 > K_1$ and, for each $i$, choose a partial isometry $v_i \in e_{K_2}(\mathcal{A}\otimes\mathcal{K})e_{K_2}$, such that

(3.1) (i) $e_{K_1}v_i$ is within $\delta_1$ of $e_{K_1}U_i$,
(ii) $v_ie_{K_1}$ is within $\delta_1$ of $U_ie_{K_1}$,
(iii) both the initial and range projections of $v_i$ contain $e_{K_1}$.

Now, for $1 \leq i \leq N$, $e_{K_2} - v_i(v_i)^*$ and $e_{K_2} - (v_i)^*v_i$ give the same class (of a projection) in $K_0(\mathcal{A}\otimes\mathcal{K})$. Hence, since $\{e_k\}_{k=1}^{\infty}$ is an approximate unit for $\mathcal{A}\otimes\mathcal{K}$, let $K_3 > K_2$ be such that $e_{K_3} - v_i(v_i)^*$ is Murray–von Neumann equivalent to $e_{K_3} - (v_i)^*v_i$ in $\mathcal{A}\otimes\mathcal{K}$, for all $i$. Hence, for $1 \leq i \leq N$, let $w_i$ be a partial isometry in $\mathcal{A}\otimes\mathcal{K}$ such that $w_i$ has initial projection $e_{K_3} - (v_i)^*v_i$ and range projection $e_{K_3} - v_i(v_i)^*$. Note that for $1 \leq i \leq N$, $v_i + w_i$ is a unitary of $e_{K_3}(\mathcal{A}\otimes\mathcal{K})e_{K_3}$.

Recall that $e_{K_3}$ has the form $1_{\mathcal{A}}\otimes p_{K_3}$ where $p_{K_3}$ is a projection in $\mathcal{K}$. Hence, $e_{K_3}(\mathcal{A}\otimes\mathcal{K})e_{K_3}$ is ∗-isomorphic to $\mathbb{M}_{\mathrm{rank}(p_{K_3})}(\mathcal{A})$, where $\mathrm{rank}(p_{K_3})$ is the rank of $p_{K_3}$.

To simplify notation, we henceforth let "$m$" denote the rank of the projection $p_{K_3}$.

Let $\phi\colon \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a unital ∗-representation. Extend this in the natural way to a unital ∗-homomorphism $\phi_m\colon \mathbb{M}_m(\mathcal{A}) \to \mathbb{M}_m\big(\mathbb{B}(\mathcal{H})\big)$. Since $\mathbb{M}_m(\mathcal{A}) \cong e_{K_3}(\mathcal{A}\otimes\mathcal{K})e_{K_3}$ and $\mathbb{M}_m\big(\mathbb{B}(\mathcal{H})\big) \cong \mathbb{B}(\mathcal{H}^m) \cong \mathbb{B}(\mathcal{H})$, we get a unital ∗-representation $\phi_m\colon e_{K_3}(\mathcal{A}\otimes\mathcal{K})e_{K_3} \to \mathbb{B}(\mathcal{H})$ (which we denote by the same symbol). We then get the unital ∗-homomorphism $1_{\mathcal{A}}\otimes\phi_m\colon e_{K_3}(\mathcal{A}\otimes\mathcal{K})e_{K_3} \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that

(3.2) (i) $P_n \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ for all $n \geq 1$,
(ii) $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = \sum_{n=1}^{\infty} P_n$, where the sum converges in the strict topology on $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

For each $n \geq 1$, let $S_n \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be an isometry with range projection $P_n$. Let $\psi\colon e_{K_3}(A \otimes \mathcal{K})e_{K_3} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the trivial extension given by $\psi(a) := \sum_{l=1}^{\infty} S_{2l} a S_{2l}^*$. Also, let $T \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be an isometry with range projection $TT^* = \sum_{l=0}^{\infty} S_{2l+1}S_{2l+1}^*$. Let $\tilde{\phi}_m\colon e_{K_3}(\mathcal{A} \otimes \mathcal{K})e_{K_3} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the trivial extension given by $\tilde{\phi}_m(a) := T\big(1_{\mathcal{A}} \otimes \phi_m(a)\big)T^*$.

Now by hypothesis, the Kasparov extension $1_{\mathcal{A}}\otimes\phi$, of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$, is absorbing. Hence, the extension $1_{\mathcal{A}} \otimes \phi_m$, of $\mathcal{A} \otimes \mathcal{K}$ by $\mathbb{M}_m(\mathcal{A})$, is absorbing. Hence, let $V \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be a unitary and for each $1 \leq i \leq N$, let $b_i \in \mathcal{A}\otimes\mathcal{K}$ be such that

$$\psi(v_i + w_i) + \tilde{\phi}_m(v_i + w_i) + b_i = V\big(1_{\mathcal{A}} \otimes \phi_m(v_i + w_i)\big)V^*.$$

Since $\sum_{n=1}^{\infty} P_n = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, where the sum converges in the strict topology, there exists $M \geq 1$ such that for all $n \geq M$ and for $1 \leq i \leq N$, $\|P_n b_i\|$, $\|b_i P_n\|$ both have norm strictly less than $\delta_1$. Hence, it follows that for $1 \leq i \leq N$,

$$(**) \qquad S_{2M}(v_i + w_i)S_{2M}^* \\ = P_{2M}\big(\psi(v_i + w_i) + \tilde{\phi}_m(v_i + w_i)\big) \\ = \big(\psi(v_i + w_i) + \tilde{\phi}_m(v_i + w_i)\big)P_{2M}, P_{2M}V\big(1_{\mathcal{A}} \otimes \phi_m(v_i + w_i)\big)V^*,$$

and $V\big(1_{\mathcal{A}} \otimes \phi_m(v_i + w_i)\big)V^*P_{2M}$ are all within $2\delta_1$ of each other in norm.

Multiplying the operators in $(**)$ by $S_{2M}e_{K_3}$ and multiplying by $e_{K_3}S_{2M}^*$, we get that for $1 \leq i \leq N$,

(3.3) (i) $S_{2M}(v_i + w_i)e_{K_3}$ is within $2\delta_1$ of $V\big(1 \otimes \phi_m(v_i + w_i)\big)V^*S_{2M}e_{K_3}$,
(ii) $e_{K_3}(v_i + w_i)S_{2M}^*$ is within $2\delta_1$ of $e_{K_3}S_{2M}^*V\big(1 \otimes \phi_m(v_i + w_i)\big)V^*$.

From (3.3)(i), we have that for $1 \leq i \leq N$, $P_{2M}V\big(1\otimes\phi_m(v_i+w_i)\big)V^*S_{2M}e_{K_3}$ is within $4\delta_1$ of $V\big(1 \otimes \phi_m(v_i + w_i)\big)V^*S_{2M}e_{K_3}$. Hence,

$$\big\|(1 - P_{2M})V\big(1 \otimes \phi_m(v_i + w_i)\big)V^*S_{2M}e_{K_3}\big\| < 4\delta_1. \tag{3.4}$$

From (3.3)(i) again, we have that for $1 \leq i \leq N$, $(v_i + w_i)e_{K_3}$ is within $2\delta_1$ of $S_{2M}^*V\big(1\otimes\phi_m(v_i+w_i)\big)V^*S_{2M}e_{K_3}$. Hence, $e_{K_3}S_{2M}^*V\big(1\otimes\phi_m(v_i+w_i)\big)V^*S_{2M}e_{K_3}$ is within $4\delta_1$ of $S_{2M}^*V\big(1\otimes\phi_m(v_i+w_i)\big)V^*S_{2M}e_{K_3}$. Hence, recalling that $P_{2M} = S_{2M}S_{2M}^*$, we have that

$$\big\|(P_{2M} - S_{2M}e_{K_3}S_{2M}^*)V\big(1 \otimes \phi_m(v_i + w_i)\big)V^*S_{2M}e_{K_3}\big\| < 4\delta_1.$$

From this and (3.4), we have that

$$\left\|(1 - S_{2M}e_{K_3}S_{2M}^*)V\big(1 \otimes \phi_m(v_i + w_i)\big)V^* S_{2M}e_{K_3}\right\| < 8\delta_1. \tag{3.5}$$

Applying a similar argument to (3.3)(ii), we have that

$$\left\|e_{K_3}S_{2M}^* V\big(1 \otimes \phi_m(v_i + w_i)\big)V^*(1 - S_{2M}e_{K_3}S_{2M}^*)\right\| < 8\delta_1. \tag{3.6}$$

Now, $e_{K_3}S_{2M}^*V$ is a partial isometry in $\mathcal{A}\otimes\mathcal{K}$. Hence, by Lemma 3.3, we can extend $e_{K_3}S_{2M}^*V$ to a unitary $W \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. From $(*)$, (3.1), (3.3), and (3.5), we have that for $1 \leq i \leq N$,

$$\begin{aligned}
&\left\|U_i b - W\big(1 \otimes \phi_m(v_i + w_i)\big)W^* b\right\| \\
&\quad\leq \left\|U_i e_{K_1} b - W\big(1 \otimes \phi_m(v_i + w_i)\big)W^* e_{K_1} b\right\| \\
&\qquad + \left\|\Big(U_i - W\big(1 \otimes \phi_m(v_i + w_i)\big)W^*\Big)(b - e_{K_1}b)\right\| \\
&\quad< \left\|U_i e_{K_1} b - W\big(1 \otimes \phi_m(v_i + w_i)\big)W^* e_{K_1} b\right\| + 2\delta_1 \\
&\quad\leq \|U_i e_{K_1} b - v_i e_{K_1} b\| \\
&\qquad + \left\|(v_i + w_i)e_{K_1} b - S_{2M}^* V\big(1 \otimes \phi_m(v_i + w_i)\big)V^* S_{2M} e_{K_1} b\right\| \\
&\qquad + \Big\|S_{2M}^* V\big(1 \otimes \phi_m(v_i + w_i)\big)V^* S_{2M} e_{K_1} b \\
&\qquad\qquad - e_{K_3} S_{2M}^* V\big(1 \otimes \phi_m(v_i + w_i)\big)V^* S_{2M} e_{K_1} b\Big\| \\
&\qquad + \left\|(e_{K_3} S_{2M}^* V - W)\big(1 \otimes \phi_m(v_i + w_i)\big)V^* S_{2M} e_{K_1} b\right\| + 2\delta_1 \\
&\quad< \delta_1 + 2\delta_1 + 8\delta_1 + 8\delta_1 + 2\delta_1 \\
&\quad= 21\delta_1.
\end{aligned}$$

By a similar argument, we have that for $1 \leq i \leq N$,

$$\|bU_i - bW(1 \otimes \phi_m)(v_i + w_i)W^*\| < 21\delta_1.$$

Hence, if we choose $\delta_1 = \epsilon/42$, then for $1 \leq i \leq N$,

$$d(U_i, W(1 \otimes \phi_m)(v_i + w_i)W^*) < \epsilon.$$

Since $\epsilon$ is arbitrary, we see that the family $\{g(1_{\mathcal{A}} \otimes U(\mathcal{M}(\mathcal{K})))g^{-1}\}_{g\in G}$ naturally gives a locally dense net of subgroups of $G$. Moreover, since $1_{\mathcal{A}} \otimes U(\mathcal{M}(\mathcal{K})) \cong U\big(\mathbb{B}(\mathcal{H})\big)$ is an extremely amenable topological subgroup of $G$ ([15]), $g(1_{\mathcal{A}}\otimes U(\mathcal{M}(\mathcal{K})))g^{-1}$ is an extremely amenable topological subgroup of $G$ for every $g \in G$. (Note that the strict topology on $1_{\mathcal{A}}\otimes U\big(\mathcal{M}(\mathcal{K})\big) \subseteq \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ agrees with the strict topology on $U(\mathcal{M}(\mathcal{K})) \subseteq \mathcal{M}(\mathcal{K})$ since $U(\mathcal{M}(\mathcal{K}))$ is bounded.) Therefore, by Lemma 3.1, $G$ is an extremely amenable topological group. This completes the proof of (d) implies (c).

The proof that (e) implies (c) is similar to that of (d) implies (c). We sketch the argument for the benefit of the reader. Let $b$ and $d(\cdot,\cdot)$ be as in the argument of (d) implies (c). Let $\epsilon > 0$ be given. Let $U_1, U_2, \ldots, U_N$ be a finite set of unitaries in $G =_{df} U\big(\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\big)$. As in the proof of (d) implies (c), we prove that $U_1, U_2, \ldots, U_N$ can be approximated within $\epsilon$ by an extremely amenable subgroup of $G$, with respect to the metric $d$.

Let $\{e_k\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$, $K_1$, $K_2$, $K_3$, $v_i$, $w_i$ $(1 \leq i \leq N)$ and $m$ be as in the proof of (d) implies (c). Let $\{q_k\}_{k=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $\mathcal{A}\otimes\mathcal{K}$ such that

(i) $q_1 = p_{K_3}$,
(ii) $\sum_{k=1}^{\infty} q_k = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ where the sum converges in the strict topology on $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, and
(iii) $q_k \sim q_1$ for all $k \geq 1$.

For each $i \geq 1$, let $s_i \in \mathcal{A}\otimes\mathcal{K}$ be a partial isometry with initial projection $q_1$ and range projection $q_i$. In particular, let us choose $s_1 = q_1$.

Consider the unital $*$-homomorphism

$$\tilde{\psi}\colon p_{K_3}(\mathcal{A}\otimes\mathcal{K})p_{K_3} \cong \mathbb{M}_m(\mathcal{A}) \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K}) \cong \mathcal{M}\big(\mathbb{M}_m(\mathcal{A})\otimes\mathcal{K}\big)$$

that is given by $\tilde{\psi}(a) =_{df} \sum_{k=1}^{\infty} s_i a s_i^*$ for all $a \in q_1(\mathcal{A}\otimes\mathcal{K})q_1 = p_{K_3}(\mathcal{A}\otimes\mathcal{K})p_{K_3} \cong \mathbb{M}_m(\mathcal{A})$. Note that $\tilde{\psi}$ is a Lin extension of $\mathbb{M}_m(\mathcal{A})\otimes\mathcal{K}$ by $\mathbb{M}_m(\mathcal{A})$. Now consider the Kasparov extension $1_{\mathcal{A}}\otimes\phi_m$, of $\mathbb{M}_m(\mathcal{A})\otimes\mathcal{K}$ by $\mathbb{M}_m(\mathcal{A})$, that is defined in the proof of (b) implies (a). Since the Lin and Kasparov extensions of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$ are approximately unitarily equivalent, the Lin and Kasparov extensions of $\mathbb{M}_m(\mathcal{A})\otimes\mathcal{K}$ by $\mathbb{M}_m(\mathcal{A})$ are approximately unitarily equivalent. Hence, $\tilde{\psi}$ and $1_{\mathcal{A}}\otimes\phi_m$ are approximately unitarily equivalent. The rest of the argument is similar to that of (b) implies (a), except that here the maps $\tilde{\psi}$ and $1_{\mathcal{A}}\otimes\phi_m$ replace the maps $\psi+\tilde{\phi}_m$ and $1_{\mathcal{A}}\otimes\phi_m$ (from the proof of (b) implies (a)) respectively.

We now prove that (f) implies (d). Let $K\colon \mathcal{A}\to\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be a Kasparov extension and let $\tau\colon\mathcal{A}\to\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be a trivial extension, of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$. Referring to the definition of Kasparov extension, let $\sigma\colon\mathcal{A}\to\mathbb{B}(\mathcal{H})$ be a unital $*$-homomorphism such that $K = 1_{\mathcal{A}}\otimes\sigma$. Then consider the Kasparov extension $SK\colon\mathcal{A}\to\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})$ that is given by $SK = 1_{\mathcal{M}(S\mathcal{A})}\otimes\sigma$. Since $S\mathcal{A}\otimes\mathcal{K}$ is an ideal of $C[0,1]\otimes\mathcal{A}\otimes\mathcal{K}$, there is a natural unital $*$-homomorphism $\mathcal{M}(C[0,1]\otimes\mathcal{A}\otimes\mathcal{K})\to\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})$. Composing this with the natural unital $*$-homomorphism $\mathcal{M}(A\otimes\mathcal{K})\to\mathcal{M}(C[0,1]\otimes\mathcal{A}\otimes\mathcal{K})$, we get a natural unital $*$-homomorphism $\iota\colon\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\to\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})$. From this, we get a unital trivial extension $S\tau := \iota\circ\tau\colon\mathcal{A}\to\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})$ of $S\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{A}$. Fix $s\in(0,1)$. Let $\Psi_s\colon\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})\to\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ denote the unital, surjective $*$-homomorphism gotten by point evaluation at $s$. Clearly, $K = \Psi_s\circ SK$ and $\tau = \Psi_s\circ S\tau$. By hypothesis, $SK$ is absorbing. Hence, let $S_1, S_2\in\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be isometries with $S_1S_1^* + S_2S_2^* = 1$ and let $U\in\mathcal{M}(S\mathcal{A}\otimes\mathcal{K})$ be a unitary such that for every $a\in\mathcal{A}$,

$$S_1SK(a)S_1^* + S_2S\tau(a)S_2^* - USK(a)U^* \in S\mathcal{A}\otimes\mathcal{K}.$$

Applying $\Psi_s$ to all expressions, we obtain that for every $a \in \mathcal{A}$,

$$\Psi_s(S_1)K(a)\Psi_s(S_1)^* + \Psi_s(S_2)\tau(a)\Psi_s(S_2)^* - \Psi_s(U)K(a)\Psi_s(U)^* \in \mathcal{A} \otimes \mathcal{K}.$$

Since $\tau$ is arbitrary, we have that $K$ is an absorbing extension.

The proof of (g) implies (e) is similar to the proof of (f) implies (d). Roughly speaking, by a point evaluation argument similar to that of (f) implies (d), we can show that since the Lin and Kasparov extensions of $S\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$ are approximately unitarily equivalent, the Lin and Kasparov extensions of $\mathcal{A} \otimes \mathcal{K}$ by $\mathcal{A}$ are approximately unitarily equivalent. □

## 4. Nuclear $C^*$-algebras are exactly the $C^*$-algebras with lots of absorbing extensions.

From the work of the previous section, it is clear that characterizations of absorbing extensions, for nuclear $C^*$-algebras, actually give characterizations of nuclearity using absorbing extensions.

First, recall that the results of [11] give characterizations of nuclearly absorbing extensions. To state them, we first recall two definitions:

Definition 4.1. Let $0 \to \mathcal{B} \to \mathcal{C} \to \mathcal{D} \to 0$ be an essential, unital extension of separable $C^*$-algebras.

(i) We say that this extension is *purely large* if for every $c \in \mathcal{C}^+$ that is not in $\mathcal{B}$, the hereditary subalgebra that it generates, $\overline{c\mathcal{B}c}$, contains a stable subalgebra that is full in $\mathcal{B}$.
(ii) We say that this extension is *properly purely large* if for every $c \in \mathcal{C}^+$ that is not in $\mathcal{B}$, $\overline{c\mathcal{B}c}$ is itself stable and full in $\mathcal{B}$.

We need to recall the definition of Kasparov extension in the nonsimple case. Let $\mathcal{A}$ be a unital separable $C^*$-algebra (which need not be simple) and let $\mathcal{B}$ be a separable $C^*$-algebra. An extension $\tau$ of $\mathcal{B} \otimes \mathcal{K}$ by $\mathcal{A}$ is called a *Kasparov extension* if it is defined as in Definition 2.2, except that, additionally, the $*$-homomorphism $\phi$ is required to have the property that if $a \in \mathcal{A}$ is nonzero then $\phi(a)$ is not an element of $\mathcal{K}$.

The following result essentially follows from [11].

Theorem 4.1. *Let $0 \to \mathcal{B} \otimes \mathcal{K} \to \mathcal{C} \to \mathcal{D} \to 0$ be an essential, unital extension of separable $C^*$-algebras. The following conditions are equivalent:*

(i) *The extension is nuclearly absorbing.*
(ii) *The extension is purely large.*
(iii) *The extension is properly purely large.*

Proof. That (i) is equivalent to (ii) is the main result of [11]. That (iii) implies (ii) is immediate.

We shall now show that (i) and (ii) together imply (iii). Denote by $\tau$ the given extension. Suppose that $\tau$ satisfies (i) and (ii). Since $\tau$ is nuclearly absorbing and since Kasparov extensions are nuclearly trivial, $\tau$ absorbs every Kasparov extension. Choosing an appropriate Kasparov extension (say, an infinite repeat) and using the Hjelmborg–Rørdam characterization of stability ([16, Theorem 2.1]; see also [24, Theorem 2.2]), one can show that for every $a \in \mathcal{D}^+ \setminus \{0\}$, if $c \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K})$ is a positive element which is a lifting of $\tau(a)$ then $\overline{c(\mathcal{B} \otimes \mathcal{K})c}$ is stable. From this and (2), for all $c \in \mathcal{C}^+ \setminus \mathcal{B} \otimes \mathcal{K}$, $\overline{c(\mathcal{B} \otimes \mathcal{K})c}$ is a stable full hereditary $C^*$-subalgebra of $\mathcal{B} \otimes \mathcal{K}$. □

From Theorem 4.1 and the results of the previous section, it follows that nuclear $C^*$-algebras are exactly the ones with lots and lots of absorbing extensions. Indeed, nuclear $C^*$-algebras are, in a sense, "maximal" with respect to absorbing extensions:

THEOREM 4.2. *Let $\mathcal{A}$ be a unital, separable, simple $C^*$-algebra. The following conditions are equivalent:*
(i) *$\mathcal{A}$ is a nuclear $C^*$-algebra.*
(ii) *Every purely large extension of $\mathcal{A} \otimes \mathcal{K}$ is absorbing.*
(iii) *Every properly purely large extension of $\mathcal{A} \otimes \mathcal{K}$ is absorbing.*

PROOF. It remains in view of Theorems 4.1 and 3.4 to note that, by [11, Theorem 17 (iii)], the Kasparov extension of $\mathcal{A} \otimes \mathcal{K}$ (for that matter, also the Lin extension) is always purely large. (Alternatively, by Theorem 2.2, the Kasparov and Lin extensions are nuclearly absorbing.) □

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